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CREATING EXACT MULTIPOLAR FIELDS IN ACCELERATING RF CAVITIES VIA AN AZIMUTHALLY MODULATED DESIGN

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Abstract

In this paper, we present a novel method for designing RF structures with specifically tailored multipolar field contributions. This has a range of applications, including the suppression of unwanted multipolar fields or the introduction of wanted terms, such as for quadrupole focusing. In this article, we outline the general design methodology and compare the expected results to 3D CST simulations.

INTRODUCTION

Many RF cavities are designed to operate in the transverse magnetic modes TM_{m10}, where the integer m denotes the azimuthal order, or multipolar component, of the EM field. Examples include RF cavities that operate in the monopolar TM_{010} mode to longitudinally accelerate beams [1], crab cavities that operate in a dipolar TM_{110} mode to provide head-tail rotations to a particle bunch [2], and cavities that operate in a quadrupolar TM_{210} mode to Landau damp transverse oscillations [3].

We define the notation $TM_{\{0m\}10}$ to refer to the case where the mode consists of a TM_{010} (monopole) field component as well as a TM_{m10} (multipolar) component. If an RF cavity operated in such a $TM_{\{0m\}10}$ mode, it would allow for a fine control over transverse beam dynamics whilst simultaneously accelerating the beam longitudinally. Such a cavity could introduce desired multipolar components (e.g. quadrupole multipolar terms to provide transverse focusing in CLIC's [4] accelerating RF cavities [5]) or negate unwanted multipolar components (e.g. unwanted dipole terms that are introduced by RF cavity systems using a single-port power coupler [6]) for little extra cost in terms of space.

This paper presents a systematic method for designing such RF cavities that operate in $TM_{\{0m\}10}$ modes and the comparison of simulation results from modelling such cavities in the 3D simulation software CST [7] to theory.

THEORY

The allowed form of electromagnetic modes supported by a general RF cavity can be determined from Maxwell's equations. Assuming the vacuum form of Maxwell's equations and that the E- and B-fields have harmonic time dependence, then the spatial component of the E-field of each of the allowed modes, $\vec{E}^{(l)}(\vec{r})$, must satisfy the Helmholtz equation [8]:

$$\left(\nabla^2 + \frac{\omega_l^2}{c^2}\right)\vec{E}^{(l)}(\vec{r}) = 0, \tag{1}$$

where ω_l is the resonant angular frequency of the allowed mode l, and c is the speed of light.

Equation 1 can be solved in the cylindrical coordinate system (r, θ, z). The general transverse magnetic (TM) solution of the longitudinal electric field, E_z, that has no longitudinal variation and solves the z-component of Eq. (1) is:

$$E_z(r,\theta,z) = \sum_{m=0}^{\infty}\left(\tilde{e}_m \cos m\theta + \tilde{f}_m \sin m\theta\right) J_m\left(\frac{\omega}{c}r\right), \tag{2}$$

where the coefficients $\tilde{e}_m$ and $\tilde{f}_m$ respectively denote the magnitude of the normal and skew terms of order m, and J_m represents the Bessel function of order m. In this paper, the simplification of ignoring skew terms is taken by setting all $\tilde{f}_m = 0$ such that Eq. (2) becomes:

$$E_z(r,\theta,z) = \sum_{m=0}^{\infty}\tilde{e}_m J_m\left(\frac{\omega}{c}r\right)\cos m\theta. \tag{3}$$

Panofsky-Wenzel Theorem

At this stage, all components of the B-field as well as the radial and azimuthal components of the E-field have been ignored. The reason for doing so is the Panofsky-Wenzel theorem [9] which states that the transverse beam dynamics resulting from a beam passing through an RF cavity can be exactly determined from complete knowledge of the longitudinal electric field throughout the cavity:

$$\Delta\vec{p}_\perp = -i\frac{q}{\omega}\int_0^L \nabla_\perp E_z dz, \tag{4}$$

where $\Delta\vec{p}_\perp$ is the change in transverse momentum, $\nabla_\perp$ is the transverse components of the del operator, q is the charge of the particle and L is the length of the RF cavity. Explicitly, the Panofsky-Wenzel theorem states that if E_z only contains certain azimuthal orders (e.g. all $\tilde{e}_m = 0$ except for the two orders $m = 1, 2$), then for small deviations from the beam axis the change in transverse momentum will also only consist of those same orders.

Boundary Conditions

In addition to having a longitudinal electric field of the form in Eq. (3), the allowed EM modes supported by an RF cavity of a specific shape must also satisfy the boundary

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conditions at the cavity surface. The general form of the boundary conditions are $\hat{n} \times \vec{E} = 0$ and $\hat{n} \cdot \vec{B} = 0$, where $\hat{n}$ is the normal vector at the surface.

In this paper, the RF cavity is taken to be sealed, of longitudinal length L and, rather than being of a circular shape whereby the radial distance from the origin to the cavity wall is a constant, the RF cavity has an azimuthally modulated shape whereby the radial distance from the origin to the RF cavity wall, $r_0(\theta)$, varies with the azimuthal angle. For such a cavity, the E_z boundary condition becomes:

$$E_z(r_0(\theta), \theta, z) = 0. \tag{5}$$

Circular Pillbox

Setting all normal coefficients to zero except for a single order $\tilde{e}_m$ means the longitudinal E-field in Eq. (3) now contains a single multipolar component of order m and thus E_z takes the form of a TM_{mn0} mode:

$$E_z(r, \theta, z) = \tilde{e}_m J_m\left(\frac{\omega}{c} r\right) \cos(m\theta). \tag{6}$$

Applying the boundary condition in Eq. (5) to Eq. (6) determines the cavity shape that will support this mode. In this case, the cavity shape that solves the boundary condition is clearly one of a constant radius, r_0, with the radius and the resonant frequency directly related to j_{mn} (the n^{th} solution to the m^{th} order Bessel function) by:

$$\tilde{e}_m J_m\left(\frac{\omega}{c} r_0\right) \cos(m\theta) = 0 \rightarrow \frac{\omega}{c} r_0 = j_{mn}. \tag{7}$$

Therefore, as expected for a circular pillbox of given radius, the TM_{mn0} modes all have distinct frequencies and thus it is impossible for a circular pillbox to support a mode that contains more than one multipolar component. Additionally, as j_{01} is the smallest of the set of j_{mn}, the TM_{010} mode is the fundamental TM mode for circular pillboxes of any size.

Azimuthally Modulated Cavity

Setting all normal coefficients to zero except for the two orders $\tilde{e}_0$ and $\tilde{e}_m$ means the longitudinal E-field in Eq. (3) now only consists of two multipolar components of orders 0 and m and thus E_z takes the form of a $\mathrm{TM}_{\{0m\}n0}$ mode:

$$E_z(r, \theta, z) = \tilde{e}_0 J_0\left(\frac{\omega}{c} r\right) + \tilde{e}_m J_m\left(\frac{\omega}{c} r\right) \cos(m\theta). \tag{8}$$

In this case, applying the boundary condition to Eq. (8) gives:

$$J_0\left(\frac{\omega}{c} r_0(\theta)\right) + \frac{\tilde{e}_m}{\tilde{e}_0} J_m\left(\frac{\omega}{c} r_0(\theta)\right) \cos m\theta = 0. \tag{9}$$

Equation (9) is not solved for all θ by a constant radius, r_0. Instead it must be numerically solved for every θ value to determine the azimuthally-varying cavity shape, $r_0(\theta)$, that will support the $\mathrm{TM}_{\{0m\}n0}$ mode. Note that for the specific angle $\theta = \frac{\pi}{2m}$, the cosine term in Eq. (9) is zero. Thus, for this specific angle, Eq. (9) can be solved analytically:

$$\frac{\omega}{c} r_0\left(\frac{\pi}{2m}\right) = j_{0n}. \tag{10}$$

As j_{01} is the smallest of the set of j_{mn}, Eq. (10) seems to infer that the $\mathrm{TM}_{\{0m\}10}$ mode will be the fundamental TM mode of that cavity. Although not explicitly mathematically proven, this has so far been the case for each continuously shaped cavity that has been modelled in CST.

SIMULATIONS

Four example RF cavity shapes that support $\mathrm{TM}_{\{0m\}10}$ modes are presented in Fig. 1: Fig. 1a shows a cavity shape with a dipole to monopole ratio of 2; Fig. 1b a quadrupole to monopole ratio of 5; Fig. 1c a sextupole to monopole ratio of 0.5; and Fig. 1d a sextupole to monopole ratio of 0.98. The cavity shapes were determined by numerically solving Eq. (9) with $\omega = 2\pi * 3$ GHz and each shape was modelled in the 3D simulation software CST with the supported modes then solved for. The cavity shapes in Fig. 1 are overlaid with the magnitude of E_z of the fundamental mode calculated by CST. Each fundamental mode had a resonant frequency of 3 GHz. The origin of the $[x, y]$ coordinate axis indicates where the beam would pass through.

A sense of scale as to the size of the different cavities is given in Fig. 2 which compares the simulation results of E_z along the positive horizontal and vertical axes to the theoretically predicted value from Eq. (8). The predicted and simulated results for E_z are individually normalised so that the maximum value is 1. The simulation results agree well with theory; the root mean square error (that is the square root of the average squared difference between theory and simulation and thus a measure of the average difference between theoretical and simulation results) is under 0.003 for both x and y directions in all four examples.

Figure 2 shows there is good agreement between the simulation results and theory at all radial positions along two specific θ angles: 0° (along the positive x axis) and 90° (along the positive y axis). It is also useful to determine the level of agreement between simulation results and theory for all θ values at specific radii. This can be done by performing a Helmholtz decomposition [10] which decomposes the longitudinal electric field into the form given by Eq. (3) and returns the magnitude of the normal ($\tilde{e}_m$) and skew ($\tilde{f}_m$) terms for every m. Helmholtz decompositions were performed on the longitudinal electric field of each cavity shape by exporting E_z along cylinders made up of 5 linearly spaced circles, each formed of 360 linearly spaced θ values, and Fig. 3 shows the results of performing a Helmholtz decomposition at each radius. The individual subfigures each show that the multipolar content is indeed as designed: for example, Fig. 3a shows that the dipole component is two times larger than the monopole component whilst the quadrupole, sextupole and octupole components are all approximately zero.

In all four subfigures in Fig. 3, the octupole component is non-zero for small radius values. This is an error that arises because the data is divided by r^m to determine $\tilde{e}_m$. Therefore, simulation errors arising from the finite accuracy of the finite mesh and numerical errors arising from the circles

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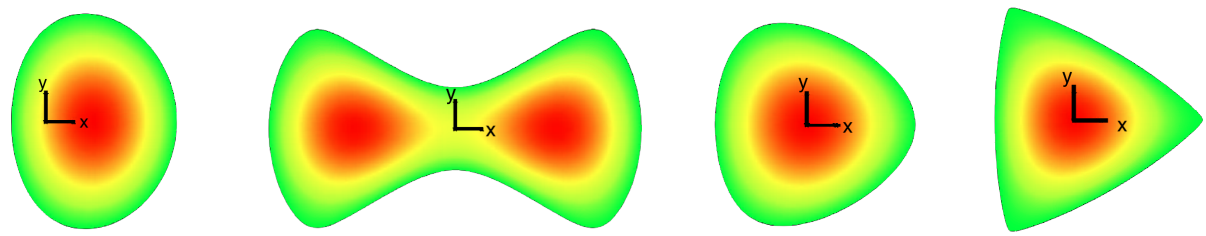
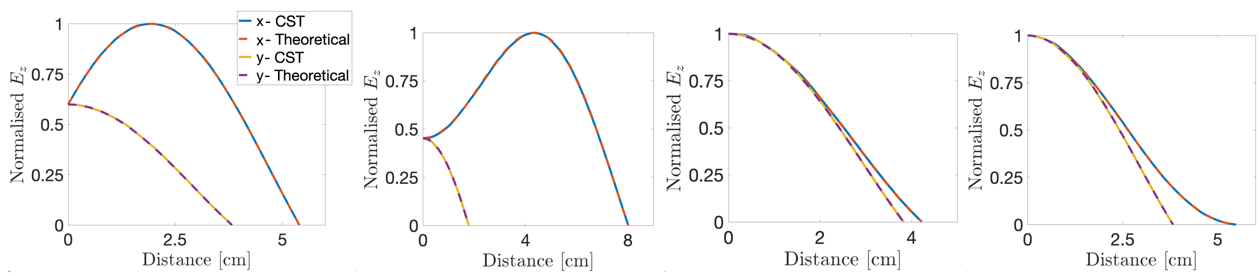


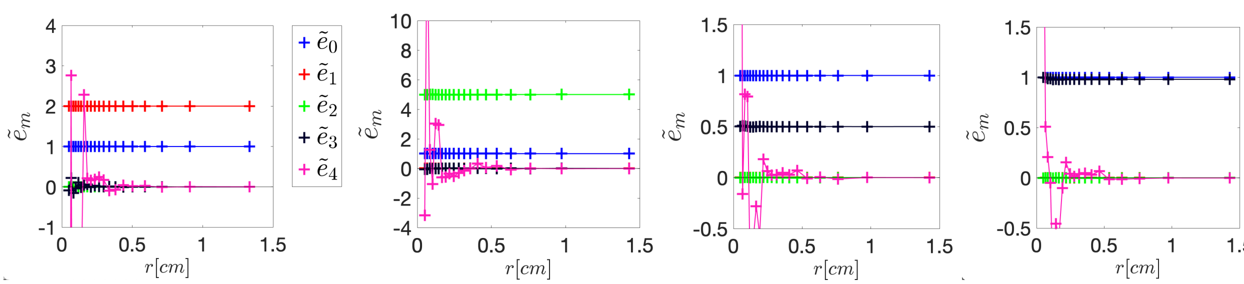
(a) $TM_{\{0,1\}10}$, $\tilde{e}_1/\tilde{e}_0 = 2$ (b) $TM_{\{0,2\}10}$, $\tilde{e}_2/\tilde{e}_0 = 5$ (c) $TM_{\{0,3\}10}$, $\tilde{e}_3/\tilde{e}_0 = 0.5$ (d) $TM_{\{0,1\}10}$, $\tilde{e}_3/\tilde{e}_0 = 0.98$

Figure 1: Colour map of the E-fields in four different $TM_{\{0m\}10}$ cavities (from red as largest magnitude, to green as zero field). The notation $TM_{\{0m\}10}$ denotes the cavity has been designed to support a mode that contains 0^{th} and m^{th} order multipolar terms and the value of $\tilde{e}_m/\tilde{e}_0$ is the multipole to monopole ratio of that cavity design. The origin of the [x, y] coordinate axis is where the beam passes through.



(a) $\tilde{e}_1/\tilde{e}_0 = 2$ (b) $\tilde{e}_2/\tilde{e}_0 = 5$ (c) $\tilde{e}_3/\tilde{e}_0 = 0.5$ (d) $\tilde{e}_3/\tilde{e}_0 = 0.98$

Figure 2: Comparison between simulation (full) to theoretical (dashed) E_z values normalised to the maximum along the positive, horizontal x (blue and red) and vertical y (yellow and purple) axes.



(a) $\tilde{e}_1/\tilde{e}_0 = 2$ (b) $\tilde{e}_2/\tilde{e}_0 = 5$ (c) $\tilde{e}_3/\tilde{e}_0 = 0.5$ (d) $\tilde{e}_3/\tilde{e}_0 = 0.98$

Figure 3: Helmholtz decompositions of E_z along cylinders of radii r, with the first five orders of skew terms displayed.

being only composed of 360 distinct points are amplified for smaller radii and larger m. Accounting for this error, all multipolar components are constant for all radius for each cavity shape and thus simulation results agree with the radial Bessel dependence given by Eq. (8).

CONCLUSION

A systematic method has been presented for designing azimuthally modulated RF cavities that operate at specified frequencies and support $TM_{\{0m\}n0}$ modes that contain both a monopolar and a multipolar term of specified magnitudes. A strong agreement between the theoretical prediction of the longitudinal electric field and simulation results from modelling four different example cavities in CST has been shown, both for different radial measurements along the positive horizontal and vertical axes and for different azimuthal measurements at different radii. These example cavities also have the $TM_{\{0m\}10}$ mode as their fundamental TM mode.

Such RF cavities operating in a $TM_{\{0m\}n0}$ mode have application where the introduction or suppression of exact multipolar terms is desired. As these cavities simultaneously provide acceleration to a beam and are similarly sized to cavities designed to solely accelerate a beam whilst operating in the TM_{010} mode, they could provide a similar function whilst providing the additional benefit of having an extra multipole component, all for little extra cost in space.

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