EQUILIBRIUM BUNCH DENSITY DISTRIBUTION WITH MULTIPLE ACTIVE AND PASSIVE RF CAVITIES

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Abstract
This paper describes a method to get the equilibrium bunch density distribution with an arbitrary number of active or passive RF cavities in uniform filling. This method is an extension of the one presented by M. Venturini which assumes a passive harmonic cavity and no beam loading in the main RF cavity [1].

INTRODUCTION
In the last decade, following the growing interest for the usage of harmonic cavities (HC) in the light source community, there were many attempts to develop analytic methods to obtain the equilibrium bunch density. First by assuming a uniform filling pattern [1, 2] and then with non-uniform filling pattern [3–5].

All these methods assume that the RF system is composed of two cavities: the main RF cavity (MC) is assumed to be a perfect sine wave (active cavity with no beam loading) and the HC is a passive one (no generator voltage). For most of these methods, it should be straightforward to extend them to a general case of a RF system with an arbitrary number of cavities with both beam loading and generator voltage, which can then be applied to all cases: MC, passive and active HC.

This paper provides the steps needed to first treat the active HC case and then to add the beam loading of the main cavity to the method developed by Venturini in his paper [1]. Because the IPAC paper format is quite short and that the method described here is an extension, we will use the same notation as in [1] and the reader should refer to that paper for further information on the base method.

The method described here with the general case, considering N active or passive cavities, has been added to mbtrack2 [6]. Finally, the equilibrium bunch profile obtained with this method for the SOLEIL upgrade case [7] is compared to tracking using mbtrack2.

EQUILIBRIUM BUNCH PROFILE
In this section, the method used (corresponding to appendix B of [1]) to obtain the equilibrium bunch profile $\rho_0(z)$ is succinctly presented.

The equations of motions for the synchrotron oscillations are given by:

$$\frac{dz}{dt} = acc_0,$$

$$\frac{d\delta}{dt} = \frac{eV_{rf}(z) - U_0}{E_0T_0},$$

where $z$ is the longitudinal distance from the synchronous particle, $\alpha$ is the momentum compaction factor, $E_0$ is the reference energy, $\delta = \frac{E - E_0}{E_0}$ is the relative energy deviation, $U_0$ the losses per turn and $V_{rf}$ is the total RF voltage. From these equations of motions, a scaled potential $u(z)$ can be defined as:

$$u(z) = \frac{-1}{ac\sigma_0^2 E_0 T_0} \int_0^z (eV_{rf}(z') - U_0) dz',$$

where $e$ is the elementary charge, $\sigma_0$ is the energy spread and $T_0$ is the revolution period. As we are looking for the equilibrium conditions, we can write the bunch profile at equilibrium $\rho_0(z)$ as:

$$\rho_0(z; F, \Phi) = \int_{-\infty}^{\infty} e^{-i\omega(z; F, \Phi)} dz',$$

where $F$ and $\Phi$ are bunch form factors which depends on the bunch profile. For which the following normalisation condition is verified:

$$\int_{-\infty}^{\infty} \rho_0(z) dz = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} e^{i\omega(z; F, \Phi)} dz dz'=1.$$

In [1], the author assumes the total RF voltage $V_{rf}$ to be the sum of a perfect main RF cavity (neglecting beam loading) providing a voltage $V_1(z)$ and a passive 3rd harmonic cavity so the total RF voltage $V_{rf}$ is:

$$V_{rf}(z) = V_1(z) + V_3(z) = V_1 \sin(k_1 z + \phi_1) + V_3(z),$$

where $k_1 = \frac{\omega_1}{c}$ is the RF wave number, $\omega_1$ is the RF frequency and $\phi_1$ is the phase of the MC. The expression $V_3(z)$ for the beam loading voltage in the passive 3rd harmonic cavity $V_3(z)$ is detailed in Appendix A of [1] and is given by:

$$V_3(z) = -2I_{avg} R_s F \cos \psi \cos(k_3 z + \psi - \Phi),$$

where $I_{avg}$ is the total beam current, $R_s$ is the shunt impedance of the cavity, $\psi$ is the cavity tuning angle and $k_3 = \frac{3\omega_1}{c}$ is the third harmonic wave number. This expression assumes that the beam has a uniform filling and that the cavity impedance is narrow impedance described by the resonator model. So the scaled potential $u(z)$ is expressed as:

$$u(z) = \frac{U_0 z}{ac\sigma_0^2 E_0 T_0} - \frac{e}{ac\sigma_0^2 E_0 T_0} \left[ V_1 \int_0^z \sin(k_1 z' + \phi_1) dz' \right. - 2I_{avg} R_s F \cos \psi \left. \int_0^z \cos(k_3 z' + \psi - \Phi) dz' \right].$$
After integration, and using the condition \( u(z = 0) = 0 \) to find the value of the integration constant, it gives:

\[
u(z; F, \Phi) = u_0 z + u_1 \cos (k_1 z + \phi_1) - \cos \phi_1 + u_3 \cos F \left[ \sin (k_3 z + \psi - \Phi) - \sin (\psi - \Phi) \right],
\]

\[
(9)
\]

where \( u_0 = \frac{U_0}{(\alpha \sigma_2^2 E_0 C_1)} \), \( u_1 = \frac{e V_1}{(\alpha \sigma_2^2 E_0 C_k)} \), \( u_3 = \frac{2 e I_{avg} R_s}{(\alpha \sigma_2^2 E_0 C_3)} \) and \( C = CT_0 \) is the ring circumference. The condition for the energy balance of the synchronous particle, \( eV_{rf}(z = 0) = U_0 \), also gives us the following equation:

\[
V_1 \sin \phi_1 = U_0 + 2I_{avg} R_s F \cos \psi \cos (\psi - \Phi).
\]

(10)

From there, to be able to obtain \( \rho_0 \), one needs to find an expression for \( F \) and \( \Phi \) as \( u(z) = u(z, F, \Phi) \). To do this, one can compute the Fourier transform of \( \rho_0 \) at the third harmonic \( k_3 \):

\[
\tilde{\rho}_0(k_3) = F e^{i \Phi} = \int_{-\infty}^{\infty} e^{i k_3 z} \rho_0(z) dz.
\]

(11)

Taking the real part and the imaginary part of the last equations, together with the condition for the energy balance of the synchronous particle, it allows to numerically compute \( F, \Phi \) and \( \phi_1 \) for a given \( R_s \) and \( \psi \). Then the bunch equilibrium profile \( \rho_0 \) is obtained using Eq. (4).

ADAPTATION FOR ACTIVE CAVITIES

The idea of this section is to adapt Venturini’s method to be able to use it for active cavities. To achieve this, the results obtained in the previous section are modified in order to take into account this addition. The voltage of the harmonic cavity, now active, is the sum of the generator voltage and of the beam loading part:

\[
V_3(z) = V_g \cos (k_3 z + \theta_g) - \frac{2I_{avg} R_s F}{1 + \beta} \cos \psi \cos (k_3 z + \psi - \Phi),
\]

(12)

where \( V_g \) is the generator voltage amplitude and \( \theta_g \) the generator phase of the active HC in the cosine definition (seen from the beam axis in phasor space).

The impedance \( Z(\omega) \) of the active HC is slightly changed compared to the impedance of a passive HC, it is now expressed in terms of the loaded quality factor \( Q_L = Q/(1 + \beta) \) instead of the unloaded quality factor \( Q \) to take into account the coupling coefficient \( \beta \):

\[
Z(\omega) = \frac{R_g}{1 + j Q_L \left( \frac{\omega}{\omega_g} - \frac{\omega_g}{\omega} \right)},
\]

(13)

which reduces the amplitude of the beam loading voltage by a factor \( \frac{1}{1 + \beta} \) and also impact the definition of the tuning angle \( \psi \) and:

\[
\tan \psi = Q_L \left( \frac{\omega_g}{\omega} - \frac{\omega}{\omega_g} \right).
\]

(14)

This change introduces a modification of the scaled potential \( u(z) \) to a new scaled potential taking into account active HC \( u_a \):

\[
u_a(z) = \frac{U_0 z}{\alpha \sigma_2^2 E_0 T_0} - \frac{e}{\alpha \sigma_2^2 E_0 T_0} \left[ V_0 \int_{-\infty}^{\infty} \sin (k_1 z') d' \right] - \frac{V_g}{2I_{avg} R_s} \int_{-\infty}^{\infty} \cos (k_3 z' + \theta_g) d'.
\]

(15)

Keeping the same definitions for \( u_0, u_1 \) and \( u_3 \), after integration and using the condition \( u_a(z = 0) = 0 \) to find the value of the integration constant, it gives:

\[
u_a(z) = u_0 z + u_1 \left[ \cos (k_1 z + \phi_1) - \cos \phi_1 \right] + \frac{u_3}{1 + \beta} F \cos \psi \left[ \sin (k_3 z + \psi - \Phi) - \sin (\psi - \Phi) \right] - \frac{u_3 V_g}{2I_{avg} R_s} \left[ \sin (k_3 z + \theta_g) - \sin \theta_g \right].
\]

(16)

The new energy balance condition for the synchronous particle, \( (z = 0, \delta = 0) \), is given by:

\[
V_1 \sin \phi_1 + V_g \cos \theta_g = \frac{U_0}{e} + \frac{2I_{avg} R_s F}{1 + \beta} \cos \psi \cos (\psi - \Phi).
\]

(17)

Then using the equations \( \tilde{\rho}_0(k_3) = F_3 e^{i \Phi} \), it is possible to solve numerically to find \( \theta_g, F, \Phi \).

GENERAL CASE OF N CAVITIES WITH BEAM LOADING

In this section, we generalize to the case of \( N \) active (or passive) cavities with beam loading. The total RF voltage can be expressed as the sum of the generator voltage and of the beam voltage for each cavity:

\[
V_{rf}(z) = \sum_{i=1}^{N} V_i(z)
\]

(18)

\[
= \sum_{i=1}^{N} V_{g_i} \cos (k_i z + \theta_{g_i}) - \frac{2I_{avg} R_s F_i}{1 + \beta_i} \cos \psi_i \cos (k_i z + \psi_i - \Phi_i),
\]

where the index \( i \) corresponds to quantities for the cavity of the \( i \)th cavity.
This change introduces a modification of the scaled potential \( u(z) \) to a new scaled potential \( u_b \):

\[
 u_b(z) = \frac{U_0 z}{\alpha \sigma^2 E_0 I_0} - \frac{e}{\alpha \sigma^2 E_0 I_0} \sum_{i=1}^{N} \left[ V_{R_i} \int_C \cos(k_i z' + \theta_{R_i}) dz' - 2I_{av} R_{s_i} F_i \cos \psi_i \int_C \cos(k_i z' + \psi_i - \Phi_i) dz' \right],
\]

(19)

which gives, keeping the same definition of \( u_0 \), after integration:

\[
 u_b(z) = u_0 z + \sum_{i=1}^{N} -u_i \left[ \sin(k_i z + \theta_{R_i}) - \sin \theta_{R_i} \right] + u_i F_i \cos \psi_i \left[ \sin(k_i z + \psi_i - \Phi_i) - \sin \psi_i - \Phi_i \right],
\]

(20)

where we introduced the constants \( u_i = eV_{R_i}/(\alpha \sigma^2 E_0 C k_i) \) and \( u_i = 2eI_{av} R_{s_i}/(\alpha \sigma^2 E_0 C k_i (1 + \beta_i)) \).

The new energy balance condition for the synchronous particle, \( (z = 0, \delta = 0) \), is given by:

\[
 \sum_{i=1}^{N} V_{R_i} \cos \theta_{R_i} = \frac{U_0}{e} + \sum_{i=1}^{N} \frac{2I_{av} R_{s_i} F_i}{1 + \beta_i} \cos \psi_i \cos(\psi_i - \Phi_i)
\]

(21)

Then using the system of equations \( \forall i \in N, \rho_{0}(k_i) = F_i e^{i \phi_i} \), it is possible to solve numerically to find \( \theta_i \) and all the form factors \( F_i \) and \( \Phi_i \). This algorithm has been added to the mbtrack2 collective effect library [6].

**COMPARISON WITH TRACKING**

Now we compare the bunch profiles obtained by using this method and by tracking in the SOLEIL upgrade case [7]. We consider a normal conducting (active) main cavity and a superconducting passive 3rd harmonic cavity with the parameters in Table 1. The phasor diagram of the main RF cavity is shown in Fig. 1.

The tracking is done using \( 10^6 \) macro-particles with mbtrack2 using the algorithm described in [8]. As shown in Fig. 2, there is a very good agreement between both methods for the two detuning values shown here.

However it is important to stress that finding a solution with this analytical method does not say anything about its stability. The stability of the solution should be checked with a self-consistent method like tracking or Vlasov analysis. But if no solution exists, then the beam cannot be stable with this parameter set.

**ACKNOWLEDGEMENTS**

The authors thank R. Nagaoka the many useful discussions and for proofreading this paper.

**REFERENCES**


Table 1: RF Cavities Parameters

<table>
<thead>
<tr>
<th>Parameters</th>
<th>MC</th>
<th>HC</th>
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<tbody>
<tr>
<td>Harmonic number ( m )</td>
<td>1</td>
<td>3</td>
</tr>
<tr>
<td>Shunt impedance ( R_s ) (MΩ)</td>
<td>19.6</td>
<td>9000</td>
</tr>
<tr>
<td>Quality factor ( Q_0 )</td>
<td>34,000</td>
<td>10^8</td>
</tr>
<tr>
<td>Loaded quality factor ( Q_L )</td>
<td>6,000</td>
<td>10^8</td>
</tr>
<tr>
<td>Detuning ( f - f_2 ) (kHz)</td>
<td>-100</td>
<td>90 and 85</td>
</tr>
<tr>
<td>Generator voltage ( V_G ) (MV)</td>
<td>1.05</td>
<td>0</td>
</tr>
<tr>
<td>Generator phase ( \theta_G ) (°)</td>
<td>44.2</td>
<td>0</td>
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<tr>
<td>Beam voltage ( V_B ) (MV)</td>
<td>0.97</td>
<td>0.52 and 0.56</td>
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<tr>
<td>Beam phase ( \theta_B ) (°)</td>
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<td>89.9</td>
</tr>
<tr>
<td>Cavity voltage ( V_C ) (MV)</td>
<td>1.7</td>
<td>0.52 and 0.56</td>
</tr>
<tr>
<td>Cavity phase ( \theta_C ) (°)</td>
<td>72.4</td>
<td>89.9</td>
</tr>
</tbody>
</table>

Figure 1: Phasor diagram of the main RF cavity, representing the cavity phasor \( V_c = V_b + V_g \), the beam phasor \( V_b \) and the generator phasor \( V_g \).

Figure 2: Equilibrium bunch profile for SOLEIL upgrade parameters at 500 mA in uniform filling for two different detuning, \( f_r - f_3 = 85 \) kHz and 90 kHz. SOLEIL upgrade parameters can be found in [7].


