Isolated Period 3 Implies Chaos

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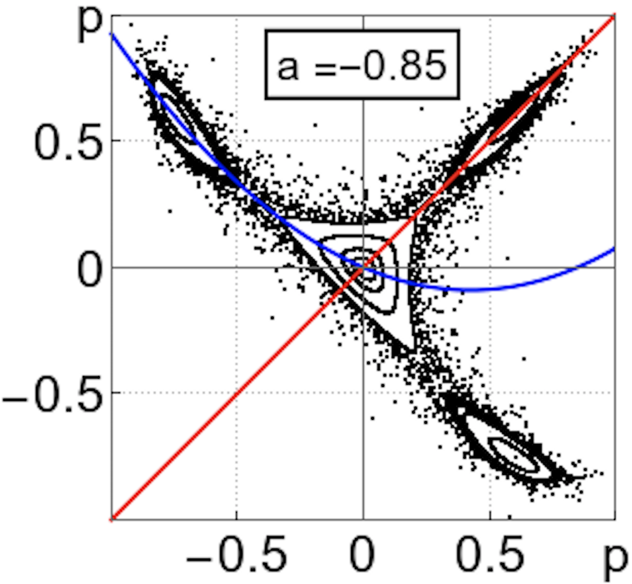
May 23, 2019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a =−0.85
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Basic definitions

Consider a **mapping** (**map**) $\mathrm{T} : \mathrm{M} \to \mathrm{M}$ defined by a function f

$$\zeta_{n+1} = f(\zeta_n), \qquad \zeta_i \in \mathrm{M}.$$

Manifold M can be $\mathbb{R}^n$, $\mathbb{C}^n$, $\mathbb{S}^n$, $\mathbb{T}^n$, etc..

The **trajectory** of ζ_0 is the finite set

$$\{\zeta_0, \mathrm{T}(\zeta_0), \mathrm{T}^2(\zeta_0), \ldots, \mathrm{T}^n(\zeta_0)\}$$

The **orbit** of ζ_0, is a set of all points that can be reached

$$\{\ldots, \mathrm{T}^{-2}(\zeta_0), \mathrm{T}^{-1}(\zeta_0), \zeta_0, \mathrm{T}(\zeta_0), \mathrm{T}^2(\zeta_0), \ldots\}$$

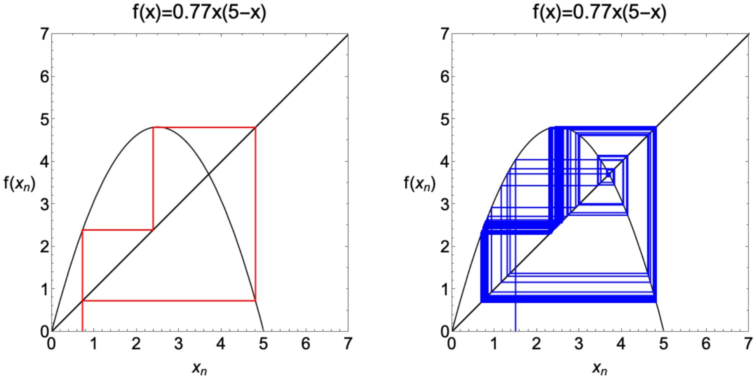
The ***n*-cycle** (or **periodic orbit** of periof n) is a solution of

$$\mathrm{T}^n(\zeta_0) = \zeta_0$$

Cobweb plot: periodic vs. chaotic orbits

Example: Logistic map

$$x_{n+1} = f(x_n) = r\,x_n\,(\alpha - x_n)$$



Sharkovskii's theorem (1964)

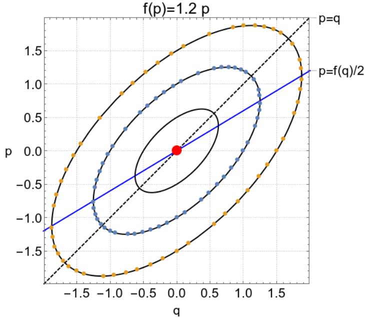
Suppose $\mathrm{T} : \mathbb{R} \to \mathbb{R}$ is continuous. Suppose that T has a periodic point of period n and that n precedes k in the **Sharkovski ordering**. Then T also has a period point of prime period k.

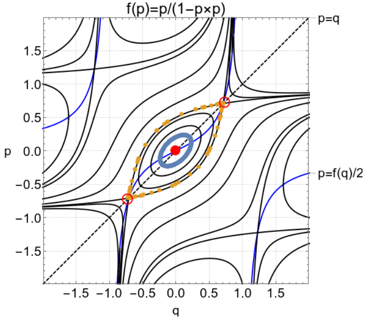
$$3 \times 2^0 \prec 5 \times 2^0 \prec 7 \times 2^0 \prec 9 \times 2^0 \prec \dots$$
$$3 \times 2^1 \prec 5 \times 2^1 \prec 7 \times 2^1 \prec 9 \times 2^1 \prec \dots$$
$$3 \times 2^2 \prec 5 \times 2^2 \prec 7 \times 2^2 \prec 9 \times 2^2 \prec \dots$$
$$\dots$$
$$\dots \prec 2^4 \prec 2^3 \prec 2^2 \prec 2^1 \prec 1$$

Li and Yorke (1975) proved that any one-dimensional system which exhibits a regular cycle of period 3 will also display regular cycles of every other length as well as completely chaotic cycles.

Generalization of Sharkovskii's theorem

Sharkovskii's theorem does not immediately apply to dynamical systems on other topological spaces. It is easy to find a circle map with periodic points of period 3 only: take a rotation by 120 degrees, for example. But some generalizations are possible, typically involving the mapping class group of the space minus a periodic orbit. For example, Peter Kloeden showed that Sharkovskii's theorem holds for triangular mappings.

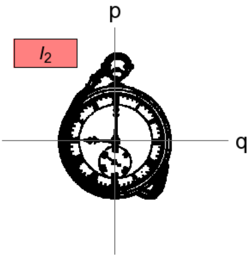




Symplectic mappings of the plane

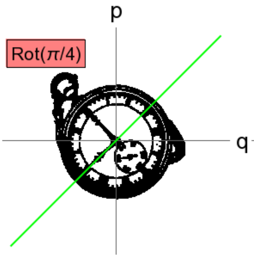
We will consider area-preserving mappings of the plane

$$\begin{aligned} q' &= q'(q,p), \\ p' &= p'(q,p), \end{aligned} \qquad \det \begin{bmatrix} \partial q'/\partial q & \partial q'/\partial p \\ \partial p'/\partial q & \partial p'/\partial p \end{bmatrix} = 1.$$



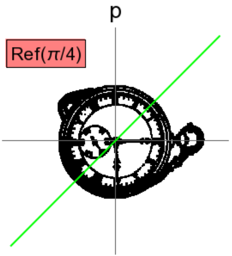
Identity, Id

$$\begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}$$



Rotation, Rot

$$\begin{bmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{bmatrix}$$



Reflection*,**, Ref

$$\begin{bmatrix} \cos 2\theta & \sin 2\theta \\ \sin 2\theta & -\cos 2\theta \end{bmatrix}$$

Integrable systems

A map T in the plane is called **integrable**, if there exists a non-constant real valued continuous functions $\mathcal{K}(q,p)$, called **integral**, which is invariant under T:

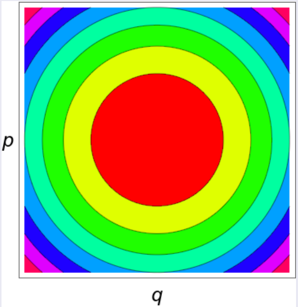
$$\forall\,(q,p): \qquad \mathcal{K}(q,p) = \mathcal{K}(q',p')$$

where primes denote the application of the map, $(q',p') = \mathrm{T}(q,p)$.

Example: Rotation transformation

$$\mathrm{Rot}(\theta): \quad \begin{aligned} q' &= q\,\cos\theta - p\,\sin\theta \\ p' &= q\,\sin\theta + p\,\cos\theta \end{aligned}$$

has the integral $\mathcal{K}(q,p) = q^2 + p^2$.

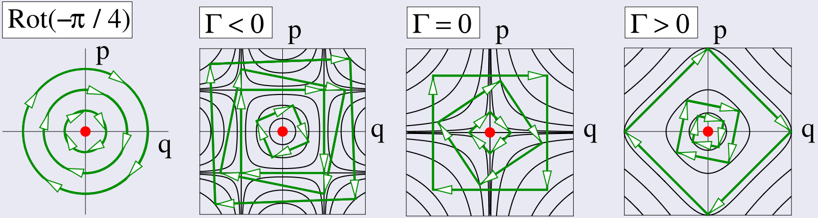


Superintegrable systems

If θ and π are commensurable, then transformation $\mathrm{Rot}(\theta)$ has infinitely many invariants of motion.

Example: Rotations through angles $\pm\pi/4$ has another invariant

$$\mathcal{K}(q,p) = q^2 p^2 + \Gamma(q^2 + p^2), \qquad \forall\,\Gamma.$$



McMillan form of the map

McMillan considered a special form of the map

$$\boxed{\mathrm{M}: \quad \begin{aligned} q' &= p, \\ p' &= -q + f(p), \end{aligned}}$$

where $f(p)$ is called *force function* (or simply *force*).

a. Fixed point

$$p = q \ \cap \ p = \frac{1}{2} f(q).$$

b. 2-cycles

$$q = \frac{1}{2} f(p) \ \cap \ p = \frac{1}{2} f(q).$$

1D accelerator lattice with thin nonlinear lens, $\mathrm{T} = \mathrm{F} \circ \mathrm{M}$

$$\mathrm{M}: \quad \begin{bmatrix} y \\ \dot{y} \end{bmatrix}' = \begin{bmatrix} \cos\Phi + \alpha \sin\Phi & \beta \sin\Phi \\ -\gamma \sin\Phi & \cos\Phi - \alpha \sin\Phi \end{bmatrix} \begin{bmatrix} y \\ \dot{y} \end{bmatrix},$$

$$\mathrm{F}: \quad \begin{bmatrix} y \\ \dot{y} \end{bmatrix}' = \begin{bmatrix} y \\ \dot{y} \end{bmatrix} + \begin{bmatrix} 0 \\ F(y) \end{bmatrix},$$

where α, β and γ are Courant-Snyder parameters at the thin lens location, and, Φ is the betatron phase advance of one period.

Mapping in McMillan form after CT to (q,p), $\mathrm{T} = \widetilde{\mathrm{F}} \circ \mathrm{Rot}(-\pi/2)$

$$q = y,$$
$$p = y\,(\cos\Phi + \alpha \sin\Phi) + \dot{y}\,\beta \sin\Phi,$$

$$\boxed{\widetilde{\mathrm{F}}(q) = 2\,q\,\cos\Phi + \beta\,F(q)\,\sin\Phi}.$$

Turaev theorem

INSTITUTE OF PHYSICS PUBLISHING NONLINEARITY

Nonlinearity **16** (2003) 123–135 PII: S0951-7715(03)35323-X

Polynomial approximations of symplectic dynamics and richness of chaos in non-hyperbolic area-preserving maps

Dmitry Turaev

Recommended by C Liverani

Abstract

It is shown that every symplectic diffeomorphism of R^{2n} can be approximated, in the C^∞-topology, on any compact set, by some iteration of some map of the form $(x, y) \mapsto (y + \eta, -x + \nabla V(y))$ where $x \in R^n$, $y \in R^n$, and V is a polynomial $R^n \to R$ and $\eta \in R^n$ is a constant vector. For the case of area-preserving maps (i.e. $n = 1$), it is shown how this result can be applied to prove that C^r-universal maps (a map is universal if its iterations approximate dynamics of all C^r-smooth area-preserving maps altogether) are dense in the C^r-topology in the Newhouse regions.

Suris theorem and recurrence $x_{n+1} + x_{n-1} = f(x_n)$.

INTEGRABLE MAPPINGS OF THE STANDARD TYPE

Yu. B. Suris

UDC 517.9

$$x_{n+1} - 2x_n + x_{n-1} = \varepsilon F(x_n, \varepsilon), \tag{1}$$

$$F(x, \varepsilon) = \sum_{k=0}^{\infty} \varepsilon^k f_k(x), \quad |\varepsilon| < \varepsilon_0. \tag{2}$$

<u>THEOREM</u>. Equation (1) has a nontrivial symmetric integral of the form

$$\Phi(x, y, \varepsilon) = \Phi_0(x, y) + \varepsilon \Phi_1(x, y), \tag{4}$$

holomorphic in the domain $|x - y| < \delta_0$, in the following and only in the following three cases:

a) $F(x, \varepsilon) = (A + Bx + Cx^2 + Dx^3)/(1 - \varepsilon(E + Cx/3 + Dx^2/2))$,

$$\Phi_0(x, y) = (x - y)^2/2, \quad \Phi_1(x, y) = -A(x + y)/2 - Bxy/2 - Cxy(x + y)/6 - Dx^2y^2/4 - E(x - y)^2/2.$$

б) $$F(x, \varepsilon) = \frac{2}{\omega\varepsilon} \operatorname{arctg} \left\{ \frac{\frac{\omega\varepsilon}{2}(A \sin \omega x + B \cos \omega x + C \sin 2\omega x + D \cos 2\omega x)}{1 - \frac{\omega\varepsilon}{2}(A \cos \omega x - B \sin \omega x + C \cos 2\omega x - D \sin 2\omega x + E)} \right\},$$

$$\Phi_0(x, y) = (1 - \cos \omega(x - y))/\omega^2, \quad \Phi_1(x, y) = (A(\cos \omega x + \cos \omega y) - B(\sin \omega x + \sin \omega y) + C \cos \omega(x + y) - D \sin \omega(x + y) + E \cos \omega(x - y))/2\omega$$

в) $$F(x, \varepsilon) = \frac{1}{\alpha\varepsilon} \ln \frac{1 + \alpha\varepsilon(B \exp(-\alpha x) + D \exp(-2\alpha x) - E)}{1 - \alpha\varepsilon(A \exp(\alpha x) + C \exp(2\alpha x) + E)},$$

$$\Phi_0(x, y) = (\operatorname{ch} \alpha(x - y) - 1)/\alpha^2, \quad \Phi_1(x, y) = (-A(e^{\alpha x} + e^{\alpha y}) + B(e^{-\alpha x} + e^{-\alpha y}) - Ce^{\alpha(x+y)} + De^{-\alpha(x+y)} - 2E \operatorname{ch} \alpha(x - y))/2\alpha.$$

A. Iatrou and J. Roberts [Nonlinearity **15** (2002)]

Appendix A. Fixed points of maps possessing an integral

Suppose $L : \boldsymbol{x} \mapsto \boldsymbol{x}'$ with $\boldsymbol{x} \in \mathbb{R}^n$ is a diffeomorphism with a smooth integral (or invariant) $K(\boldsymbol{x})$ satisfying $K(\boldsymbol{x}') = K(\boldsymbol{x})$ for all $\boldsymbol{x} \in \mathbb{R}^n$.

(1) a critical point is mapped to a critical point (since L is a diffeomorphism so that $\mathrm{d}L(\boldsymbol{x})$ is non-singular);

(2) isolated critical points always belong to n-cycles of L, $n \geqslant 1$ ($n = 1$ giving fixed points); in particular, if there are a finite number of critical points of K, they all belong to n-cycles of L;

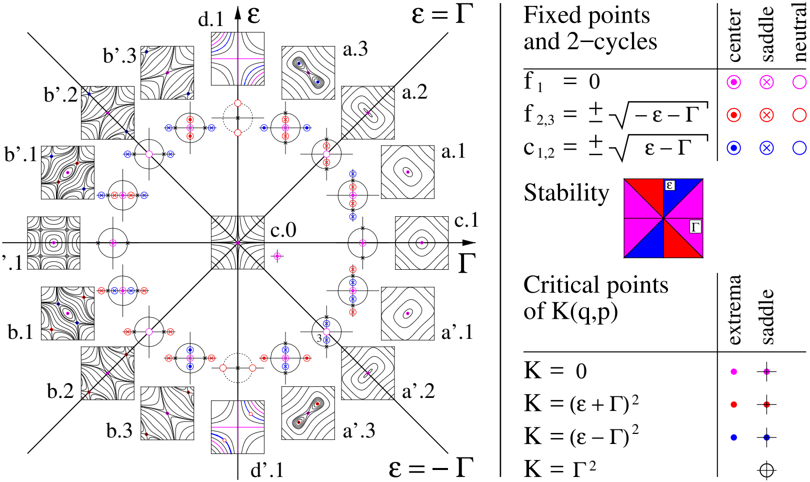
(3) if x is a point of an n-cycle (i.e. $\boldsymbol{x}^{(n)} = \boldsymbol{x}$), we find

$$(\mathrm{d}L^n(\boldsymbol{x})^{\mathrm{T}} - \mathbb{1})\frac{\partial K}{\partial \boldsymbol{x}}(\boldsymbol{x}) = \boldsymbol{0}, \tag{138}$$

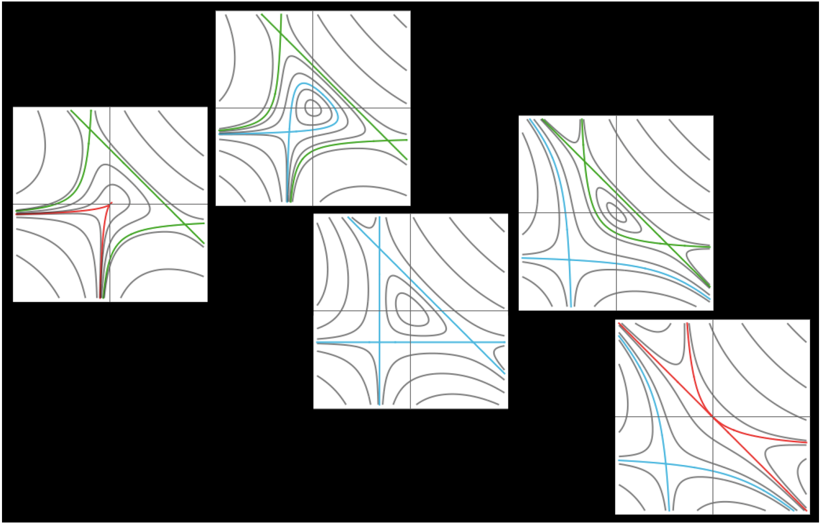
so if $(\mathrm{d}L^n(\boldsymbol{x})^{\mathrm{T}} - \mathbb{1})$ is non-singular, then the point is also a critical point. The non-singularity condition is equivalent to saying $\mathrm{d}L^n(\boldsymbol{x})$ has no eigenvalue equal to 1, which in turn means that the n-cycle containing $\boldsymbol{x}$ is isolated from other n-cycles.

One way to summarize this is as follows: isolated critical points of the integral belong to (isolated) cycles of the map and the points of isolated cycles of the map are (isolated) critical points of the integral. However, it should be noted that in integrable maps (e.g. $n = 2$ when existence of one integral suffices) n-cycles generically are *not* isolated and come in one-parameter families (the points of which are not critical points of the integral).

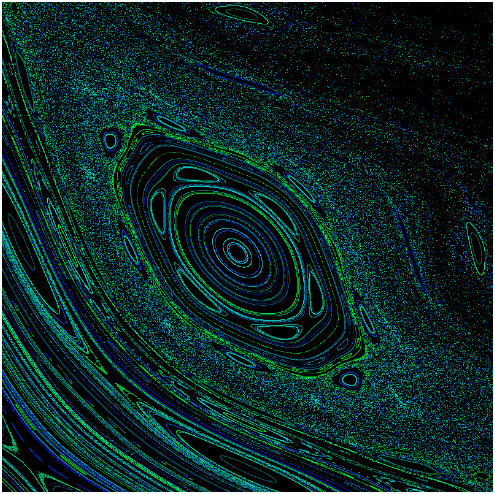
Example 1: Octupole McMillan map, $f(q) = -\frac{2\,\epsilon\, q}{q^2+\Gamma}$
$\mathcal{K}(q,p) = p^2\,q^2 + \Gamma\,(p^2+q^2) + 2\,\epsilon\,p\,q$

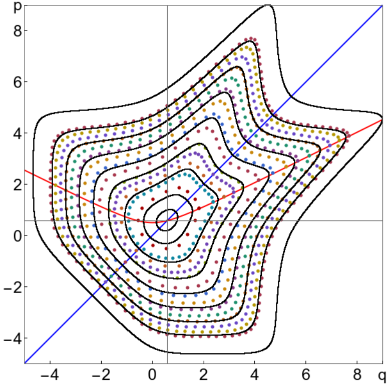


Example 2: Sextupole McMillan map, $f(q) = -\frac{q\,(q+2\,\epsilon)}{q+\Gamma}$ $\mathcal{K}(q,p) = p^2\,q + p\,q^2 + \Gamma\,(p^2+q^2) + 2\,\epsilon\,p\,q$



Example 3: Hénon, $f(q) = a\,q + b\,q^2$, and Cohen, $f(q) = \sqrt{q^2+1}$, chaotic maps

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8
6
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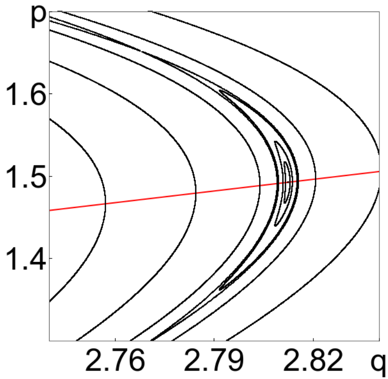
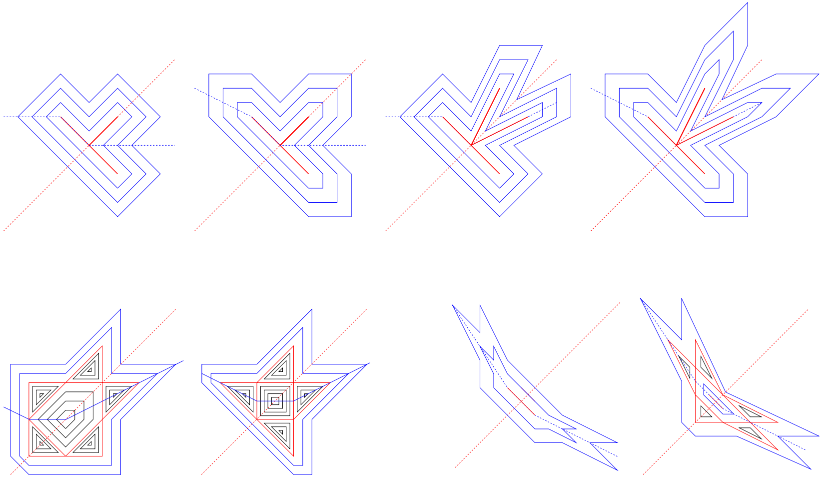
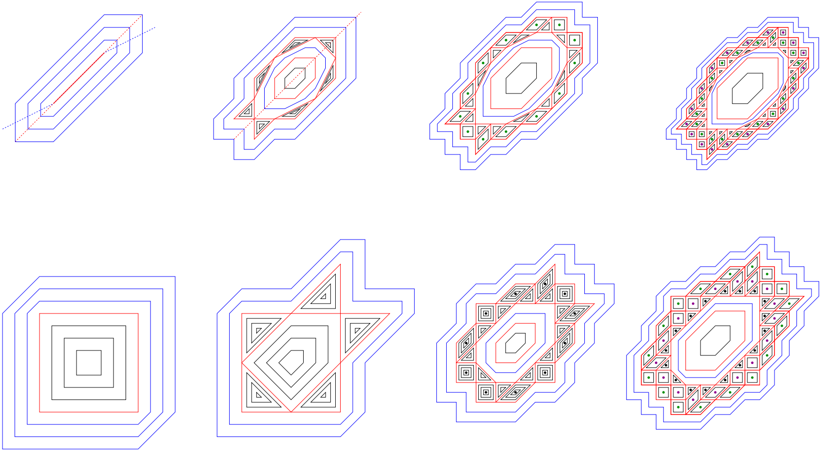
2
0
−2
−4
−4
−2
0
2
4
6
8
q


p
1.6
1.5
1.4
2.76
2.79
2.82
q

Example 4: Picewise linear continous maps



Example 5: Picewise linear continous maps (layers of linear islands)



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Thank you for your attention!

Questions?