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NEW FEATURE OF THE OSCILLATING SYNCROTRON MOTION DERIVED FROM THE HAMILTONIAN COMPOSED OF THREE MOTIONS

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Abstract

Equations for the synchrotron motion are derived from the Hamiltonian, which was composed of coasting, betatron and synchrotron motions. The synchrotron oscillation is not only an oscillation of the revolution frequency but also an oscillation of the average radius. The synchrotron oscillation, which is both longitudinal and horizontal oscillations, can exchange energy with the betatron oscillation, which is a horizontal oscillation. The synchrotron oscillation occurs under a constant particle velocity and the s description is equivalent to the t description. Coriolis like force acting on its horizontal oscillation brings out its longitudinal oscillation.

INTRODUCTION

We discuss the oscillating synchrotron motion [1] using the Hamiltonian composed of coasting, synchrotron and betatron motions. We call it the synchrotron oscillation. The Hamiltonian, which clarified the synchro-betatron resonant coupling mechanism in a storage ring, revealed that the energy exchange between the synchrotron and betatron oscillations was possible [2]. The synchrotron and betatron oscillations are obtained with s as an independent variable [3]. The betatron oscillation is an oscillation in the horizontal direction. We call it a horizontal oscillation. Since the synchrotron oscillation is accompanied by orbiting particles and occurs in the orbital direction, it is an oscillation in the longitudinal direction. We call it a longitudinal oscillation. Unless a horizontal component of the synchrotron oscillation exists, the energy exchange between those two oscillations is impossible. We show that the synchrotron oscillation derived from the Hamiltonian is not only a longitudinal oscillation but also a horizontal one and discuss about its new feature.

THE HAMILTONIAN FOR ORBITING PARTICLES

In the right-handed curvilinear coordinate system (x,s,z), $A_{x,s,z}$ is the vector potential, $p_{x,s,z}$ is the momentum. Here s is the orbital length. For an orbital momentum $-p_s$, the particle is moving in a counterclockwise direction. We assume that an on-momentum particle of mass m and momentum p_0 is revolving (without oscillating motion) on the reference closed orbit of the average radius R under the dipole magnetic field $-B_0$. Around the reference closed orbit, x is the horizontal coordinate and p_x is the horizontal momentum. We have the following relations for the on-momentum particle: the velocity $v=\frac{ds}{dt}$ which satisfies $v=\beta c$, the orbit angle θ which satisfies $\theta=\frac{s}{R}$ and the angular revolution frequency ω which satisfy $\omega=\frac{d\theta}{dt}$.

Since we have $\omega=\frac{d}{dt}\left(\frac{s}{R}\right)=\frac{\beta c}{R}$,

$$v=R\omega. \tag{1}$$

Here q is the elementary charge, c is the velocity of light, t is time and ρ is the bending radius (curvature) where $p_0c=-qB_0\rho$ is satisfied.

We have the following relations:

$$E_0^{\;2}=\left(p_0c\right)^2+\left(mc^2\right)^2,\ p_0=m\gamma\beta c \text{ and } E_0=m\gamma c^2, \tag{2}$$

where $\gamma=\frac{1}{\sqrt{1-\beta^2}}$ and E_0 is the total energy of the on momentum particle.

We have $\vec{p}_0=\left(p_x,p_s,p_z\right)$, $p_0^{\;2}=p_x^2+p_s^2+p_z^2$ and $p_0=\left|\vec{p}_0\right|\approx p_s$. We neglect vertical motion and put $z=0$ and $p_z=0$.

Then define $E=E_0+\Delta E$ and $p=p_0+\Delta p$. Δp is the momentum deviation and ΔE is the deviation of the kinetic energy of the off-momentum (orbiting) particle from the on-momentum (revolving) particle. Define a symbol of rationalized fractional deviation $\delta\equiv\frac{\Delta E}{\beta^2E_0}$. We transform coordinate system of $-E$ onto $-\Delta E$ then to $-\delta$.

$$\left(x,p_x;t,-E\right)\ \rightarrow\ \left(\bar{x},\bar{p}_x;\bar{t},-\Delta E\right)\ \rightarrow\ \left(\bar{x},\frac{\bar{p}_x}{p_0};\phi,-\delta\right) \tag{3}$$

Around the off-momentum closed orbit, $\bar{x}$ is the horizontal coordinate and $\bar{p}_x$ is the horizontal momentum. $\bar{t}$ is the arrival time of the off-momentum particle. They satisfy the following relations [4]:

$$x=\bar{x}+D\delta, \tag{4}$$

$$\frac{p_x}{p_0}=\frac{\bar{p}_x}{p_0}+D'\delta, \tag{5}$$

$$t=\bar{t}-\frac{D}{\beta c}\frac{\bar{p}_x}{p_0}+\frac{D'}{\beta c}\bar{x}, \tag{6}$$

where D is the dispersion function.

The phase of the orbiting particle ϕ is given as follows:

$$\phi=\omega\bar{t}-\frac{\Delta s}{R}. \tag{7}$$

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Define $\tau = \frac{1}{\beta c}\left(-D\frac{\bar{p}_x}{p_0} + D'\bar{x}\right)$ and $\phi_D = -\frac{D}{R}\left(\frac{\bar{p}_x}{p_0}\right) + \frac{D'}{R}\bar{x}$ where the prime denotes differentiation with respect to s . Then $t = \bar{t} + \tau$ and $\phi = \bar{\phi} + \phi_D$ where $\phi = \omega t$ and $\bar{\phi} = \omega \bar{t}$. In fact ϕ_D 1 . The delay phase ϕ_D is equivalent to the phase delay $\Delta\phi = \phi - \bar{\phi} = \omega\tau$.

The delay time τ is the time delay of the off-momentum particle from the on-momentum particle and Δs is corresponding orbital length. Actually oscillations are periodic deviations of the orbit of the off-momentum particle from the orbit of the on-momentum particle. In fact the (rationalized) fractional deviation δ consists of two components: $\delta = \delta_C + \delta_S$. Keeping up to the 2nd order to describe an orbiting particle with coasting, betatron and synchrotron motions, the Hamiltonian composed of three motions is given as follows from Eq. (21) of Ref. [2]:

$$\begin{aligned} H &= -\left(1+\delta_C+\delta_S\right) + \frac{1}{2}\left(\frac{\bar{p}_x}{p_0}\right)^2 + \frac{1}{2}K_x\bar{x}^2 + \frac{1}{2}(-\eta)\left(\delta_c+\delta_s\right)^2 \\ &- \frac{hqV}{2\pi\beta^2 E_0}\left\{\cos\left(\phi+\phi_D\right) - \cos\left(\phi_S+\phi_D\right) + \left(\phi-\phi_S\right)\sin\left(\phi_S+\phi_D\right)\right\} \end{aligned} \quad .(8)$$

where δ_S (oscillating component) is the (rationalized) fractional deviation of the kinetic energy caused by the synchrotron oscillation and δ_C (DC component) is the (rationalized) fractional deviation of the kinetic energy caused by the dispersion. The coasting motion consists of the 0th (on-momentum particle) and 1st (δ_S and δ_C) order effects. h is the harmonic number. η is the phase slip factor.

OSCILLATIONS OBTAINED FROM THE HAMILTONIAN

Since on-momentum particles are revolving on the reference closed orbit, both the betatron and the synchrotron oscillations are excited by orbiting off-momentum particles. We discuss about the synchrotron oscillation which is obtained when $\phi \to \phi_S$ is satisfied. Hamilton's equations of motion for $(\phi, -\delta_S)$ are obtained from $\bar{H}$ as follows:

$$\frac{d\delta_S}{d\theta} = -\frac{\partial \bar{H}}{\partial \phi} = -\frac{hqV}{2\pi\beta^2 E_0}\left\{\sin\left(\phi+\phi_D\right) - \sin\left(\phi_S+\phi_D\right)\right\} , \quad (9)$$

$$\frac{d\phi}{d\theta} = \frac{\partial \bar{H}}{\partial \delta_S} = -1 + (-\eta)\left(\delta_C + \delta_S\right) \cdot \quad (10)$$

We consider a small amplitude oscillation of δ_S .

Putting $\phi \to \phi_S$, we can differentiate RHS of Eq. (9),

$$\frac{d\delta_S}{d\theta} = -\frac{hqV\cos\left(\phi_S+\phi_D\right)}{2\pi\beta^2 E_0}\left(\phi - \phi_S\right) \cdot \quad (11)$$

From Eqs. (10) and (11),

$$\begin{aligned} \frac{d^2\delta_S}{d\theta^2} &= -\frac{hqV\cos\left(\phi_S+\phi_D\right)}{2\pi\beta^2 E_0}\frac{d\phi}{d\theta} \\ &= -\frac{hqV\cos\left(\phi_S+\phi_D\right)}{2\pi\beta^2 E_0}\left\{-1+(-\eta)\left(\delta_C+\delta_S\right)\right\} \end{aligned} . \quad (12)$$

Then,

$$\frac{d^2}{d\theta^2}\left(\delta_S - \delta_0\right) = -\nu_S^2\left(\delta_S - \delta_0\right) \cdot \quad (13)$$

We obtain the following equations.

$$\delta_s = \hat{\delta}\cos\left\{\nu_s\left(\theta-\theta_0\right)\right\} + \delta_0 , \quad (14)$$

$$\nu_s^2 = \frac{\omega_s^2}{\omega^2} = \frac{hqV\left|\eta\cos\left(\phi_S+\phi_D\right)\right|}{2\pi\beta^2 E_0} , \quad (15)$$

where $\delta_0 = -\delta_C + \frac{1}{(-\eta)} + C$ and C is an integration constant. ω_s is the angular synchrotron frequency, ν_s is the synchrotron tune and $\hat{\delta}$ is the amplitude of oscillation. Generally $\delta_C << 1$ and we can choose $\delta_C = \hat{\delta}$ at $\theta = \theta_0$. We can neglect this term. From Eq. (14)

$$\delta_S = \hat{\delta}\cos\left\{\nu_s\left(\theta-\theta_0\right)\right\} + \frac{1}{(-\eta)} + C \cdot \quad (16)$$

And,

$$-\eta\delta_S = -\eta\hat{\delta}\cos\left\{\nu_s\left(\theta-\theta_0\right)\right\} + 1 + (-\eta)C \cdot \quad (17)$$

We have the following relation [5]:

$$\frac{\Delta\omega}{\omega} = -\eta\delta_s , \quad (18)$$

where $\Delta\omega$ is the deviation of angular revolution frequency caused by the synchrotron oscillation. Keeping up to the 1st order,

$$\frac{\Delta\omega}{\omega} = \frac{\Delta\hat{\omega}}{\omega}\cos\left\{\nu_s\left(\theta-\theta_0\right)\right\} + 1 + (-\eta)C , \quad (19)$$

where $\Delta\hat{\omega}$ is the amplitude of oscillation which satisfies $\frac{\Delta\hat{\omega}}{\omega} = -\eta\hat{\delta}$.

If we choose $C = 0$, from Eq. (19),

$$\frac{\Delta\omega}{\omega} = \frac{\Delta\hat{\omega}}{\omega}\cos\left\{\nu_s\left(\theta-\theta_0\right)\right\} + 1 . \quad (20)$$

$\Delta\omega$ is an oscillation around ω . However, $\Delta\omega$ can be larger than ω . It is embracing. So we choose $C = \frac{1}{\eta}$ and we obtain a rationalized equation.

$$\frac{\Delta\omega}{\omega} = \frac{\Delta\hat{\omega}}{\omega}\cos\left\{\nu_s\left(\theta-\theta_0\right)\right\} . \quad (21)$$

Now $\Delta\omega$ is a standing wave oscillation on the kinetic frame revolving with ω .

The angular frequency of the revolving particle is changed periodically but very slowly in the longitudinal oscillation. In practical situation, however, the particle revolves many times for one longitudinal oscillation and it is not easy to detect the synchrotron tune in that direction. So why the synchrotron tune is detectable in experiments?

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We have dealt with the coordinate $-\delta_S$. Since the coordinate system is $(\phi, -\delta_S)$, we now consider the coordinate ϕ. We again consider a small amplitude oscillation of ϕ. Putting $\phi \to \phi_S$, from Eqs. (10) and (11),

$$\begin{aligned}\frac{d^2\phi}{d\theta^2} &= \frac{d^2}{d\theta^2}(\phi-\phi_S) = -\eta\frac{d\delta_S}{d\theta} \\ &= -\frac{hqV\left|\eta\cos(\phi_S+\phi_D)\right|}{2\pi\beta^2 E_0}(\phi-\phi_S)\end{aligned}, \quad (22)$$

Using Eq. (15), we have

$$\frac{d^2}{d\theta^2}(\phi-\phi_S) = -\nu_s^2(\phi-\phi_S) \ , \quad (23)$$

$$\phi-\phi_S = \hat{\phi}\cos\{\nu_s(\theta-\theta_0)\}+C, \quad (24)$$

where $\hat{\phi}$ is the amplitude of oscillation and C is an integration constant.

From the definition of the phase in Eq. (7),

$$\phi = \omega\overline{t} - \frac{\Delta s}{R}. \quad (25)$$

Putting $\phi_S = \overline{\phi} = \omega\overline{t}$ and $\Delta s = 2\pi\Delta R$ since $2\pi R$ is the circumference, we have

$$\phi-\phi_S = -\frac{2\pi\Delta R}{R}, \quad (26)$$

where ΔR is the deviation of average radius caused by the synchrotron oscillation.

Then we have, from Eq. (24),

$$-\frac{\Delta R}{R} = -\frac{\Delta\hat{R}}{R}\cos\{\nu_S(\theta-\theta_0)\}+\frac{C}{2\pi}, \quad (27)$$

where $-\Delta\hat{R}$ is the amplitude of oscillation which satisfies $\hat{\phi} = -2\pi\frac{\Delta\hat{R}}{R}$.

If we choose $C=0$, from Eq. (27), we obtain another rationalized equation.

$$-\frac{\Delta R}{R} = -\frac{\Delta\hat{R}}{R}\cos\{\nu_s(\theta-\theta_0)\}. \quad (28)$$

ΔR is a standing wave oscillation around R.

From Eq. (26), we have the phase delay $\Delta\phi = -\frac{2\pi\Delta R}{R}$. [4

After τ, the oscillating off-momentum particle deviates $\Delta\phi$ from the on-momentum particle.

DISCUSSION

The synchrotron motion turns to be a simple harmonic oscillation when $\phi \to \phi_S$ is satisfied. The synchrotron oscillation is the oscillation of the angular revolution frequency (Eq. (21)) and the oscillation of the average radius (Eq. (28)) at the same time. Two pictures are equivalent but represent oscillations of two different directions. Since the first one occurs in the orbital direction, it is a longitudinal oscillation. For the second one, the average radius of the orbiting particle oscillates also around the reference closed orbit. Since it occurs in the radial direction, it is a horizontal oscillation. So the synchrotron tune of the horizontal oscillation is detectable in ordinary experiments.

The synchrotron oscillation is a standing wave oscillation in both longitudinal and horizontal directions. Since they are an equivalent oscillation, we can assume

$$\frac{\Delta\omega}{\omega} = -\frac{\Delta R}{R} \text{ and } \frac{\Delta\hat{\omega}}{\omega} = -\frac{\Delta\hat{R}}{R}.$$

From Eq. (1), $\frac{\Delta\omega}{\omega} = -\frac{\Delta R}{R} \ \to \ \frac{d\omega}{\omega} = -\frac{dR}{R}$.

Then, $0 = \omega dR + Rd\omega = d(R\omega) = dv$.

We have

$$v = \text{const} \ . \quad (29)$$

The particle is oscillating around the reference closed orbit under a constant velocity. In the synchrotron oscillation, as the particle circles the outer orbit, R increases and ω decreases so that that the velocity is kept constant. On the other hand, as the particle circles the inner orbit, R decreases and ω increases so that that the velocity is kept constant. Therefore, Coriolis like force acting on the horizontal oscillation brings out the longitudinal oscillation.

Since $v = \frac{ds}{dt}$, we have $ds = v\,dt$. Now v is a constant of proportionality. It shows that the special difference is proportional to the time difference for all particles of the same velocity v. There is a coherence between oscillation of particles. Therefore, we can observe a clear synchrotron oscillation as the coherent synchrotron mode in experiments. The synchrotron motion have been discussed in either the s description [3] or the t description [5]. In fact, the s description is equivalent to the t description in the oscillating synchrotron motion.

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