ISBN: 978-3-95450-184-7 doi:10.18429/JACoW-IPAC2018-THPAK134

DYNAMIC EQUATIONS: THE MATRIX REPRESENTATION OF BEAM DYNAMIC EQUATIONS INSTEAD OF TENSOR DESCRIPTION *

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Abstract

In the paper, we are considering methods of mathematical and computer modeling of nonlinear dynamics of particle beams in cyclic accelerators in terms of the matrix representation of the corresponding nonlinear differential equations. In the paper, we use the coefficients representation in the form of two-dimensional matrices. The similar approach allows significantly reduce the time spent on modeling beam dynamics, and also use symbolic mathematics to calculate the necessary two-dimensional matrices. This method demonstrates the effectiveness for problems of dynamics and optimization of the corresponding control systems, and for evaluating the influence of various effects on the dynamics of the beam (including taking into account the spin). Using the tools of symbolical computations not only significantly increases the computational efficiency of the method, but also allows you to create databases of "ready-made" transformations (Lego-objects), what greatly simplify the process of modeling particle dynamics. Examples of solving practical problems are given.

INTRODUCTION

It should be noted that in practice there is an inevitable deviation between the design of the system, physical or mathematical models, on the one hand, and the realized system, on the other. It should also be noted that the development and operation of all higher-intensity and high-energy accelerators leads to the need to take into account new complex and non-linear processes. This leads to the need for upgrading both traditional mathematical methods and the creation and implementation of new, more efficient mathematical approaches and algorithms. In particular, this is what led to the search for new, computationally effective methods, in particular, such as machine learning [1], artificial intelligence [2], and multi-agent systems [3]. It should also be noted that these methods can be used both at the design stage and in the process of controlling the operation of accelerators. In particular, it should be noted that methods based on the use of neural networks are well adapted to modeling, control, and diagnostic problems of complex systems and systems with the large parameter volumes. The effectiveness of using the above methods is largely determined by the choice of mathematical modeling tools and their implementation in the form of appropriate software. In this article, we will consider the theoretical (mathematical) methods on which the proposed software is based, and also demonstrate the effectiveness of both the proposed mathematical tools and the corresponding software. One of the main problems solved in accelerator physics is the stage of mathematical and computer modeling the nonlinear dynamics of particle beams (including taking into account spin dynamics) for a number of practical problems. In this article, we will review the fundamental mathematical provisions on which to base the relevant software, and also demonstrate the effectiveness of the proposed mathematical tools and corresponding software.

BEAM LINES MATHEMATICAL MODELS

One of the concerns of designers of a cyclic accelerator is to estimate the influence of nonlinear forces on the motion of an ensemble of beam particles. These nonlinear forces can manifest as systematic and random errors generated by control elements, for example, such as the sextupoles used for correction the energy and octupoles energy spread used to stabilize the collective motion of particles, in particular using the nonlinear interaction forces of the beam. Usually, the nonlinear forces are sufficiently small in comparison with the linear forces. However, the amplitudes of the transverse oscillations of the particles can be quite large, which can lead to unstable beam evolution, and as a consequence to the loss of particles on the walls of the vacuum chamber.

Traditionally, in dynamic problems for cyclic accelerators, curvilinear coordinates are used instead of cartesian coordinates in some neighborhood of a closed (reference) orbit with a radius of curvature ρ (we shall assume that the torsion of a closed orbit is equal to zero everywhere) to the variable s as an independent variable. The formalization of the corresponding equations is based on the Hamiltonian formalism, which, in particular, makes it possible to track the preservation of such important properties of the dynamic system as the conservation of energy and the properties of symplecticity [4, 5].

One of the concerns of designers of a cyclic accelerator is to estimate the influence of nonlinear forces on the evolution of beam particles. These nonlinear forces can manifest both as systematic and random errors generated by control elements, for example, such as the sextupoles used for correction the energy and octupoles energy spread used to stabilize the collective motion of particles, in particular using the nonlinear interaction forces of the beam. Usually, the nonlinear forces are sufficiently small in comparison with the linear forces. However, the amplitudes of the transverse oscillations of the particles can be quite large, which can lead to unstable beam evolution, and as a consequence to

* The work is supported by a grant "Computer simulation of the dynamics of the spin-orbit motion of polarized beams in the NICA project for long times of the order of n x 1000 seconds"
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the loss of particles on the walls of the vacuum chamber. In addition, it is the Hamiltonian formalism that provides a clear mechanism for controlling the quality of the procedure for the numerical solution of the corresponding dynamical problems using the property of symplecticity of Hamiltonian systems that allows the correct use of various computational schemes – so-called symplectic integrators.

The main task of modeling the dynamics of the particle beam is the degree of influence of nonlinear (control) forces on the stability of particle motion. Traditionally, numerical methods are used to solve this problem, based mainly on the expansion of evolutionary equations in the Taylor series, see, first of all [6,7]. These methods are based on the Taylor series expansion in the vicinity of the reference trajectory. Usually, the number of particles in the beam is not less than 10^9–10^{12}, usually the dimension of the phase vector is equal to $\mathbf{X} \geq 4$. Given the number of integration steps, we are forced to take into account not only a huge number of equations but also the number of control elements. For example, the number of control elements (dipoles, quadrupoles, etc.) in Large Hadron Collider exceeds 9000 units.

It is note that the use of so-called thin elements (for example, for sextupoles and octupoles) can be used at the initial stage of modeling. For more fine-tuning, i.e. for determining the degree of influence of these or other control elements, it is necessary to more accurately investigate the influence of these elements. The use of packages based on the traditional Taylor series expansion leads to the need to perform not only a huge number of integration steps but also to do this for each trajectory, which substantially increases computational costs. In addition, the traditional methods of integrating the corresponding particle beam evolution equations do not ensure the preservation of the symplectic property, which significantly reduces the correctness of the obtained modeling results.

The Matrix Presentation of the Beam Evolution

The motion of a particle beam can be described using the Newton-Lorentz equations:

$$\frac{d\mathbf{X}}{ds} = \mathbf{F}(\mathbf{X}, s) = \sum_{k=0}^{\infty} \mathbb{P}^{1k}(\mathbf{E}, \mathbf{B}, s)\mathbf{X}^{[k]}, \tag{1}$$

where s – is an independent variable, $\mathbf{F}(\mathbf{X}, s) = \mathbf{F}(\mathbf{X}, \mathbf{E}, \mathbf{B}, s)$ is the Newton-Lorentz force, where $\mathbf{E} = \mathbf{E}(\mathbf{X}, s)$ – the electric field vector, $\mathbf{B} = \mathbf{B}(\mathbf{X}, s)$ – the vector of the magnetic induction. Solution of the motion Equation 1 can be written in the following matrix form:

$$\mathbf{X}(s) = \sum_{\mathbf{k}=0}^{\infty} \mathbb{R}^{1k}(\mathbf{E}, \mathbf{B}, s)\mathbf{X}_0^{[k]}, \tag{2}$$

where $\mathbf{X}_0 = \mathbf{X}(s_0)$ is the initial value of the phase vector. After fulfilling the basic requirements for the constructed linear model, various nonlinear effects are introduced into the system up to the necessary nonlinear order (2). Here we also note that the Equation 1 can be rewritten in the form

$$\mathbb{X}(s) = \sum_{k=0}^{\infty} \mathbb{R}^{1k}(\mathbf{E}, \mathbf{B}, s)\mathbb{X}_0^{[k]},$$

where $\mathbb{X} = (\mathbf{X}_1, \ldots, \mathbf{X}_N)$ is the matrix consisting of phase vectors for N particles.

Here and further, $\mathbf{X}$ is a Kronecker degree of the k-th order of the vector $\mathbf{X}$. The matrices $\mathbb{P}^{1k}(\mathbf{E}, \mathbf{B}, s)$, $k \geq 1$ correspond to control elements that depend on the control electromagnetic field, the matrices $\mathbb{R}^{1k}(\mathbf{E}, \mathbf{B}, s)$, $k \geq 1$ – represent the solutions of the corresponding equations of evolution. It should be noted that the traditional tensor representation of equations and the corresponding solutions that are used in practically all modern packages in beam physics from a computational point of view is not effective enough for two main reasons. First, these are numerical methods, which leads to the need to perform computational procedures for each trajectory separately, and when changing the parameters of the control elements, recalculate the transformations generated by them. When solving problems of searching for optimal solutions (in very different situations) in multiparameter problems, the number of operations performed often does not allow us to perform a search efficiently. The proposed approach with the use of Kronecker operations allows us to build not only numerical (matrix) mappings generated by the control system of the accelerator, but also to construct appropriate solutions in both numerical and symbolic form. We should note that the symbolic representation allows you not only use the corresponding solutions repeatedly but also create the corresponding databases. We should note that the very use of Kronecker operations allows us describing control systems (dipoles, quadrupoles, etc.) as a certain set of special elements (Lego-objects), use of which one can assemble from individual modules complex particle beam control systems using individual modules [8].

As an example, we give the equation for the evolution of particles (up to the second order of nonlinearity) to the second order of nonlinearity with respect to the phase coordinates $x, x', y, y', \delta p$, including taking into account the sextupole field. In this case we have

$$\begin{aligned} x'' + (1-n)\,h^2\,x = -h^3(1-2n+\beta)x^2 + h'(xx' + yy') + \\ + \frac{h}{2}(x'^2 - y'^2) + \\ + \frac{1}{2}(h'' - h^3 n + 2h^3\beta)y^2 + h\delta p - \\ - h\delta p^2 - 3C(x^2 - y^2) + O(3), \end{aligned}$$

$$\begin{aligned} y'' + n\,h^2\,y = 2h^3(\beta - n)xy - h'(xy' - xy') + hx'y' + \\ + h^2 ny\delta p - 3Cxy + O(3), \end{aligned}$$

where C is the coefficient characterizing the sextupole field and

$$n = -\frac{1}{h^2}\frac{\partial B_y(0,0,s)}{\partial x}, \quad \beta = \frac{1}{2h^3}\frac{\partial^2 B_y(0,0,s)}{\partial x^2};$$

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$$\mathbf{X}^2 = (x, x', \delta p, x^2, xx', x'^2, x\delta P, x'\delta P, \delta P^2, \\ xy, x'y, xy', x'y', y^2, yy', y'^2),$$

$$\mathbf{Y}^2 = (y, y', \delta p, y^2, yy', y'^2, y\delta P, y'\delta P, \delta P^2, \\ yx, yx', y'x, y'x', x^2, xx', x'^2),$$

The solution of equations

$$\frac{d\mathbf{X}^2}{ds} = \mathbb{P}_x\mathbf{X}^2, \quad \frac{d\mathbf{Y}^2}{ds} = \mathbb{P}_y\mathbf{Y}^2,$$

where

$$\mathbb{P}_x = \begin{pmatrix} \mathbb{P}_x^{11} & \mathbb{P}_x^{12} & \mathbb{P}_x^{13} & \mathbb{O} & \mathbb{P}_x^{15} \\ \mathbb{O} & \mathbb{P}_x^{22} & \mathbb{O} & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & \mathbb{P}_x^{33} & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{P}_x^{44} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{P}_x^{55} \end{pmatrix},$$

$$\mathbb{P}_y = \begin{pmatrix} \mathbb{P}_y^{11} & \mathbb{O} & \mathbb{P}_y^{13} & \mathbb{P}_y^{14} & \mathbb{O} \\ \mathbb{O} & \mathbb{P}_y^{22} & \mathbb{O} & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & \mathbb{P}_x^{33} & \mathbb{O} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{P}_x^{44} & \mathbb{O} \\ \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{O} & \mathbb{P}_x^{55} \end{pmatrix},$$

will be sought with the help of non-linear matrices $\mathbb{R}_x, \mathbb{R}_y$. In this case, the matrices $\mathbb{R}_x, \mathbb{R}_y$ have the same block structure as the matrices $\mathbb{P}_x$, $\mathbb{P}_y$. The corresponding solutions for the evolution matrices (upto second order of nonlinearities) $\mathbb{R}_x^{12}, \mathbb{R}_x^{13}, \mathbb{R}_y^{14}, \mathbb{R}_x^{15}$ can be computed using computer algebra codes (for example, Mathematica). As an example, we give the first element of the matrix $\mathbb{R}_x^{12}$:

$$r_{11} = -\frac{h^3(\beta + 2 - 2n) + 3C}{6\omega_x^2} \times \\ \times \left[3(1 - \cos\varphi_x) + (\cos\varphi_x - \cos 2\varphi_x)\right] + \\ + \frac{h}{12}\left[3(1 - \cos\varphi_x) - (\cos\varphi_x - \cos 2\varphi_x)\right],$$

$$n = -\frac{1}{hB_y^0}\frac{\partial B_y(x,y,s)}{\partial x}\bigg|_{\substack{x=0\\y=0}},$$

$$\beta = \frac{1}{2!h^2B_y^0}\frac{\partial^2 B_y(x,y,s)}{\partial x^2}\bigg|_{\substack{x=0\\y=0}},$$

$$B_y^0 = \frac{1}{L}\int_0^L B_y(0,0,\tau)d\tau,$$

where L is the length of equilibrium trajectory, h is the curvature of the equilibrium trajectory, C is coefficient characterizing a sextupole magnetic field.

Similar expressions can be calculated in symbolic forms and used in the process of parametric study of the influence of control parameters on the characteristics of the beam

CONCLUSION

The above reasoning and mathematical formulas demonstrate the possibility of representing information about dynamic equations in a symbolic form. It is this kind of representation that allows us to carry out parametric studies in order to find the optimal solutions for the formation of the accelerator structure. Using the method of matrix formalism to describe the dynamics of the beam in both numerical and symbolic form substantially reduces the corresponding computational pressures. In addition, the matrix formalism makes it possible to guarantee the preservation of the symplecticity property, which ensures the necessary correctness of the numerical expressions. On the Fig. 1 the main steps are presented for calculating the necessary matrices upto the required order of nonlinearity.

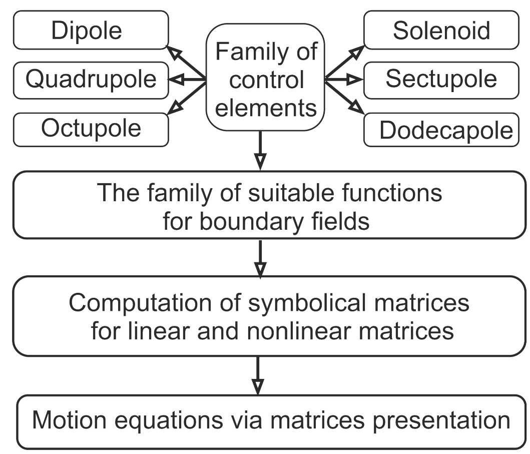

Figure 1: The components for modelling of the control systems for accelerator via

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