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FRINGE FIELD EFFECT OF SOLENOIDS*

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Abstract

We derive a precise analytical nonlinear transverse map for single particle transport through a solenoid with hard edge fringe fields. The transfer map is two dimensional for transverse coordinates and momenta with fixed longitudinal momentum. Because it is an accurate analytic map, it is also symplectic. The transfer map is compared with exact numerical tracking.

INTRODUCTION

We present ongoing research on the nonlinear transport of a single particle through a magnetic coil/solenoid. Solenoids are among the oldest optical elements that have been used for focussing charged particles, and they are widely used in high energy accelerators. The linear transfer map for charged particle dynamics in solenoids is well known and can be found for example in [1]. However, real particle transport through solenoids contains nonlinear effects due to the structure of its transverse magnetic field components at the entrance and exit regions of the solenoid. These nonlinear effects can play an important role in some accelerators that require precise tuning [2-4]. There are well-known Lie algebraic methods in the literature to find transfer maps for different optical elements by applying Lie operators and integrating Hamiltonians [1, 5]. Unfortunately, no explicit form for precise nonlinear transfer maps for solenoids was found in the literature. Only approximate, semi-numerical methods [6-8] were found.

In this paper we present an explicit precise nonlinear transfer map for solenoids with hard edge fringe fields. We derive this map from the relativistic equation of a charged particle in a magnetic field **B**. As the "hard edge" fringe field, we assume that **B** satisfies the Maxwell/Laplacian equation and that its longitudinal component is the Heaviside step function. The relativistic equation with this **B** field can be solved by an iteration/perturbation method from which it is possible to get a solution in the form of an infinite series with explicitly defined coefficients. In this form we calculated the sum of this series and obtained a final analytical formula for the nonlinear transfer map. Also, this method can be applied for precise analytical tracking in other optical elements such as quadrupoles and multipoles.

BEAM DYNAMICS IN SOLENOID

The magnetic field of a cylindrically symmetric solenoid in a Cartesian frame B(x,y,z) is completely defined by its longitudinal magnetic field on the z-axis B(0,0,z) = b(z):

$$B=\begin{bmatrix} -\frac{1}{2}xb'(z)+\frac{1}{16}x(x^2+y^2)b'''(z) \\ -\frac{1}{2}yb'(z)+\frac{1}{16}y(x^2+y^2)b'''(z) \\ b(z)-\frac{1}{4}(x^2+y^2)b''(z) \end{bmatrix} \quad (1)$$

At this point we assume b(z) to be continuous, increasing for the entrance fringe field and decreasing for the exit fringe field. We assume b(z) is constant inside the solenoid b(z)→B_0. The relativistic equation of charged particle motion in a magnetic field can be written as

$$\ddot{r}=\frac{q}{m\gamma}[\dot{r}\times B]. \quad (2)$$

Equation (2) can be simplified for the 2D transverse coordinates in terms of z, assuming constant longitudinal momentum $\dot{z}(t)=\beta c=$ const

$$\begin{bmatrix} x' \\ y' \end{bmatrix}=k_0\begin{bmatrix} +ky'+\frac{yk'}{2}-\frac{(x'^2+y'^2)(4y'k''+yk''')}{16} \\ -kx'-\frac{xk'}{2}+\frac{(x'^2+y'^2)(4x'k''+xk''')}{16} \end{bmatrix} \quad (3)$$

where $k_0=qB_0/cm\beta\gamma$ is a constant of motion and all other values x, y, k in Eq. (3) depend on z. $k(z)=b(z)/B_0$ is the normalized longitudinal magnetic field. We wish to calculate the particle transport through a solenoid by a transfer map M_{sol}:

$$M_{sol}=M_{in}M_{body}M_{out} \quad (4)$$

where M_{sol} is the full transfer map of the solenoid and M_{in} and M_{out} contain the entrance and exit fringe field effects. The whole linear map M_{sol} as well as the linear focussing effect of fringe fields can be found for example in [1].

To calculate the nonlinear fringe maps M_{in} and M_{out} we need to solve the particle motion through the solenoid using Eq. (3), and cast the result in the form of Eq. (4). Equation (4) represents the solenoid as two thin elements of fringe field surrounding a thick solenoid with so called effective length $L_{eff}=\int k(z)dz$. The differential Eq. (2) or (3) can be solved using an iteration/perturbation method with an initial nonperturbed "drift" orbit at zero magnetic field for a particle coming from z = -∞ (see Fig. 1):

$$\begin{aligned} x_0(z)&=x+x_pz \\ y_0(z)&=y+y_pz \end{aligned} \quad (5)$$

In the iterative method, the n^th iteration is obtained by integrating Eq. (3) from -∞ to some inner point of the solenoid z by using the previous iteration:

$$r_{n+1}(z)=r_0(z)+\int_{-\infty}^{z}\int_{-\infty}^{u}\frac{q}{m\gamma c\beta}\left[\frac{\partial r_n}{\partial z}\times B\right]dp\,du \quad (6)$$

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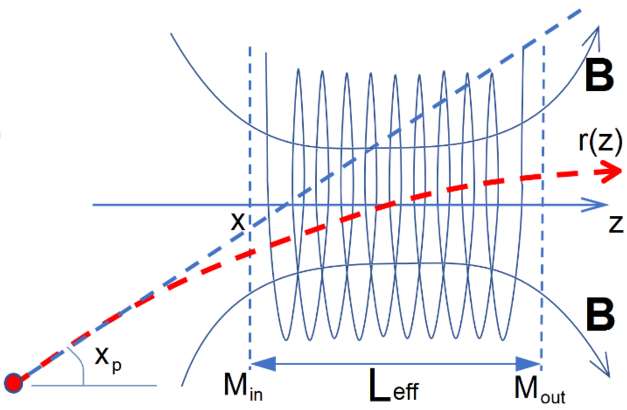

Figure 1: Schematic of particle input parameters drifting from z=-∞ and its orbit in solenoid.

After the n[th] integration (6) we find the solution expanded over the amplitude of the magnetic field B_0 or k_0:

$$\begin{aligned} x_n(z) &= x + x_p z + \textstyle\sum_{i=1}^{n} k_0^i x_i(z) \\ y_n(z) &= y + y_p z + \textstyle\sum_{i=1}^{n} k_0^i y_i(z) \end{aligned} \tag{7}$$

Although it is beyond the scope of this paper, it is possible to extract an algebraic recurrence expression from Eq. (6) for $x_i(z)$ in series (7).

Now, it also can be shown that the transfer map for the entrance fringe field M_{in} to the magnet center z=0 with input {x, x_p, y, y_p} and output transverse coordinates $\{x, x_p, y, y_p\}^f$

$$\{x, x_p, y, y_p\}^f = M_{in}\{x, x_p, y, y_p\} \tag{8}$$

can be extracted from the linear part of asymptotic orbit (7) for z→∞:

$$r(z)_f = \begin{matrix} \lim\limits_{z\to\infty}\left[x + x_p z + \sum_{i=1}^{n} k_0^i x_i(z)\right] \\ \lim\limits_{z\to\infty}\left[y + y_p z + \sum_{i=1}^{n} k_0^i y_i(z)\right] \end{matrix} \tag{9}$$

or

$$\begin{bmatrix} x \\ x_p \\ y \\ y_p \end{bmatrix}_f = \begin{bmatrix} \lim\limits_{z\to\infty}\left[x + x_p z + \sum_{i=1}^{n} k_0^i x_i(z)\right] \\ \frac{\partial}{\partial z}\lim\limits_{z\to\infty}\left[x + x_p z + \sum_{i=1}^{n} k_0^i x_i(z)\right] \\ \lim\limits_{z\to\infty}\left[y + y_p z + \sum_{i=1}^{n} k_0^i y_i(z)\right] \\ \frac{\partial}{\partial z}\lim\limits_{z\to\infty}\left[y + y_p z + \sum_{i=1}^{n} k_0^i y_i(z)\right] \end{bmatrix}_{z=0} \tag{10}$$

After calculating (10) we obtained the transfer map:

$$\begin{bmatrix} x \\ x_p \\ y \\ y_p \end{bmatrix}_f = \begin{bmatrix} x + \sum_{i=1}^{\infty} k_0^i x_i \\ x_p + \sum_{i=1}^{\infty} k_0^i x_{pi} \\ y + \sum_{i=1}^{\infty} k_0^i y_i \\ y_p + \sum_{i=1}^{\infty} k_0^i y_{pi} \end{bmatrix} \tag{11}$$

In the general case expression (11) contains complicated parameters that depend on the longitudinal magnetic field profile *k(z)* in the form of integrals. The hard edge model of the fringe field assumes *k(z)* to be a Heaviside step function with a derivative that equals a Dirac delta function *k'(z)=δ(z)*. In this case we have only one nonzero parameter in (11):

$$\int_{-\infty}^{z} k'(z)dz = 1 \tag{12}$$

From the algebraic recurrence expression for $x_i(z)$ and $y_i(z)$ we can obtain terms x_i, y_i, x_{pi}, y_{pi} for the nonlinear fringe field map (11):

$$\begin{aligned} x_i &= \frac{1}{16i}\sum_{p=0}^{i-2}\sum_{m=0}^{p} x_{i-p-2}\left(x_m x_{p-m} + y_m y_{p-m}\right) \\ &\quad + \frac{1}{8i}\sum_{p=0}^{i-1}\sum_{m=0}^{p}\Big[2x'_{i-p-1}x_m y_{p-m} \\ &\quad + y'_{i-p-1}\left(y_m y_{p-m} - x_m x_{p-m}\right)\Big] \\ x'_i &= -\frac{1}{16i}\sum_{p=0}^{i-2}\sum_{m=0}^{p}\Big[x'_{i-p-2}\left(3x_m x_{p-m} + y_m y_{p-m}\right) \\ &\quad + 2y'_{i-p-2}x_m y_{p-m}\Big] + \left(1 - \frac{1}{2i}\right) y_{i-1} \\ &\quad + \frac{1}{8i}\sum_{p=0}^{i-1}\sum_{m=0}^{p}\Big[2x_{i-p-1}x'_m y'_{p-m} \\ &\quad + y_{i-p-1}\left(y'_m y'_{p-m} - x'_m x'_{p-m}\right)\Big] \end{aligned} \tag{13}$$

$$\begin{aligned} y_i &= \frac{1}{16i}\sum_{p=0}^{i-2}\sum_{m=0}^{p} y_{i-p-2}\left(x_m x_{p-m} + y_m y_{p-m}\right) \\ &\quad - \frac{1}{8i}\sum_{p=0}^{i-1}\sum_{m=0}^{p}\Big[2y'_{i-p-1}x_m y_{p-m} \\ &\quad + x'_{i-p-1}\left(x_m x_{p-m} - y_m y_{p-m}\right)\Big] \\ y'_i &= -\frac{1}{16i}\sum_{p=0}^{i-2}\sum_{m=0}^{p}\Big[y'_{i-p-2}\left(3y_m y_{p-m} + x_m x_{p-m}\right) \\ &\quad + 2x'_{i-p-2}x_m y_{p-m}\Big] - \left(1 - \frac{1}{2i}\right) x_{i-1} \\ &\quad - \frac{1}{8i}\sum_{p=0}^{i-1}\sum_{m=0}^{p}\Big[2y_{i-p-1}x'_m y'_{p-m} \\ &\quad + x_{i-p-1}\left(x'_m x'_{p-m} - y'_m y'_{p-m}\right)\Big] \end{aligned}$$

where $i = 1 \ldots \infty$ and $\{x_0, x'_0, y_0, y'_0\} = \{x, x_p, y, y_p\}$ are input coordinates for the transfer map (11).

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Entrance Fringe Field Transfer Map

As an example, we present the second order input fringe field transfer map (n=2) in explicit form as calculated by (13):

$$
\begin{aligned}
x^f &= x + \frac{k_0}{8}\left(2x_p xy + y_p(y^2 - x^2)\right) \\
&\quad + \frac{k_0^2}{128}\left(8x(x^2+y^2) + 2x_p y_p y(y^2-3x^2) + x(3y^2 - x^2)(x_p^2 - y_p^2)\right) \\
x_p^f &= x_p + \frac{k_0}{8}\left(2x_p y_p x + y(4 - x_p^2 + y_p^2)\right) \\
&\quad + \frac{k_0^2}{128}\left(x_p(y^2-x^2)(x_p^2 - 3y_p^2) + 2y_p xy(y_p^2 - 3x_p^2 - 16) - 32x^2 x_p\right)
\end{aligned}
\tag{14}
$$

$$
\begin{aligned}
y^f &= y - \frac{k_0}{8}\left(2y_p xy + x_p(x^2 - y^2)\right) \\
&\quad + \frac{k_0^2}{128}\left(8y(x^2+y^2) + 2x_p y_p x(x^2-3y^2) + y(3x^2 - y^2)(y_p^2 - x_p^2)\right) \\
y_p^f &= y_p - \frac{k_0}{8}\left(2x_p y_p y + x(4 - y_p^2 + x_p^2)\right) \\
&\quad + \frac{k_0^2}{128}\left(y_p(x^2-y^2)(y_p^2 - 3x_p^2) + 2x_p xy(x_p^2 - 3y_p^2 - 16) - 32y^2 y_p\right)
\end{aligned}
$$

Exit Fringe Field Transfer Map

Exit fringe field transfer map can be derived in the same way as the entrance map, but it will be different from (13), (14). By using the symmetry of the solenoid field and particle dynamics it can be derived that the transfer map for the exit will be exactly inverse to the entrance one. If the transfer map (8) with explicit form (11) (13) is expressed formally as a 4D vector function, then the exit map is the inverse function:

$$\{x, x_p, y, y_p\}^f = M_{out}\{x, x_p, y, y_p\} = M_{in}^{-1}\{x, x_p, y, y_p\} \tag{15}$$

We will not present recurrence formulas for the exit map similar to Eq. (13), but we will present explicit form of the second order map:

$$
\begin{aligned}
x^f &= x - \frac{k_0}{8}\left(2x_p xy + y_p(y^2 - x^2)\right) \\
&\quad + \frac{k_0^2}{128}\left(2x_p y_p y(y^2-3x^2) - x(x^2 - 3y^2)(x_p^2 - y_p^2)\right) \\
x_p^f &= x_p - \frac{k_0}{8}\left(2x_p y_p x + y(4 - x_p^2 + y_p^2)\right) \\
&\quad + \frac{k_0^2}{128}\left(x_p(x^2-y^2)(8 - x_p^2 + 3y_p^2) + 2y_p xy(8 - 3x_p^2 + y_p^2)\right)
\end{aligned}
\tag{16}
$$

$$
\begin{aligned}
y^f &= y + \frac{k_0}{8}\left(2y_p xy + x_p(x^2 - y^2)\right) \\
&\quad + \frac{k_0^2}{128}\left(2x_p y_p x(x^2-3y^2) - y(y^2 - 3x^2)(y_p^2 - x_p^2)\right) \\
y_p^f &= y_p + \frac{k_0}{8}\left(2x_p y_p y + x(4 - y_p^2 + x_p^2)\right) \\
&\quad + \frac{k_0^2}{128}\left(y_p(y^2-x^2)(8 - y_p^2 + 3x_p^2) + 2x_p xy(8 - 3y_p^2 + x_p^2)\right)
\end{aligned}
$$

It can be verified that expressions (16) are not exactly inverse to (14), but they are inverse with accuracy of $O(k_0^3)$. It can be also verified that the whole map of the solenoid (4) is symplectic with accuracy of $O(k_0^{n+1})$, where n is expansion order of the map (11).

Remarkably, the infinite series (11) and the corresponding map can be calculated analytically and presented in the form of simple analytical functions, but we omit this here due to lack of space. Conditions for convergence of the series (11) also have been found.

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