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EFFECTS OF SYNCHROTRON MOTION ON NONLINEAR INTEGRABLE OPTICS*

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Abstract

An integrable Rapid-Cycling Synchrotron (iRCS) has been proposed as a replacement for the Fermilab Booster to achieve multi-MW beam power for the Fermilab high-energy neutrino program. The successful application of nonlinear integrable optics to proton synchrotrons requires careful examination of single-particle longitudinal effects, especially synchrotron motion. For example, synchrobetatron coupling may excite transverse resonances in the ring. We will use the Synergia code to simulate the effects of this synchrobetatron coupling on the iRCS design with nonlinear inserts. Assuming the synchrotron tune is sufficiently small, we have identified one or more adiabatic invariants of the motion. These invariants suggest that integrable optics with synchrobetatron coupling retains integrability when averaged over a synchrotron period.

ADIABATIC RF AND INTEGRABLE DYNAMICS

Nonlinear integrable optics has been thoroughly studied for on-momentum, single-particle dynamics [1]. There has been some work on off-momentum dynamics [2] – chromaticity and dispersion – but for many applications we must consider synchrotron oscillations. Because the synchrotron tune is small for trajectories far from the separatrix, $\nu_s(J) << 1$ for synchrotron action variable J, we can treat the synchrotron oscillations as a slow-oscillating perturbation to the integrable optics Hamiltonian. In terms of a single turn map [3] with RF, we can write this as

$$\mathcal{M}_{ring} = \mathcal{M}_0 \mathcal{M}_{rf} \tag{1}$$

where the initial particle coordinates $z^{in.}$ are given by $z^{fin.} = \mathcal{M}_{ring} \circ z^{in.}$ after a single turn in the ring. Here $\mathcal{M}_0$ is the single-turn map for the transverse dynamics, including chromatic effects, and $\mathcal{M}_{rf}$ is the map for the pure synchrotron motion.

In general, we write the maps in terms of Lie operators [4]

$$\mathcal{M}_0 = \exp\{-:\mathcal{H}(\vec{p}_\perp, \vec{q}_\perp; \delta(J,\theta):\} \tag{2a}$$

$$\mathcal{M}_{rf} = \exp\{-:\epsilon\mathcal{V}(J):\} \tag{2b}$$

with δ the energy offset. The synchrotron tune is given by

$$\epsilon\nu_s(J) = \epsilon\frac{\partial\mathcal{V}}{\partial J} \tag{3}$$

and $\mathcal{H}$ would be integrable if δ were kept constant – i.e. if there were no synchrotron oscillations. In principle, the longitudinal dynamics changing δ should break the transverse invariants. However, if the synchrotron tune is sufficiently small, we will be able to recover a Hamiltonian that is integrable over many turns.

N-TURN HAMILTONIAN

The map for taking N turns is

$$\left(\mathcal{M}_{ring}\right)^N = \left(\mathcal{M}_0\mathcal{M}_{rf}\right)^N \tag{4}$$

which we can exactly rewrite as[1]

$$\left(\mathcal{M}_{ring}\right)^N = \left(\prod_{j=0}^{N-1}\mathcal{M}_0^{(j)}\right)\left(\mathcal{M}_{rf}\right)^N \tag{5a}$$

$$\mathcal{M}_0^{(j)} = \left(\mathcal{M}_{rf}\right)^j \mathcal{M}_0 \left(\mathcal{M}_{rf}\right)^{-j} \tag{5b}$$

From the similarity transformation property of symplectic maps [5], we can rewrite the Hamiltonian for $\mathcal{M}_0^{(m)}$ as

$$\mathcal{M}_0^{(m)} = \exp\{-:\mathcal{H}(\vec{p}_\perp, \vec{q}_\perp; \delta(J, \theta + m\epsilon\nu_s(J)):\} \tag{6}$$

Using the Baker-Campbell-Hausdorff series, we can compute to leading order the product as an exponential operator

$$\left(\prod_{j=0}^{N-1}\mathcal{M}_0^{(j)}\right) \approx \exp\left\{-:\sum_{m=0}^{N-1}\mathcal{H}(\vec{p}_\perp, \vec{q}_\perp; \delta(J, \theta + m\epsilon\nu_s(J)):\right\} \tag{7}$$

where we have dropped the higher order terms. So long as $\epsilon\nu_s(J) \ll 2\pi$ we can approximate this sum as an integral over θ, so our N-turn Hamiltonian for the transverse dynamics is

$$\mathcal{H}^{(N)} \approx \frac{1}{\epsilon\nu_s(J)}\int_0^{N\epsilon\nu_s(J)} d\theta\mathcal{H}(\vec{p}_\perp, \vec{q}_\perp; \delta(J,\theta)). \tag{8}$$

Because the integrable system is periodic in θ, we can rewrite $\mathcal{H}$ as a Fourier series

$$\mathcal{H}(\vec{p}_\perp, \vec{q}_\perp; \delta(J,\theta)) = \sum_k H^{(k)}(\vec{p}_\perp, \vec{q}_\perp; J)e^{ik\theta} \tag{9}$$

which, after integrating over one or more synchrotron periods, leaves the N-turn Hamiltonian as

$$\mathcal{H}^{(N)} \approx N \times H^{(0)}(\vec{p}_\perp, \vec{q}_\perp; J) + \text{oscillating terms.} \tag{10}$$

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[1] The details of this calculation are beyond the scope of this paper, but will be in a paper submitted to the IPAC '18 Special Edition of PRAB.

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The N-turn map can then be given, approximately, as

$$\left(\mathcal{M}_{ring}\right)^N \approx \exp\left\{-N :H^{(0)}(\vec{p}_\perp, \vec{q}_\perp; J) + \epsilon\mathcal{V}(J):\right\}. \quad (11)$$

If $H^{(0)}$ is integrable, then there is a canonical transformation that allows us to transform to

$$H^{(0)}(\vec{p}_\perp, \vec{q}_\perp; J) \mapsto \tilde{H}^{(0)}(\vec{I}_\perp; J) \quad (12)$$

for the transverse action variables $\vec{I}_\perp$.

This applies to the integrable optics Hamiltonian described in [2], where the single-turn Hamiltonian with off-momentum terms was calculated to be

$$\begin{aligned}\mathcal{H} =& \mu_0\left\{(1 - C_x(\delta))\,\frac{p_x^2 + x^2}{2} + \right.\\ &\left.(1 - C_y(\delta))\,\frac{p_y^2 + y^2}{2} + tU_{DN}(x, y)\right\}\end{aligned} \quad (13)$$

for the fractional phase advance μ_0 across the integrable optics insert drift, tU_{DN} the potential derived by Danilov and Nagaitsev, and $C(\delta)$ the chromaticity. The dressed chromaticity $\overline{C}(J) = \int d\theta C(\delta(J, \theta))$ lets us rewrite the N-turn Hamiltonian as

$$\begin{aligned}H^{(0)} =& \mu_0\left\{\left(1 - \overline{C}_x(J)\right)\frac{p_x^2 + x^2}{2} + \right.\\ &\left.\left(1 - \overline{C}_y(J)\right)\frac{p_y^2 + y^2}{2} + tU_{DN}(x, y)\right\}\end{aligned} \quad (14)$$

which, if the chromaticities are equal, we can rewrite as[2]

$$\begin{aligned}H^{(0)} = \mu_0\left(1 - \overline{C}(J)\right)&\left\{\frac{p_x^2 + x^2}{2} + \frac{p_y^2 + y^2}{2} + \right.\\ &\left. t\left(1 - \overline{C}_x(J)\right)^{-1} U_{DN}(x, y)\right\}\end{aligned} \quad (15)$$

which is integrable with a renormalized t value and with adjusted phase advances.

From this result, we can expect the Danilov-Nagaitsev invariants to be conserved every N turns, with oscillations with the synchrotron motion around a nominal value.

NUMERICAL EVIDENCE – IRCS TRACKING DATA

Preliminary studies of an iRCS design [6] support this conclusion. The relevant parameters for the iRCS are given in Table 1. In this case, the vertical and horizontal chromaticity are close but not equal (Fig. 1), giving a small perturbation to the integrable system.

As we can see in Fig. 1, the chromaticities for positive δ are very close, with differences in the tune well under 0.001, while for negative δ the chromaticity difference is much larger. We expect to see this as an asymmetry in the fast oscillations in the invariant – during the phase of the synchrotron period where $\delta > 0$ we will see larger variation in the invariants than in the phase where $\delta < 0$.

We see this correlation between the energy deviation and the level of deviation from the invariant in Fig. 2. As the energy deviation goes negative, the envelope of the oscillations of the two invariants grows, suggesting a stronger perturbing term at negative energy deviation. The synchrotron period is very close to twelve turns. If we only look every synchrotron

[2] If the chromaticities are not equal, there remains a perturbation to the Danilov-Nagaitsev Hamiltonian involving the linear transverse dynamics of the lattice proportional to $\overline{C}_x(J) - \overline{C}_y(J)$.

Table 1: Relevant Parameters of iRCS Lattice

Parameter	Value
Periodicity	12
Betatron Tune	21.6
Synchrotron Tune	0.08
Phase-advance over insert	$0.3 \times 2\pi$
Nonlinear Strength t-value	0.3
Elliptic Distance c-value	$0.14\ \mathrm{m}^{1/2}$

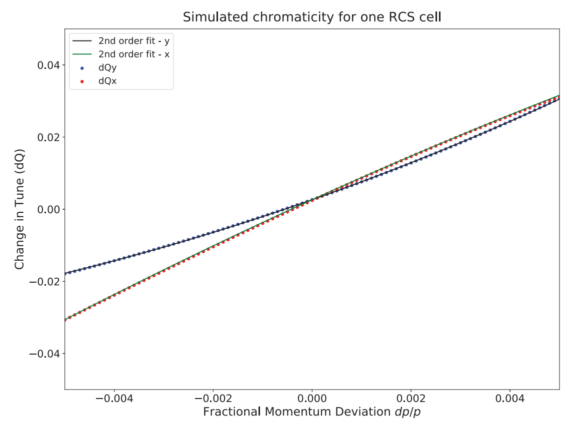

Figure 1: iRCS vertical and horizontal chromaticities.

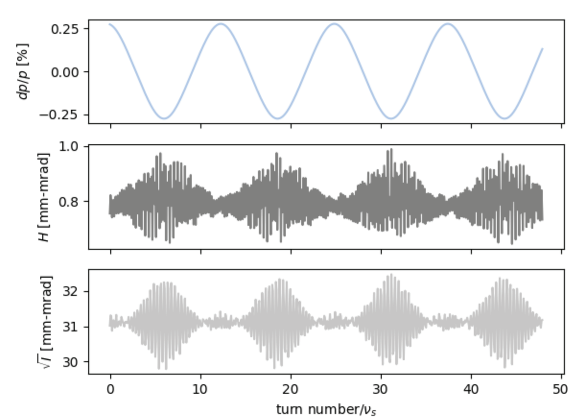

Figure 2: Envelope oscillations of H and $\sqrt{I}$ with the energy offset, but the system is periodic with the synchrotron motion.

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period, we see that H and $\sqrt{I}$ [3] are very well conserved, and are much better-conserved near when the chromaticities are equal. The region with equal chromaticities is the region where the single-turn Hamiltonian is closest to the modified Danilov-Nagaitsev Hamiltonian in [2]. This pattern persists over many synchrotron oscillations in simulations, thereby suggesting the adiabatic invariant described by the map approach above.

We can also visualize how H and I vary with $\delta p/p$ in the 3D plot in Fig. 3. This plot includes data over approximately twenty synchrotron oscillations. The H and $\sqrt{I}$ values fill in a trapezoidal area at each fixed slice of $\delta p/p$, indicating that the motion is bounded and that there are two nearby invariants even when the chromatic perturbations are stronger.

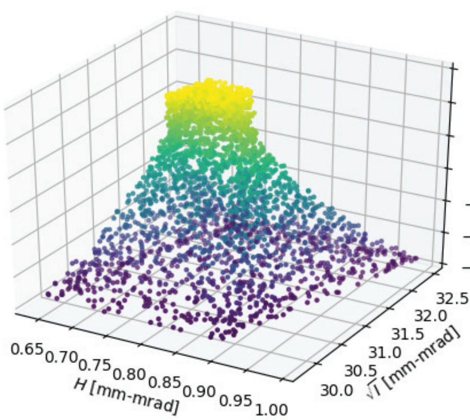

Figure 3: Scatter plot of H and $\sqrt{I}$ versus $\delta p/p$. The color map is coordinated with the momentum offset, so that yellow is very positive and purple is very negative.

These plots are all for a single trajectory, but the same basic pattern forms regardless of the particle for the iRCS lattice considered.

CONCLUSION & FUTURE WORK

We have presented a high-level theoretical calculation of the N-turn map for an integrable optics ring with synchrotron motion and momentum-dependence in the ring. That map suggests a phase-averaged Hamiltonian over many turns which contains the primary dynamics, and is approximately conserved. We have found evidence for this Hamiltonian in tracking simulations of an integrable Rapid-Cycling Synchrotron. This suggests that synchrobetatron coupling retains integrability in an averaged sense over many turns.

This analysis assumes that we average over N periods, so that $N\epsilon\nu_s(J) \gg 2\pi$, which requires increasingly large N as we approach the rf separatrix. If we sample data for $N > 2\pi/\nu_s(J)$ turns, we expect to see the invariants changing slowly from turn to turn with no periodic return – this is what we mean by saying that the integrable optics remains integrable on the average for synchrobetatron coupling.

Future work will focus on comparing the theoretical prediction that the transverse Hamiltonian shifts the synchrotron tune to computational data. We expect that particles with large initial $H^{(0)}$ will have a stronger effect on the synchrotron tune. We also will work on understanding how the perturbing term due to the unequal chromaticities affects the existence of an invariant, and how unequal the chromaticities can be before we lose dynamic aperture. This will lead to important guidelines for designing integrable optics lattices.

[3] We use $\sqrt{I}$ instead of the I derived in [1] because $\sqrt{I}$ has the same units as H.

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