LONGITUDINAL AND QUADRUPOLAR COUPLING IMPEDANCE OF AN ELLIPTICAL VACUUM CHAMBER WITH FINITE CONDUCTIVITY IN TERMS OF MATHIEU FUNCTIONS*

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Abstract

The resistive wall impedance of an elliptical vacuum chamber in the classical regime with infinite thickness is known analytically for ultra-relativistic beams by means of the Yokoya form factors.

Starting from the longitudinal electric field of a point charge moving at arbitrary speed in an elliptical vacuum chamber, which we express in terms of Mathieu functions, in this paper we take into account the finite conductivity of the beam pipe walls and evaluate the longitudinal and quadrupolar impedance for any beam velocity. We also obtain that the quadrupolar impedance of a circular pipe is different from zero, approaching zero only for ultra-relativistic particles.

Even if some of the results, in particular in the ultra-relativistic limit, are already known and expressed in terms of form factors, this approach is the first step towards the calculation of the general problem of a multi-layer vacuum chamber with different conductivities and of elliptic cross section.

INTRODUCTION

The coupling impedance [1,2] of a resistive vacuum chamber represents an important contribution to the total machine impedance, in particular for large particle accelerators [3,4]. Among several geometries of the beam pipe, the elliptic cross section is very common [5,6].

The impedance of an elliptical lossy vacuum chamber, and more in general with an arbitrary cross section, has been derived in the ultra-relativistic limit in refs. [7–10]. It is expressed in terms of form factors, known as Yokoya form factors, which depend on the ellipticity of the beam pipe and correspond to the ratio between the impedance with elliptic cross section and circular one with radius equal to the minor semi-axis of the ellipse. For a perfectly conducting elliptic pipe, an equivalent radius at low frequency has also been derived in ref [11].

The extension to the non relativistic case, for the elliptic cross section, has been obtained in ref. [12], where, however, a Gaussian beam distribution, and not a point charge, has been used, leading to a quite complicated formulation of the field. Another formulation, expressed as an integral form, has been also derived in ref. [13] in the classical regime for a good conductor.

In this paper, starting from the longitudinal electromagnetic field of a point charge in an elliptical perfectly conducting beam pipe obtained in ref. [14] as expansion of Mathieu functions, we first derive the indirect, or scattered, field due to the finite conductivity of the beam pipe. This field, valid in the classical regime of infinite thickness and for a good conductor, allows to derive the longitudinal and the quadrupolar resistive wall impedance for arbitrary beam velocities.

This represents a first step towards the derivation of the resistive wall impedance for a multilayer vacuum chamber with elliptical cross section.

LONGITUDINAL ELECTRIC FIELD IN A PERFECTLY CONDUCTING ELLIPTICAL PIPE

Let's consider a point charge travelling with velocity $v = \beta c$ along the axis of an elliptical pipe. To describe the geometry we use confocal elliptical coordinates φ, describing a set of hyperbolas having the same foci, and μ, describing a set of confocal ellipses, as shown in Fig. 1.
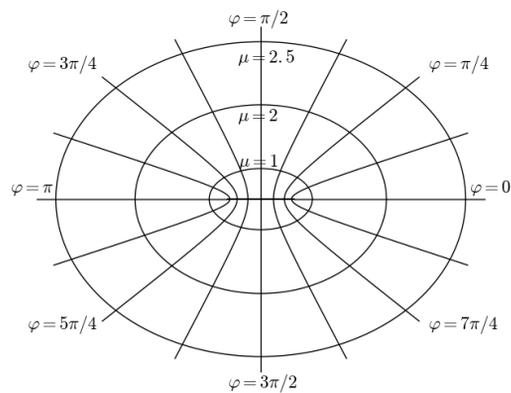

Figure 1: Elliptic coordinates.

The relation between elliptical and Cartesian coordinates is given by

$$\begin{cases} x = F \, \cosh \mu \, \cos \varphi, \\ y = F \, \sinh \mu \, \sin \varphi, \end{cases} \tag{1}$$

where F is the focal distance of the ellipse, related to the major and minor semi-axis a and b by

$$F = \sqrt{a^2 - b^2}. \tag{2}$$

These coordinates are useful to express the Mathieu functions [15]. In particular we define the elliptic cosine even function of negative argument $-q$ as

$$ce_{2l}(\varphi, -q) = (-1)^l \sum_{r=0}^{\infty} (-1)^r A_{2r}^{(2l)} \cos(2r\varphi), \quad (3)$$

and the corresponding radial modified Mathieu functions of the first and second kind respectively as

$$\begin{aligned} Ce_{2l}(\mu, -q) &= (-1)^l \sum_{r=0}^{\infty} (-1)^r A_{2r}^{(2l)} \cosh(2r\mu), \\ Fek_{2l}(\mu, -q) &= \frac{p'_{2l}}{\pi A_0^{(2l)}} \sum_{r=0}^{\infty} A_{2r}^{(2l)} I_r(\nu_1) K_r(\nu_2), \end{aligned} \quad (4)$$

with

$$p'_{2l} = (-1)^l \frac{ce_{2l}(0, q)\, ce_{2l}(\frac{\pi}{2}, q)}{A_0^{(2l)}}, \quad (5)$$

and $\nu_1 = \sqrt{q}e^{-\mu}$ and $\nu_2 = \sqrt{q}e^{\mu}$. Here $I_r(x)$ and $K_r(x)$ are the modified Bessel functions of first and second kind respectively. The expansion coefficients $A_{2r}^{(2l)}$ can be obtained by solving an eigenvalue problem of a truncated matrix [14].

With the above expressions, the longitudinal electric field produced by a point charge moving on the axis of a perfectly conducting elliptic vacuum chamber has been written in ref. [14] as an infinite series of Mathieu functions:

$$\begin{aligned} E_z^0 = 2\pi G \sum_{l=0}^{\infty} \frac{A_0^{(2l)}}{p'_{2l}} ce_{2l}\left(\frac{\pi}{2} - \varphi, q\right) \\ \left(Fek_{2l}(\mu, -q) - \frac{Fek_{2l}(\mu_0, -q)}{Ce_{2l}(\mu_0, -q)} Ce_{2l}(\mu, -q)\right), \end{aligned} \quad (6)$$

with

$$G = j\frac{Z_0 Q k_0}{2\pi \beta^2 \gamma^2}, \quad q = \left(\frac{k_0 F}{2\beta\gamma}\right)^2, \quad \cosh \mu_0 = \frac{a}{F}. \quad (7)$$

Here Q is the point charge, β and γ the relativistic factors, Z_0 the vacuum impedance, and k_0 the wave number in free space, equal to ω/c.

LONGITUDINAL ELECTRIC FIELD IN A FINITE CONDUCTIVITY ELLIPTICAL PIPE

The longitudinal electric field given by Eq. (6) allows to derive the azimuthal magnetic field inside the perfectly conducting pipe as

$$H_\varphi = \frac{\beta\gamma}{k_t Z_0} \frac{\partial E_z^0}{\partial n} = \frac{\sqrt{2}\beta\gamma}{k_t Z_0 F \sqrt{\cosh 2\mu - \cos 2\varphi}} \frac{\partial E_z^0}{\partial \mu}, \quad (8)$$

where $k_t = jk_0/\beta\gamma$, Z_0 is the vacuum impedance, and n the coordinate normal to the iso-azimuthal lines.

This magnetic field, evaluated at the boundary $\mu = \mu_0$, can be written in terms of the Wronskian

$$W_{2l}(-q) = (-1)^{l+1} \frac{p'_{2l}}{\pi A_0^{(2l)}} ce_{2l}\left(\frac{\pi}{2}, -q\right) ce_{2l}(0, -q), \quad (9)$$

as

$$\begin{aligned} H_\varphi = \frac{2\pi\beta\gamma G\sqrt{2}}{k_t Z_0 F} \\ \sum_{l=0}^{\infty} \frac{(-1)^l A_0^{(2l)} ce_{2l}(\varphi, -q)}{p'_{2l}\sqrt{\cosh 2\mu_0 - \cos 2\varphi}} \frac{W_{2l}(-q)}{Ce_{2l}(\mu_0, -q)}. \end{aligned} \quad (10)$$

In case of a vacuum chamber with a finite conductivity σ, we use the approximation that Eq. (10) remains valid, for a good conductor, also at the boundary in the conducting material. By applying then the Leontovich [16] condition, from the magnetic field we can obtain the electric field induced in the pipe wall as

$$E_z^\sigma(\varphi, \mu_0, q) = \frac{1 + j}{\delta\sigma} H_\varphi = Z_s H_\varphi, \quad (11)$$

with δ the skin depth and Z_s the surface impedance. This relation is valid for a wall of infinite thickness.

We suppose now that the electric field in the vacuum has the same configuration as that inside the perfectly conducting pipe of Eq. (6) plus an additional term due to the scattered field of the finite conducting wall, which we write as

$$\begin{aligned} E_z^i(\varphi, \mu, q) = \frac{2\sqrt{2}\pi\beta\gamma G Z_s}{k_t Z_0 F} \\ \sum_{p=0}^{\infty} (-1)^p D_{2p} ce_{2p}(\varphi, -q) Ce_{2p}(\mu, -q), \end{aligned} \quad (12)$$

with unknown coefficients D_{2p}. The total field $E_z^0 + E_z^i$ evaluated at the boundary $\mu = \mu_0$ must be equal to the field given by Eq. (11). Since Eq. (6) is zero at $\mu = \mu_0$, we then remain with $E_z^i(\varphi, \mu_0, q) = E_z^\sigma(\varphi, \mu_0, q)$. This equation can be used to obtain the coefficients D_{2p}.

By using the orthogonality properties of the $ce_{2l}(\varphi, -q)$, and after some manipulations, we obtain

$$\begin{aligned} D_{2p} = \frac{1}{\pi Ce_{2p}(\mu_0, -q)} \sum_{l=0}^{\infty} \frac{A_0^{(2l)} W_{2l}(-q)}{p'_{2l} Ce_{2l}(\mu_0, -q)} \\ \sum_{r=0}^{\infty} \sum_{t=0}^{\infty} (-1)^{(r+t)} A_{2r}^{(2p)} A_{2t}^{(2l)} L_{r,t}(\mu_0), \end{aligned} \quad (13)$$

where

$$\begin{aligned} L_{r,t}(\mu_0) = \frac{\sqrt{2}\pi e^{-(2|r-t|+1)\mu_0} \Gamma\left(\frac{1}{2} + |r-t|\right)}{\Gamma\left(\frac{1}{2}\right)|r-t|!} \\ F\left(\frac{1}{2}, |r-t| + \frac{1}{2}; |r-t| + 1; e^{-4\mu_0}\right) + \\ \frac{\sqrt{2}\pi e^{-(2r+2t+1)\mu_0} \Gamma\left(\frac{1}{2} + r + t\right)}{\Gamma\left(\frac{1}{2}\right)(r+t)!} \\ F\left(\frac{1}{2}, r + t + \frac{1}{2}; r + t + 1; e^{-4\mu_0}\right) \end{aligned} \quad (14)$$

with Γ the gamma function and $F(a, b; c; z)$ the hypergeometric function.

For sake of completeness, by using the same method for a circular pipe, the electric field can be expressed as [17]

$$E_{z,circ}^i = -\frac{G\beta\gamma Z_s}{k_t b Z_0 I_0^2\left(\frac{k_0 b}{\beta\gamma}\right)} I_0\left(\frac{k_0 r}{\beta\gamma}\right), \qquad (15)$$

with $I_0(x)$ the zero order modified Bessel function of the first kind.

LONGITUDINAL AND QUADRUPOLAR RESISTIVE WALL IMPEDANCE

The longitudinal electric field given by Eq. (12), with the coefficients of Eq. (13), can be used to determine the longitudinal and the quadrupolar impedance. Indeed the longitudinal impedance per unit of length is defined as [17]

$$\frac{dZ_\parallel}{dz} = -\frac{E_z^i\left(\mu = 0, \varphi = \frac{\pi}{2}\right)}{Q}. \qquad (16)$$

The same expression (electric field in the origin) can be used also for the circular case of Eq. (15). In Fig. 2 we show the longitudinal impedance for $\beta = 0.5$, $b = 35$ mm, and $\sigma = 4 \times 10^7$ S/m as a function of frequency for the extreme cases when the elliptic pipe tends to the circular and the flat one. For the flat case we have used, as comparison, the IW2D
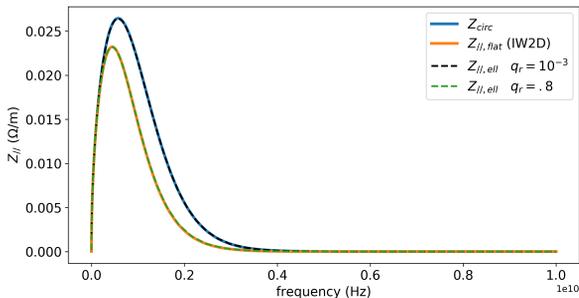

Figure 2: Longitudinal impedance vs frequency in the extreme cases of circular and flat pipe.

code [18]. When comparing the impedance of the elliptic pipe with the circular one, it is possible to define its ratio at relativistic energies, known as Yokoya form factor [7], which depends only on the coefficient $q_r = (a - b)/(a + b)$.

With the above expressions we are able to obtain the form factor for any beam velocity. Indeed, it is possible to demonstrate that this factor depends now on q_r and on the parameter $k_p = k_0 b/\beta\gamma$. When this last term tends to zero we obtain the Yokoya form factor. In Fig. 3 we show the form factor as a function of q_r for different values of k_p compared with the Yokoya form factor.

The same longitudinal field can also be used to obtain the quadrupolar impedance. For the vertical case, for example, we have

$$\frac{dZ_{y,quad}}{dz} = -\frac{\beta}{k_0 Q} \left.\frac{\partial^2 E_z^i}{\partial y^2}\right|_{\mu=0, \varphi=\frac{\pi}{2}}, \qquad (17)$$
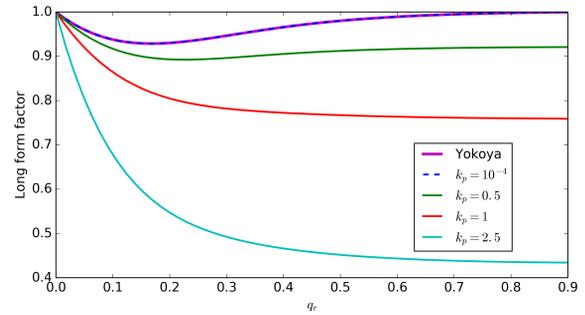

Figure 3: Longitudinal form factor vs q_r for different values of $k_p = k_0 b/\beta\gamma$.

where y is given by Eq. (1) or, for the cylindrical pipe, $y = r$.

This impedance is zero for the circular pipe only at ultra-relativistic velocity. When $\beta < 1$, also in cylindrical symmetry, a quadrupolar impedance appears, as shown in Fig. 4, where the quadrupolar impedance at different q_r, for $\beta = 0.5$, $b = 35$ mm, and $\sigma = 4 \times 10^7$ S/m, is compared with the circular and the flat cases [18].

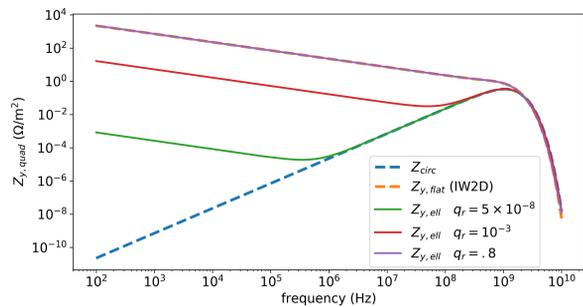
Figure 4: Quadrupolar impedance vs frequency for different values of q_r.

CONCLUSION AND OUTLOOK

Starting from the expression of the longitudinal electric field inside a perfectly conducting elliptic vacuum chamber given as infinite series of Mathieu functions, we have derived the longitudinal and the quadrupolar impedance in the classic resistive wall regime taking into account for the finite conductivity of the beam pipe.

We have obtained very good agreement in the extreme cases of circular and flat beam pipe, and extended the longitudinal Yokoya form factor at any beam energy.

We have also shown that a quadrupolar, not negligible impedance appears also with circular symmetry in the non ultra-relativistic regime.

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