NOTES ON RELATIONS BETWEEN SLICE AND PROJECTED BEAM PARAMETERS

V. Balandin*, N. Golubeva, DESY, Hamburg, Germany

Abstract

We consider some aspects of the relations between slice and projected beam parameters.

INTRODUCTION

Connections between properties of a set of particles and properties of its subsets are of certain interest in several areas of the beam physics. One uses, for example, the concept of multibunch emittance in an accelerator operating not with the single bunch but with the bunch train, or the concept of slice and projected emittances in the design and optimization of SASE FEL performance. Rather general relationships between slice and projected beam quantities were calculated in [1] on the basis of some results of the theory of conditional probability and with main attention paid to the case of a particle beam which can be described by continuous density function. In this paper, we limit our considerations to the situation when the original beam is separated into finite set of sub-beams, which makes the concepts of aligned and granulated beams almost obvious. We also avoid usage of the theory of conditional probability (at least in explicit form) so that all our calculations can be elementary checked.

ORIGINAL BEAM, ALIGNED BEAM AND GRANULATED BEAM

We describe a particle beam in the n-dimensional phase space $\mathbf{z} = (z_1, \ldots, z_n)^\top$ by means of the non-negative in the distribution sense density function (distribution function) $\rho(\mathbf{z})$ satisfying the normalization condition

$$\int \rho(\mathbf{z})\, d\mathbf{z} = 1, \tag{1}$$

and say that the beam is decomposed into m portions (slices) if there are m given non-negative in the distribution sense functions $\rho_k(\mathbf{z})$ such that

$$w_k = \int \rho_k(\mathbf{z})\, d\mathbf{z} > 0, \quad k = 1, \ldots, m, \tag{2}$$

and

$$\rho(\mathbf{z}) = \rho_1(\mathbf{z}) + \ldots + \rho_m(\mathbf{z}). \tag{3}$$

With each beam portion we associate the slice distribution function

$$\hat{\rho}_k(\mathbf{z}) = \frac{1}{w_k} \rho_k(\mathbf{z}) \tag{4}$$

and position of the slice centroid

$$\bar{\mathbf{z}}^{(k)} = \langle \mathbf{z} \rangle_k = \int \mathbf{z} \hat{\rho}_k(\mathbf{z})\, d\mathbf{z}. \tag{5}$$

With these notations $\rho(\mathbf{z})$ can be rewritten in the form

$$\rho(\mathbf{z}) = w_1 \hat{\rho}_1(\mathbf{z}) + \ldots + w_m \hat{\rho}_m(\mathbf{z}), \tag{6}$$

and for position of the beam centroid one obtains

$$\bar{\mathbf{z}} = \langle \mathbf{z} \rangle = \int \mathbf{z} \rho(\mathbf{z})\, d\mathbf{z} = w_1 \bar{\mathbf{z}}^{(1)} + \ldots + w_m \bar{\mathbf{z}}^{(m)}. \tag{7}$$

Separation of the particle beam into slices naturally leads to the concepts of aligned and granulated beams, which are defined as follows.

Aligned Beam

Aligned beam is the beam that would be obtained if all slice centroids would be aligned to the same value, which, without loss of generality, we will take equal to zero.

The distribution function $\rho_a(\mathbf{z})$ of the aligned beam can be expressed through the slice distribution functions and positions of the slice centroids as follows

$$\rho_a(\mathbf{z}) = w_1 \hat{\rho}_1 \left(\mathbf{z} + \bar{\mathbf{z}}^{(1)}\right) + \ldots + w_m \hat{\rho}_m \left(\mathbf{z} + \bar{\mathbf{z}}^{(m)}\right), \tag{8}$$

and the centroid of the aligned beam coincides with the origin of the coordinate system

$$\langle \mathbf{z} \rangle_a = \int \mathbf{z} \rho_a(\mathbf{z})\, d\mathbf{z} = \mathbf{0}_n.^1 \tag{9}$$

Granulated Beam

Granulated beam is the beam that would be obtained if all charge within each slice would be concentrated at the location of the corresponding slice centroid, and its distribution function is given by the expression

$$\rho_g(\mathbf{z}) = w_1 \delta\left(\left|\mathbf{z} - \bar{\mathbf{z}}^{(1)}\right|\right) + \ldots + w_m \delta\left(\left|\mathbf{z} - \bar{\mathbf{z}}^{(m)}\right|\right), \tag{10}$$

where δ is the Dirac delta function.

Note that centroids of the original and granulated beams are equal to each other

$$\langle \mathbf{z} \rangle_g = \int \mathbf{z} \rho_g(\mathbf{z})\, d\mathbf{z} = w_1 \bar{\mathbf{z}}^{(1)} + \ldots + w_m \bar{\mathbf{z}}^{(m)} = \bar{\mathbf{z}}. \tag{11}$$

* vladimir.balandin@desy.de

[1] Here and later on notations $\mathbf{0}_k$ and $\mathbf{1}_k$ stay for the k-dimensional vectors with all components equal to zero and to one, respectively.

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DECOMPOSITION OF THE COVARIANCE MATRIX OF THE ORIGINAL BEAM

By definition, the beam (covariance) matrix is the matrix of the second-order central moments of the beam distribution. Let Σ be the beam matrix of the original beam

$$\Sigma = \left\langle (\mathbf{z} - \langle \mathbf{z} \rangle) \cdot (\mathbf{z} - \langle \mathbf{z} \rangle)^{\top} \right\rangle, \tag{12}$$

and let Σ_a, Σ_g and Σ_k be the beam matrices of the aligned beam, of the granulated beam and of the k-th beam slice, respectively. Then, taking into account (6), (8) and (10), one obtains the following decomposition formulas

$$\Sigma = \Sigma_a + \Sigma_g \tag{13}$$

and

$$\Sigma_a = w_1 \Sigma_1 + \ldots + w_m \Sigma_m. \tag{14}$$

2D PROJECTED PARAMETERS

Let $\tilde{\Sigma}$ be the 2×2 principal submatrix of the matrix Σ whose entries are in the intersection of rows and columns specified by indices i and j. The quantity

$$\epsilon = \sqrt{\det(\tilde{\Sigma})}, \tag{15}$$

where the symbol $\sqrt{}$ denotes the principal square root, is called the (2D) projected emittance, and if ϵ is positive, then one can also define the (projected) Twiss parameters (β, α, γ) by means of the equality

$$\tilde{\Sigma} = \epsilon \begin{pmatrix} \beta & -\alpha \\ -\alpha & \gamma \end{pmatrix}. \tag{16}$$

Introducing by analogy submatrices $\tilde{\Sigma}_a$, $\tilde{\Sigma}_g$ and $\tilde{\Sigma}_k$; their emittances ϵ_a, ϵ_g and ϵ_k; and, if possible, their Twiss parameters, let us consider in the following subsections relations between them coming from the basic formulas (13) and (14).

Aligned Beam

Connection of the projected emittance of the aligned beam with the properties of the beam slices can be obtained using formulas (14) and (34), and linearity of the trace function

$$\epsilon_a^2 = \det\left(\tilde{\Sigma}_a\right) = \det\left(w_1 \tilde{\Sigma}_1 + \ldots + w_m \tilde{\Sigma}_m\right)$$

$$= \frac{1}{2} \sum_{k,l=1}^{m} w_k w_l \left[\mathrm{tr}\left(\tilde{\Sigma}_k\right) \mathrm{tr}\left(\tilde{\Sigma}_l\right) - \mathrm{tr}\left(\tilde{\Sigma}_k \tilde{\Sigma}_l\right) \right]. \tag{17}$$

As the next step, the formula (17) can be transformed into the following useful form

$$\epsilon_a^2 = \left[\sum_{k=1}^{m} (w_k \epsilon_k) \right]^2$$

$$+ \frac{1}{2} \sum_{k,l=1}^{m} w_k w_l \left[\det\left(\tilde{\Sigma}_k + \tilde{\Sigma}_l\right) - (\epsilon_k + \epsilon_l)^2 \right]. \tag{18}$$

Because all matrices $\tilde{\Sigma}_k$ are symmetric and positive semidefinite, all the terms in the sums (17) and (18) are nonnegative (see appendix for explanation).

Let us assume that all slice projected emittances ϵ_k are positive. Then one can define the slice Twiss parameters $(\beta_k, \alpha_k, \gamma_k)$ and transform (17) and (18) into equalities

$$\epsilon_a^2 = \sum_{k,l=1}^{m} m_p(\beta_k, \beta_l) (w_k \epsilon_k)(w_l \epsilon_l), \tag{19}$$

$$\epsilon_a^2 = \left[\sum_{k=1}^{m} (w_k \epsilon_k) \right]^2$$

$$+ \sum_{k,l=1}^{m} \left[m_p(\beta_k, \beta_l) - 1 \right] (w_k \epsilon_k)(w_l \epsilon_l), \tag{20}$$

where

$$m_p(\beta_k, \beta_l) = \frac{\beta_k \gamma_l - 2\alpha_k \alpha_l + \gamma_k \beta_l}{2} \tag{21}$$

is the betatron mismatch parameter.

The first addend in the equality (20) is the mean slice emittance squared, and one sees that ϵ_a will be larger than the mean slice emittance if and only if there are at least two slices which are mismatched with respect to each other.

Note that the Twiss parameters of the aligned beam are connected with the slice Twiss parameters as follows

$$\beta_a = \frac{1}{\epsilon_a} \sum_{k=1}^{m} w_k \epsilon_k \beta_k, \tag{22a}$$

$$\alpha_a = \frac{1}{\epsilon_a} \sum_{k=1}^{m} w_k \epsilon_k \alpha_k, \tag{22b}$$

$$\gamma_a = \frac{1}{\epsilon_a} \sum_{k=1}^{m} w_k \epsilon_k \gamma_k. \tag{22c}$$

Granulated Beam

Let us introduce special notations (x, p) for the coordinates on the plane of interest (z_i, z_j), and let

$$\mathbf{x} = (x_1, \ldots, x_m)^{\top}, \tag{23a}$$

$$\mathbf{p} = (p_1, \ldots, p_m)^{\top} \tag{23b}$$

be the notations for the projections of the slice centroids onto this plane. Then

$$\epsilon_g^2 = \langle (x - \bar{x})^2 \rangle_g \langle (p - \bar{p})^2 \rangle_g$$

$$-\langle (x-\bar{x})(p-\bar{p})\rangle_g^2 = \frac{1}{2}\sum_{k,l=1}^{m} w_k w_l$$

$$\cdot \left[(x_k-\bar{x})(p_l-\bar{p}) - (x_l-\bar{x})(p_k-\bar{p})\right]^2, \qquad (24)$$

where

$$\bar{x} = \langle x\rangle_g = w_1 x_1 + \ldots + w_m x_m, \qquad (25a)$$

$$\bar{p} = \langle p\rangle_g = w_1 p_1 + \ldots + w_m p_m. \qquad (25b)$$

There is not much to say, in general, when ϵ_g is positive, besides that in this case the Twiss parameters of the granulated beam (β_g, α_g, γ_g) can be defined. The situation becomes more interesting, if $\epsilon_g = 0$.

The projected emittance of the granulated beam is zero if and only if the vectors $\mathbf{x} - \bar{x}\cdot\mathbf{1}_m$ and $\mathbf{p} - \bar{p}\cdot\mathbf{1}_m$ are linearly dependent, i.e. all points (x_k, p_k) are on the same straight line passing through the point $(\bar{x}, \bar{p})$ or, in other words, there exists an angle ϕ such that

$$\cos(\phi)\,(\mathbf{x} - \bar{x}\cdot\mathbf{1}_m) + \sin(\phi)\,(\mathbf{p} - \bar{p}\cdot\mathbf{1}_m) = \mathbf{0}_m. \qquad (26)$$

With (26) satisfied, the matrix $\tilde{\Sigma}_g$ takes on the following important for the further consideration special form

$$\tilde{\Sigma}_g = \langle (x-\bar{x})^2 + (p-\bar{p})^2\rangle_g$$

$$\cdot \begin{pmatrix} \sin^2(\phi) & -\cos(\phi)\sin(\phi) \\ -\cos(\phi)\sin(\phi) & \cos^2(\phi) \end{pmatrix} \qquad (27)$$

where

$$\langle (x-\bar{x})^2 + (p-\bar{p})^2\rangle_g$$

$$= \sum_{k=1}^{m} w_k \left[(x_k-\bar{x})^2 + (p_k-\bar{p})^2\right]. \qquad (28)$$

Original Beam

The projected emittance of the original beam can be connected with the properties of the matrices $\tilde{\Sigma}_a$ and $\tilde{\Sigma}_g$ using equality (35)

$$\epsilon^2 = \epsilon_a^2 + \left[\mathrm{tr}\left(\tilde{\Sigma}_a\right)\mathrm{tr}\left(\tilde{\Sigma}_g\right) - \mathrm{tr}\left(\tilde{\Sigma}_a\tilde{\Sigma}_g\right)\right] + \epsilon_g^2. \qquad (29)$$

Because

$$\mathrm{tr}\left(\tilde{\Sigma}_a\right)\mathrm{tr}\left(\tilde{\Sigma}_g\right) - \mathrm{tr}\left(\tilde{\Sigma}_a\tilde{\Sigma}_g\right) \ge 2\epsilon_a\epsilon_g, \qquad (30)$$

the projected emittance of the original beam can never be smaller than the sum $\epsilon_a + \epsilon_g$. In general, ϵ can be positive even if both emittances ϵ_a and ϵ_g are equal to zero, but it is somewhat exceptional situation which seems to be weakly related to reality. So, in the following we will assume that $\epsilon_a > 0$.

If ϵ_g is also positive, then (29) turns into equation

$$\epsilon^2 = \epsilon_a^2 + 2m_p(\beta_a, \beta_g)\epsilon_a\epsilon_g + \epsilon_g^2$$

$$= (\epsilon_a + \epsilon_g)^2 + 2[m_p(\beta_a, \beta_g) - 1]\epsilon_a\epsilon_g, \qquad (31)$$

from which one sees that the mismatch between the Twiss parameters of aligned and granulated beams is the source of an additional (for the fixed ϵ_g value) increase of ϵ with respect to the sum $\epsilon_a + \epsilon_g$.

If ϵ_g is equal to zero, then (29) turns into the equality

$$\epsilon^2 = \epsilon_a^2 + \epsilon_a\langle (x-\bar{x})^2 + (p-\bar{p})^2\rangle_g$$

$$\cdot \left[\beta_a\cos^2(\phi) - 2\alpha_a\cos(\phi)\sin(\phi) + \gamma_a\sin^2(\phi)\right]. \qquad (32)$$

Because the matrix $\tilde{\Sigma}_a$ is positive definite, the multiplier in the brackets in (32) is positive for an arbitrary value of the angle ϕ, which means that the projected emittance of the original beam will be equal to the projected emittance of the aligned beam if and only if all projections of the slice centroids onto the plane (x, p) are perfectly aligned, i.e. if and only if

$$\langle (x-\bar{x})^2 + (p-\bar{p})^2\rangle_g = 0. \qquad (33)$$

APPENDIX

In this appendix we list properties of the 2×2 real matrices, which are needed for the better understanding of our calculations.

If A is an arbitrary 2×2 matrix, then

$$\det(A) = \tfrac{1}{2}\left[\mathrm{tr}^2(A) - \mathrm{tr}\left(A^2\right)\right]. \qquad (34)$$

If A and B are arbitrary 2×2 matrices, then

$$\det(A+B) = \det(A) + \det(B) + \mathrm{tr}(A)\,\mathrm{tr}(B) - \mathrm{tr}(AB). \quad (35)$$

If determinants of the 2×2 matrices A and B are non-negative, then

$$\mathrm{tr}(A)\,\mathrm{tr}(B) - \mathrm{tr}(AB) - 2\sqrt{\det(A)}\,\sqrt{\det(B)}$$

$$= \det(A+B) - \left[\sqrt{\det(A)} + \sqrt{\det(B)}\right]^2. \qquad (36)$$

If 2×2 matrices A and B are symmetric positive semi-definite, then expressions on the both sides of the equality (36) are non-negative due to Minkowski determinant theorem.

REFERENCES

[1] Chad E. Mitchell, "A General Slice Moment Decomposition of RMS Beam Emittance", `arXiv:1509.04765 [physics.acc-ph]`, 2015.

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