APPROXIMATE MATRICES FOR MODELING THE FOCUSING OF THE UNDULATOR PERIODS AND UNDULATOR END FIELDS

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Abstract

We describe the procedure for the construction of approximate matrices for modeling the focusing of the undulator periods and undulator end fields and discuss applicability of these matrices to the European XFEL undulators.

INTRODUCTION

The so-called natural vertical focusing of the long periodic planar undulators (or undulator segments) is usually taken into account by averaging equations of motion over one undulator period, and while accuracy of this approximation is often good enough for describing period properties, it gives no answer to the question how to model the effect of the undulator entrance and exit fields.

In this paper, looking on the naive averaging method from the common in the last few decades point of view of coordinate transformations, we describe unified approach for the construction of approximate matrices for modeling focusing of both, undulator periods and undulator end fields.

We have incorporate the matrices obtained into the MAD code, which is currently a standard tool for describing the lattice of the European XFEL accelerator, and also we use these matrices in online beam dynamics applications.

PLANAR MODEL OF THE UNDULATOR FIELD AND EQUATIONS OF MOTION

We describe the undulator field and the particle motion in a Cartesian coordinate system with x, y and z as the horizontal, vertical and longitudinal direction, respectively.

Because we assume that the undulator magnetic field $B = (B_x, B_y, B_z)^\top$ is symmetric about the horizontal midplane $y = 0$ and is homogeneous along the x-axis (approximation of infinitely wide poles), it can be described in terms of a scalar potential Ψ (with $\Delta\Psi = 0$ and $B = \nabla\Psi$) which is an odd function of y, independent from x, and can be expressed as a formal power series in y in the form

$$\Psi = \sum_{m=0}^{\infty} (-1)^m \, b_0^{[2m]}(z) \, \frac{y^{2m+1}}{(2m+1)!}, \qquad (1)$$

where $b_0(z)$ is distribution of the vertical magnetic field in the horizontal midplane and the index $[n]$ indicates the n-th derivative with respect to the longitudinal variable z.

For describing the particle dynamics we take the coordinate z as an independent variable, use a pseudoparticle flying along the z-axis in the field free space as the reference particle, and adopt a complete set of symplectic variables $\boldsymbol{u} = (x, p_x, y, p_y, \sigma, \varepsilon)^\top$ as particle coordinates. Here p_x and p_y are transverse canonical monenta scaled with the

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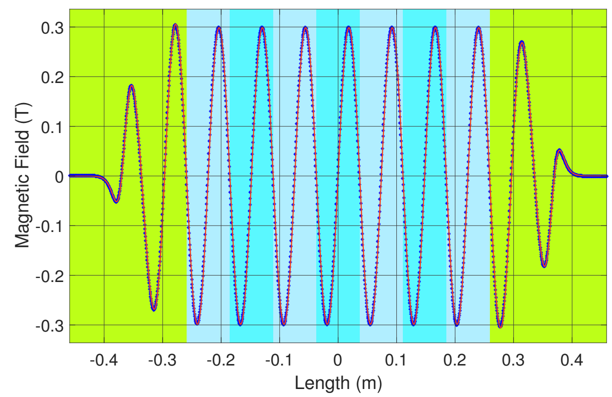

Figure 1: The vertical field in the European XFEL laser heater undulator (ULH) for a gap of 32 mm. Red curve is an analytical fit to the measured data represented in this figure by the blue dots. Rectangles of different colors indicate regions chosen for the entrance, exit and period fields.

constant kinetic momentum of the reference particle p_0 and the longitudinal variables σ and ε are

$$\sigma = c\,\beta_0\,(t_0 - t), \quad \varepsilon = (\mathcal{E} - \mathcal{E}_0)\,/\,(\beta_0^2\,\mathcal{E}_0), \qquad (2)$$

where $\mathcal{E}_0$, β_0 and $t_0 = t_0(z)$ are the energy of the reference particle, its velocity in terms of the speed of light c and its arrival time at a certain position z, respectively.

In these variables, the Hamiltonian describing the motion of a particle in the planar undulator field takes on the form

$$H = \varepsilon - \sqrt{æ^2(\varepsilon) - p_x^2 - \left(p_y - \frac{e}{p_0} x \frac{\partial\Psi}{\partial z}\right)^2} + \frac{e}{p_0} x \frac{\partial\Psi}{\partial y}, \qquad (3)$$

where

$$æ(\varepsilon) = \sqrt{(1+\varepsilon)^2 - (\varepsilon/\gamma_0)^2} = \sqrt{1 + 2\varepsilon + \beta_0^2\,\varepsilon^2}. \qquad (4)$$

INTERPOLATION OF THE MEASURED ON-AXIS VERTICAL UNDULATOR FIELD

As the first step of all further considerations, we create an analytical model of the on-axis vertical magnetic field $b_0(z)$ by interpolation of the available high accuracy magnetic field measurements. In doing so we separate undulator (or undulator segment) into entrance, exit and periodic field parts as shown, for example, in Fig.1. Then, for each gap value for which field was measured, we fit the entrance and exit fields with the current sheet-type model depending from 20 parameters and use the finite trigonometric sum of the form

$$\sum_{m=1}^{n} a_{2m-1} \sin\left[\frac{2\pi}{\lambda}(2m-1)\,z\right] \qquad (5)$$

for approximating the period field ($0 \le z \le \lambda$).

Because on-axis field of all European XFEL undulators has odd symmetry (i.e. antisymmetric with respect to the longitudinal undulator center), we keep the same property in our interpolation model, which guarantees that the so-called first field integral is automatically equal to zero. In order to have the second field integral in our model also equal to zero, we constrain our fit with the requirements that the second integral of the entrance field vanishes and that the first integral of the entrance field is equal to the minus second integral of the period field divided by the period length. Besides that we require for our field model to be continuously differentiable function.

MOTION OF THE BEAM CENTROID AND BETATRON OSCILLATIONS

By definition the beam centroid is a particle which has all coordinates equal to zero at the undulator entrance. We will denote the coordinates of this particle by $\mathring{\boldsymbol{u}}$ and its dynamics is given by the following equations

$$d\mathring{x} \,/\, dz = \tan\left(\mathring{\varphi}_z\right), \quad d\mathring{p}_x \,/\, dz = -h_0(z), \tag{6}$$

$$d\mathring{\sigma} \,/\, dz = 1 - \sec\left(\mathring{\varphi}_z\right), \tag{7}$$

$$\mathring{y}(z) = \mathring{p}_y(z) = \mathring{\varepsilon}(z) \equiv 0, \tag{8}$$

where $h_0(z) = (e \,/\, p_0)\, b_0(z)$ and $\mathring{\varphi}_z$ is the angle which the centroid trajectory makes with the z-axis, i.e.

$$\mathring{p}_x(z) = \sin\left(\mathring{\varphi}_z\right). \tag{9}$$

Let us introduce variables $\tilde{\boldsymbol{u}} = \boldsymbol{u} - \mathring{\boldsymbol{u}}$ for the deviations of the solution for an arbitrary particle from the beam centroid coordinates and then linearize the equations obtained. The resulting equations of linear betatron oscillations split into two groups, the horizontal-longitudinal and the vertical, governed by the quadratic Hamiltonians

$$\tilde{H}_2^{x\sigma} = \frac{\sec^3(\mathring{\varphi}_z)}{2}\Big\{\tilde{p}_x^2 - 2\sin(\mathring{\varphi}_z)\,\tilde{p}_x\,\tilde{\varepsilon} + \left[1 - \beta_0^2\cos^2(\mathring{\varphi}_z)\right]\tilde{\varepsilon}^2\Big\} \tag{10}$$

and

$$\tilde{H}_2^{y} = \tfrac{1}{2}\left[\sec(\mathring{\varphi}_z)\left(\tilde{p}_y - h_0^{[1]}\,\mathring{x}\,\tilde{y}\right)^2 - h_0^{[2]}\,\mathring{x}\,\tilde{y}^2\right], \tag{11}$$

respectively.

To remove from (11) the unphysical dependence from the horizontal centroid position $\mathring{x}$, let us introduce the new variables

$$\tilde{\tilde{y}} = \tilde{y}, \quad \tilde{\tilde{p}}_y = \tilde{p}_y - h_0^{[1]}\,\mathring{x}\,\tilde{y} \tag{12}$$

and obtain the new Hamiltonian

$$\tilde{\tilde{H}}_2^{y} = \frac{\sec(\mathring{\varphi}_z)}{2}\left[\tilde{\tilde{p}}_y^2 + h_0^{[1]}\sin(\mathring{\varphi}_z)\,\tilde{\tilde{y}}^2\right]. \tag{13}$$

CENTROID DYNAMICS AND FOCUSING IN THE UNDULATOR SUBPARTS

Let the interval $0 \le z \le l$ will denote some subpart of the undulator segment (for example, entrance region or one period). Then the solution of equations (6) and (7) within this subpart can be expressed in the form of the series

$$\mathring{p}_x(z) = \mathring{p}_x(0) - a(z), \tag{14}$$

$$\mathring{x}(z) = \mathring{x}(0) + \sum_{n=0}^{\infty}(n+1)D_{n+1}^{-1}\left[\tan\left(\mathring{\varphi}_0\right)\right]\sec^n\left(\mathring{\varphi}_0\right)A_n(z), \tag{15}$$

$$\mathring{\sigma}(z) = \mathring{\sigma}(0) + z - \sum_{n=0}^{\infty}D_n^{1}\left[\tan\left(\mathring{\varphi}_0\right)\right]\sec^{n+1}\left(\mathring{\varphi}_0\right)A_n(z), \tag{16}$$

where the (scaled) field integrals are defined as follows

$$a(z) = \int_0^z h_0(\tau)\,d\tau, \quad A_n(z) = \int_0^z a^n(\tau)\,d\tau, \tag{17}$$

and D_n^m are Gegenbauer polynomials in Lee-Whiting's notations [1, 2].

The subpart focusing matrix, i.e. the fundamental matrix solution of the equations with the Hamiltonian equal to the sum of the Hamiltonians (10) and (13) has the form

$$M = \begin{pmatrix} 1 & r_{12} & 0 & 0 & 0 & r_{16} \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & r_{33} & r_{34} & 0 & 0 \\ 0 & 0 & r_{43} & r_{44} & 0 & 0 \\ 0 & r_{52} & 0 & 0 & 1 & r_{56} \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}, \tag{18}$$

where r_{km} elements related to the horizontal and longitudinal motion can also be found in the form of series as follows

$$r_{12}(z) = \sum_{n=0}^{\infty}D_n^{3}\left[\tan\left(\mathring{\varphi}_0\right)\right]\sec^{n+3}\left(\mathring{\varphi}_0\right)A_n(z), \tag{19}$$

$$r_{16}(z) = r_{52}(z) = \sum_{n=0}^{\infty}(n+1)D_{n+1}^{1}\left[\tan\left(\mathring{\varphi}_0\right)\right]\sec^{n+2}\left(\mathring{\varphi}_0\right)A_n(z), \tag{20}$$

$$r_{56}(z) = r_{12}(z) - \beta_0^2\sum_{n=0}^{\infty}D_n^{1}\left[\tan\left(\mathring{\varphi}_0\right)\right]\sec^{n+1}\left(\mathring{\varphi}_0\right)A_n(z). \tag{21}$$

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PROCEDURE FOR OBTAINING APPROXIMATE FOCUSING MATRICES

Let us observe first that due to assumed field antisymmetry the matrix of the undulator exit is equal to the mirror symmetric image of the undulator entrance matrix, and therefore don't require special consideration. So let the interval $0 \le z \le l$ will be the undulator entrance region or one undulator period.

For the horizontal-longitudinal motion the drift (the field free space) approximation is typically used, and, if the better accuracy is required, one can collect leading correction terms from the series (19)-(21) with the result

$$r_{12}(l) \approx l + \tfrac{3}{2}A_2(l) - \tfrac{3}{2l}A_1^2(l), \quad r_{16}(l) = r_{52}(l) \approx 0, \tag{22}$$

$$r_{56}(l) \approx \frac{l}{\gamma_0^2} + \left(1 + \frac{1}{2\gamma_0^2}\right)\left[A_2(l) - \tfrac{1}{l}A_1^2(l)\right], \tag{23}$$

which seems to be sufficient for all practical purposes.

Looking for the vertical focusing effect we first simplify the Hamiltonian (13) to the form

$$\tilde{\tilde{H}}_2^y = \tfrac{1}{2}\left(\tilde{\tilde{p}}_y^2 + h_0^{[1]}\,\mathring{p}_x\,\tilde{\tilde{y}}^2\right) \tag{24}$$

by using small angle approximation, and then make in (24) coordinate transformation

$$\hat{y} = \tilde{\tilde{y}}, \quad \hat{p}_y = \tilde{\tilde{p}}_y - \psi(z)\,\tilde{\tilde{y}}, \tag{25}$$

which brings the Hamiltonian (24) to the form

$$\hat{H}_2^y = \tfrac{1}{2}\left[\hat{p}_y^2 + (\psi^{[1]} + h_0^{[1]}\,\mathring{p}_x)\,\hat{y}^2\right] + \psi\hat{y}\hat{p}_y + \tfrac{1}{2}\psi^2\hat{y}^2. \tag{26}$$

As the next step, let us assume that the function $\psi(z)$ is such that for some constant Ω^2

$$\psi^{[1]} + h_0^{[1]}\,\mathring{p}_x = \Omega^2 \tag{27}$$

and that the integral

$$\int_0^l \psi^2(z)\,dz \tag{28}$$

is sufficiently small. Then one can take

$$\hat{H}_2^y \approx \tfrac{1}{2}\left(\hat{p}_y^2 + \Omega^2\,\hat{y}^2\right) \tag{29}$$

and obtain approximation to the vertical focusing matrix as the matrix of the simple harmonic oscillator (29) sandwiched between entrance and exit triangular coordinate transformations (25).

From (27) it follows that the function $\psi(z)$ has the form

$$\psi(z) = \psi(0) + \Omega^2 z - F(z), \tag{30}$$

where

$$F(z) = \int_0^z h_0^{[1]}(\tau)\,\mathring{p}_x(\tau)\,d\tau, \tag{31}$$

and $\psi(0)$ and Ω^2 can be treated as free parameters which can be used for minimization of the integral (28).

Let us first make this minimization within the class of periodic functions with $\psi(0) = \psi(l)$. This problem has the unique solution, which can be expressed as follows

$$\Omega^2 = \tfrac{1}{l}F(l) = \tfrac{1}{l}\left(\tfrac{e}{p_0}\right)^2 \int_0^l b_0^2(z)\,dz, \tag{32}$$

$$\psi(0) = \tfrac{1}{l}\int_0^l F(z)\,dz - \tfrac{1}{2}F(l) = -\tfrac{1}{l}\left(\tfrac{e}{p_0}\right)^2 \cdot \left[\int_0^l (z - \tfrac{l}{2})b_0^2(z)\,dz + \tfrac{1}{2}\left(\int_0^l b_0(z)\,dz\right)^2\right]. \tag{33}$$

Note that with defined in such a way parameters $\psi(0)$ and Ω^2, the function $\psi(z)$ has zero mean value and, when calculations are made for the motion through the undulator period, has $\psi(0) = 0$, i.e. recovers usual naive averaging procedure.

When both parameters $\psi(0)$ and Ω^2 are used for optimization of the integral (28), the solution is also unique and is given by the formulas

$$\Omega^2 = \tfrac{12}{l^3}\int_0^l zF(z)\,dz - \tfrac{6}{l^2}\int_0^l F(z)\,dz, \tag{34}$$

$$\psi(0) = -\tfrac{6}{l^2}\int_0^l zF(z)\,dz + \tfrac{4}{l}\int_0^l F(z)\,dz. \tag{35}$$

Note that specific property of this solution is that for the motion through the undulator period $\psi(0) \neq 0$, but always $\psi(0) + \psi(l) = 0$.

PRACTICAL IMPLEMENTATION

Both solutions (32)-(33) and (34)-(35) show good performance in all our particular applications, and though formulas (34)-(35) always provide slightly better precision, we prefer to use (32)-(33) for the motion through the undulator periods and (34)-(35) for the transport through the entrance and exit field regions.

In both solutions the parameters $\psi(0)$ and Ω^2 can be represented in the form

$$\psi(0) = \left(\tfrac{e}{p_0}\right)^2 \hat{\psi}(0), \quad \Omega^2 = \left(\tfrac{e}{p_0}\right)^2 \hat{\Omega}^2, \tag{36}$$

where, for every particular undulator, $\hat{\psi}(0)$ and $\hat{\Omega}^2$ are the functions only of the undulator gap. This functional dependence has to be found only once, beforehand, and then can be kept in some appropriate form. It is not clear yet if it is a common property, but we found it sufficient to use for this purpose an exponential of low order polynomials.

REFERENCES

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