Adaptive two-dimensional Vlasov simulation of particle beams

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Plan

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The Vlasov-Poisson model

Consider the collisionless Vlasov-Poisson model

Vlasov equation

$$\partial_t f + v \cdot \nabla_x f + \frac{q}{m} (E + v \times B) \cdot \nabla_v f = 0$$

Poisson equation

$$-\Delta\phi = \int_{\mathbf{R}^d} f(t, \mathbf{x}, \mathbf{v}) d\mathbf{v},$$
$$E(t, \mathbf{x}) = -\nabla_{\mathbf{x}} \phi(t, \mathbf{x}).$$

We want to solve this system on a phase-space grid using the property of

Invariance along characteristics

$$\frac{d}{dt} f(X(t), V(t), t) = 0$$

where $\dot{X} = V$, $\dot{V} = \frac{q}{m}(E(X(t), t) + V(t) \times B(X(t), t))$.

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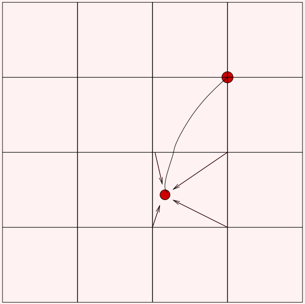
The backward semi-Lagrangian method



- f conserved along characteristics
- Find the origin of the characteristics ending at the grid points
- Interpolate old value at origin of characteristics from known grid values → High order interpolation needed

- Typical interpolation schemes.
  - Cubic spline [Chang & Knorr 1976, Sonnendrücker *et al.* 1999]
  - Cubic Hermite with derivative transport [Nakamura & Yabe 1999]

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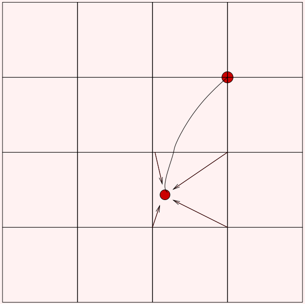

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PIC methods

Particle-In-Cell (PIC) method is the most widely used.

- Pros :
  - Good qualitative results with few particles.
  - Very good when particle dynamics dominated by fields which do not depend on particles (e.g. in accelerators when self field small compared to applied field).
  - More efficient when dimension is increased (phase-space = 6D).
- Cons : Hard to get good precision : slow convergence, numerical noise, low resolution at high velocities.

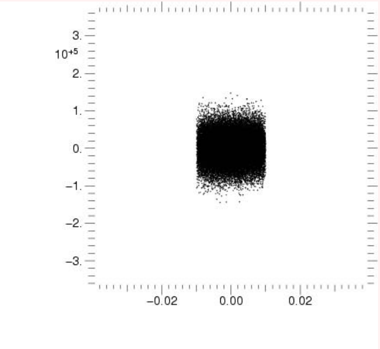
Grid based methods

Grid based Vlasov methods have been recently developped thanks to the new computationnal facilities

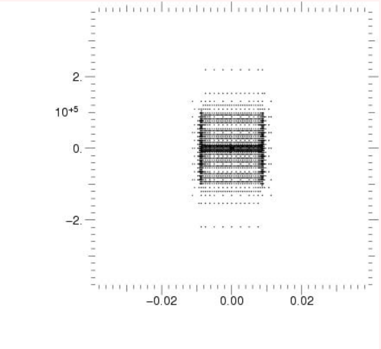
- Pros.
  - High-order method.
  - Same resolution everywhere on grid.
- Cons ;
  - Numerical diffusion
  - Curse of dimensionality : N^d grid points needed in d dimensions on uniform grids.
    Number of grid points grows exponentially with dimension
    $\longrightarrow$ killer for Vlasov equation where d up to 6.
    Memory needed
    - In 2D, 16384^2 grid $\rightarrow$ 2 GB
    - In 4D, 256^4 grid $\rightarrow$ 32 GB
    - In 6D, 64^6 grid $\rightarrow$ 512 GB

$\Rightarrow$ Adaptive algorithm in higher dimensions

Localization of points



PIC code



non linear approximation

Plan

Principle

Solve Vlasov equation using a semi-Lagrangian algorithm which consists in two stages

- Advection : follow the characteristics backward,
- Interpolation : interpolate the distribution function on a grid at the origin of the characteristics,

and a nonlinear approximation of the distribution function in order to

- have a natural criterion to refine/derefine the grid,
- possibly compute the interpolation adaptively.

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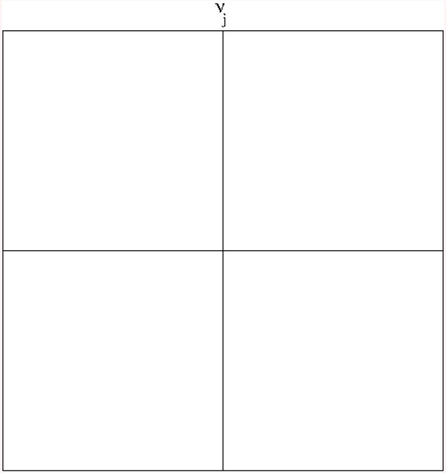
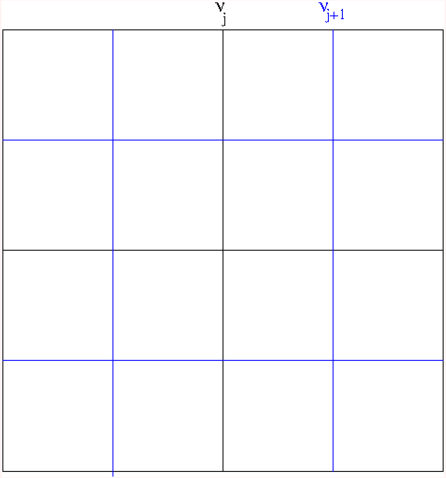
Grid refinement



Grid G_j, grid points $x_k^j = k\,2^{-j}$

- dyadic refinement of the grid
- j_0 is the coarsest level
- J is the finest level
- logical points of level j

Grid refinement



Grid G_{j+1}, grid points $x_k^{j+1} = k\,2^{-(j+1)}$

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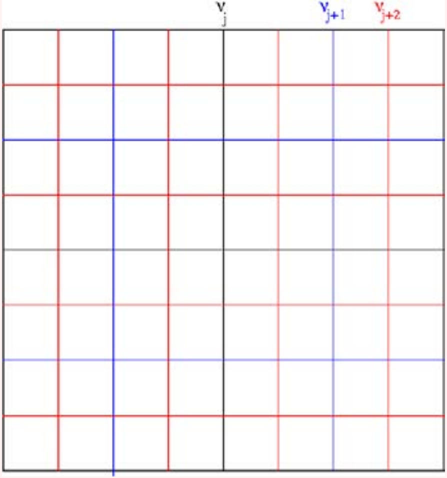

Grid G_{j+2}, grid points $x_k^{j+2} = k\,2^{-(j+2)}$

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- j_1 is the finest level
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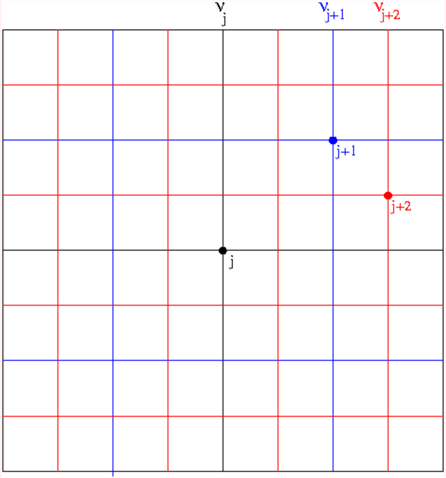
Grid refinement



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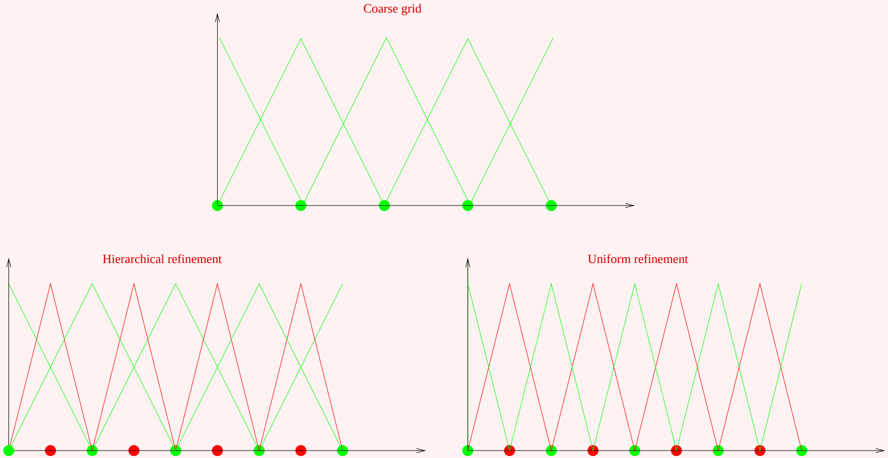
Uniform and hierarchical refinements



Nonlinear approximation

- Decomposition of f_{j+1} in uniform and hierarchical basis

$$\begin{aligned} f_{j+1} &= \sum_k c_k^{j+1} \varphi_k^{j+1} \text{ (uniform)} \\ &= \sum_k c_k^j \varphi_k^j + \sum_k d_k^j \psi_k^j \text{ (hierarchical)} \end{aligned}$$

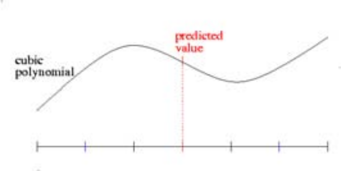
- In hierarchical decomposition coefficients d_{2i+1} at fine scale are small if f is close to affine in $[x_{2i}, x_{2i+2}]$.
- Linear (uniform) approximation consists in using a given number of basis functions independently of approximated function f.
- Nonlinear approximation consists in keeping the N highest coefficients in hierarchical decomposition (depends on f) [De Vore 1998]
  Only grid points where f varies most are kept.

Construction of a hierarchical approximation

- Hierarchical approximation is constructed by defining an interpolation method enabling to go from coarse grid to fine grid.
- Two methods have been tried :
  1. Interpolating wavelets based on Lagrange polynomial interpolation. Classical wavelet compression technique. Addressed moment conservation issues [Gutnic & Haefele & Paun & Sonnendrücker 2004, Gutnic & Haefele & Sonnendrücker 2006].
  2. Hierarchical approximation based on finite element interpolants. More local, cell based → simpler and more efficient parallelization. [Campos Pinto-Mehrenberger 2003].

Hierarchical expression of f_{j+1} of interpolating wavelets

- Consider Gridfunction f_j defined by its values c_k^j on G^j of step 2^{-j}.



- Define dyadic refinement procedure via interpolation operator, e.g. Lagrange interpolation

- Refinement procedure linear with respect to c_k^j so that on can introduce basis functions φ_k^j defined by infinite refinement of $\delta_{k,n}$: φ_k^j are the basis functions at level j such that $\varphi_k^j(x_k^j) = 1$ and $\varphi_k^j(x_l^j) = 0$ if $l \neq k$

Projection and prediction operators

To map the distribution function from one level to the next, we then have

- The projection operator ($\approx$ restriction operator) : $c_{2k}^{j+1} \mapsto c_k^j$
- The prediction operator

$$
\begin{array}{rccc}
P_j^{j+1}: & G_j & \rightarrow & G_{j+1}, \\
\text{such that} & c_{2k}^{j+1} & = & c_k^j, \\
 & c_{2k+1}^{j+1} & = & P(x_{2k+1}^{j+1}),
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where P here stands for Lagrange interpolation polynomial.

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Representation of the solution

In this approach, the representation of the solution is based on

- dyadic grid of phase space

$\Rightarrow$ Adaptive grid $\tilde{G}$

- a wavelet decomposition writing equivalently f as

$$\Rightarrow \quad f(x) = \sum_{l=-\infty}^{+\infty} c_l^{j_0}\, \varphi_l^{j_0}(x) + \sum_{j=j_0}^{j_1-1} \sum_{l=-\infty}^{+\infty} d_l^j\, \psi_l^j(x)$$

where $d_l^j = c_{2l+1}^{j+1} - P(x_{2l+1}^{j+1}) = f(x_{2l+1}^{j+1}) - P(x_{2l+1}^{j+1})$ is small when approximation at level j is good.

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Thresholding

- Consider following expression : $f_{j+1} = \sum_k c_k^j \varphi_k^j + \sum_k d_k^j \psi_k^j$.
- Adaptivity introduced by neglecting the terms in this expansion such that $|d_k^j| < \epsilon_j$.
- Error commited can be easily estimated $\|d_k^j \psi_k^j\|_{L^p} = |d_k^j| 2^{-\frac{j}{p}} \|\psi\|_{L^p} < \epsilon_j 2^{-\frac{j}{p}} \|\psi\|_{L^p}$.
- Moments of f_{j+1} can be conserved by appropriately modifying ψ : taking $\psi^m = \psi - \sum_k s_k \varphi(\cdot - k)$ with $(s_k)_k$ chosen such that $\int x^l \psi^m(x)\, dx = 0$ for $0 \leq l \leq m$.

Plan

Numerical algorithm...

Initialization :

- computation of details from analytical initial function f_0

> $\Rightarrow$ decomposition of f_0
> $\Rightarrow$ **compression** of f_0 by thresholding small details
> $\Rightarrow$ corresponding adaptive grid

Time marching step :

- forward advection in x of the adaptive grid

> $\Rightarrow$ **prediction of** the new adaptive grid $\tilde{\mathrm{G}}$
> ...with refinement procedure (one level finer)

- **Construction of** $\hat{\mathrm{G}}$: grid where we have to compute values of f^* in order to compute its wavelet transform.

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> $f^*(x,v) = f^n(x - v\,\Delta t, v)$

- **Wavelet transform of** $\mathbf{f^*}$: compute the c_k and d_k coefficients at the points of $\tilde{G}$.
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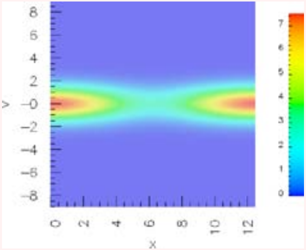
The Landau damping

The initial condition is given by

$$f_0(\mathbf{x},\mathbf{v}) = \frac{1}{2\pi}\exp\left(-\frac{v_x^2+v_y^2}{2}\right) \times (1+\alpha\cos(k_x x)\cos(k_y y)),$$

in nonlinear regimes ($k_x = k_y = 0.5$; $\alpha = 0.5$).



- Small scales well captured SLD movie
- Exponential growth of the electric energy before saturating, with an accurate damping rate $\gamma = 0.4$.

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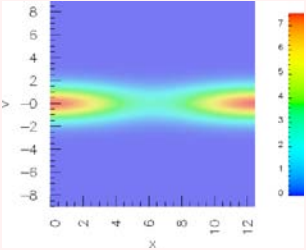

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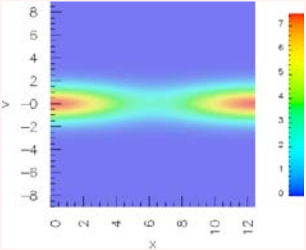

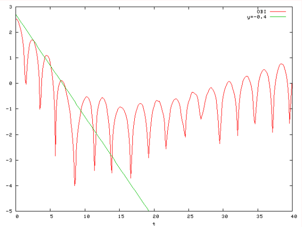

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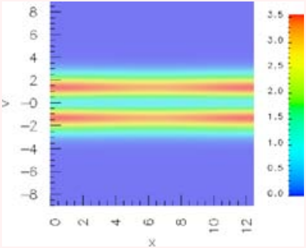
with $k_x = 0.5$ and $\alpha = 0.05$.



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- Mass is not exactly conserved but with a reasonable rate.

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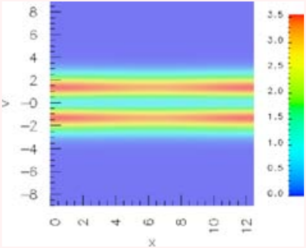

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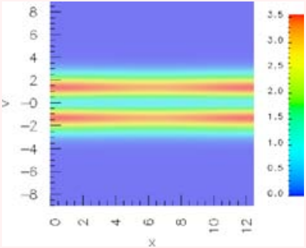
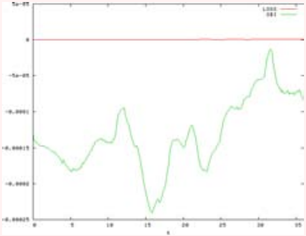
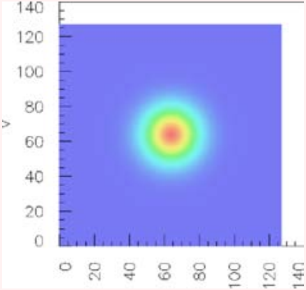




Beam focusing in solenoid lattice

Consider a Proton beam (100 mA, 5 MeV) in solenoid lattice, initial condition given by the following gaussian distribution

$$f_0(\mathbf{x}, \mathbf{v}) = \exp\left(-\frac{x^2+y^2+v_x^2+v_y^2}{2}\right),$$



- The beam is well focused SOL movie
- Points kept are less than 5%

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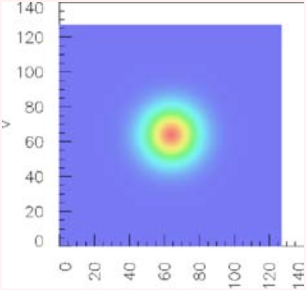

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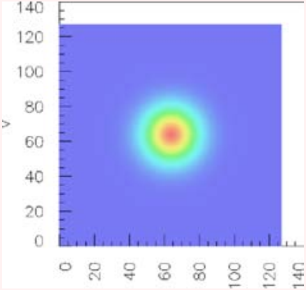

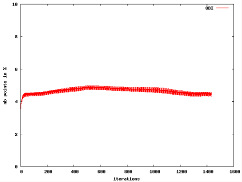

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Beam focusing in alternating gradient lattice

Consider a Potassium beam (40 mA, 1 MeV) in alternating gradient lattice, initial condition given by the following gaussian distribution

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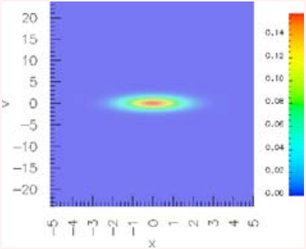

- The beam is well focused ALT movie
- Points kept are less than 10%

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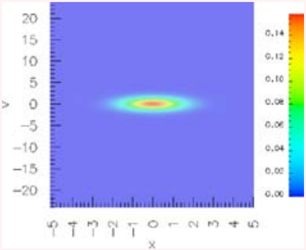

$$f_0(\mathbf{x}, \mathbf{v}) = \exp\left(-\frac{x^2+y^2+v_x^2+v_y^2}{2}\right),$$

- The beam is well focused ALT movie
- Points kept are less than 10%

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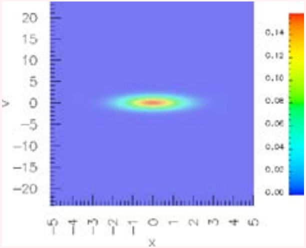
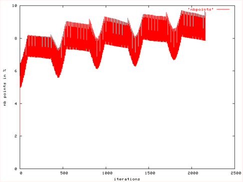


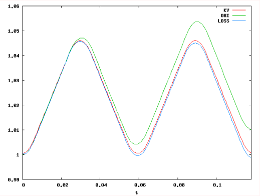


Comparison with LOSS solver

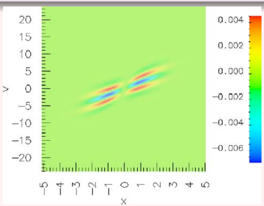
Time computation and speedup

Numbers of processors	1	2	4	8
Time (in s.) LOSS/OBI	433/408	226/206	111/105	63/55
Speedup LOSS/OBI	1	1.92/1.98	3.9/3.88	6.87/7.41

Computations conducted on an IBM Regatta machine Power5 processors. The results corresponds to 128^4 points in the phase space on one time step.



Time evolution of the X_{rms} quantity.



Difference between LOSS and OBI distribution function

Conclusion and perspectives

- Grid based Vlasov solvers are a valuable tool to have in one's simulation toolbox.
- No noise. Better representation in low density regions of phase space.
- Adaptive grid strategy can be made efficient by careful optimization.
- 2D code is now running and can perform realistic simulations of transverse phase space.

- Applications to laser-plasma interaction
- Likely that such methods can be applied to $2D\frac{1}{2}$ and $3D$ in the future, for ICAP 2008 ?
- Ready for comparison with PIC methods.

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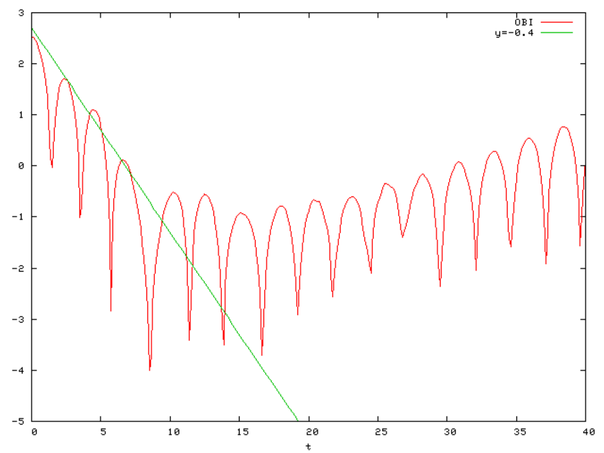
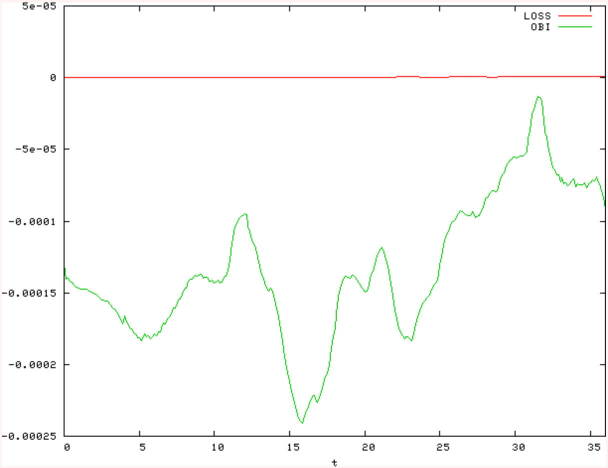
The Landau damping



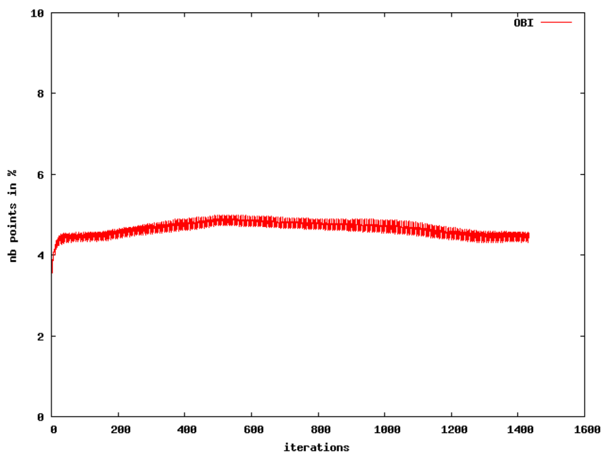
Time evolution of the mass.

The two-stream instability



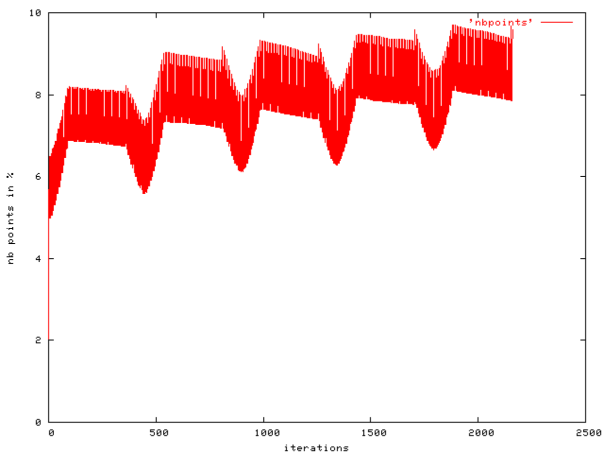
Time evolution of the mass.

Beam focusing in solenoid lattice



Kept points in percent.

Beam focusing in alternating gradient lattice



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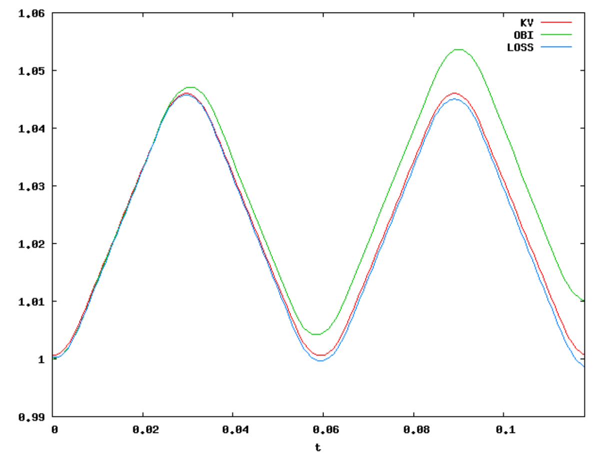
Comparison with LOSS solver



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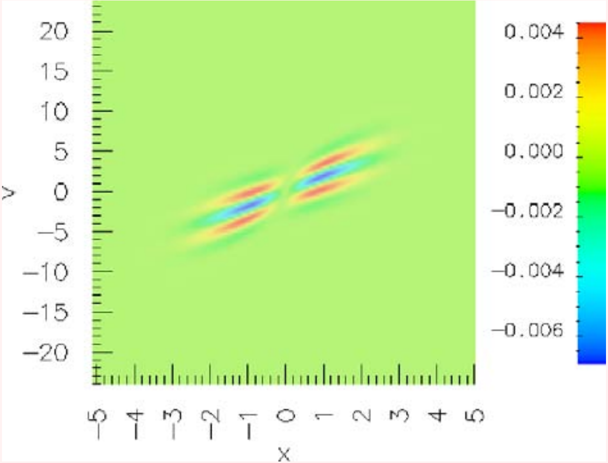

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