Low-Dispersion Wake Field Calculation Tools

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Outline

- Introduction
- Enlarged stencil for 3D curl equation
  - Numerical dispersion & stability
  - Indirect integration for 3D wake field calculations
- Potential formalism in cylindrical symmetry
  - Numerical dispersion & stability
  - Conformal scheme vs. staircasing
- Numerical examples and comparisons
- Conclusions

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Introduction

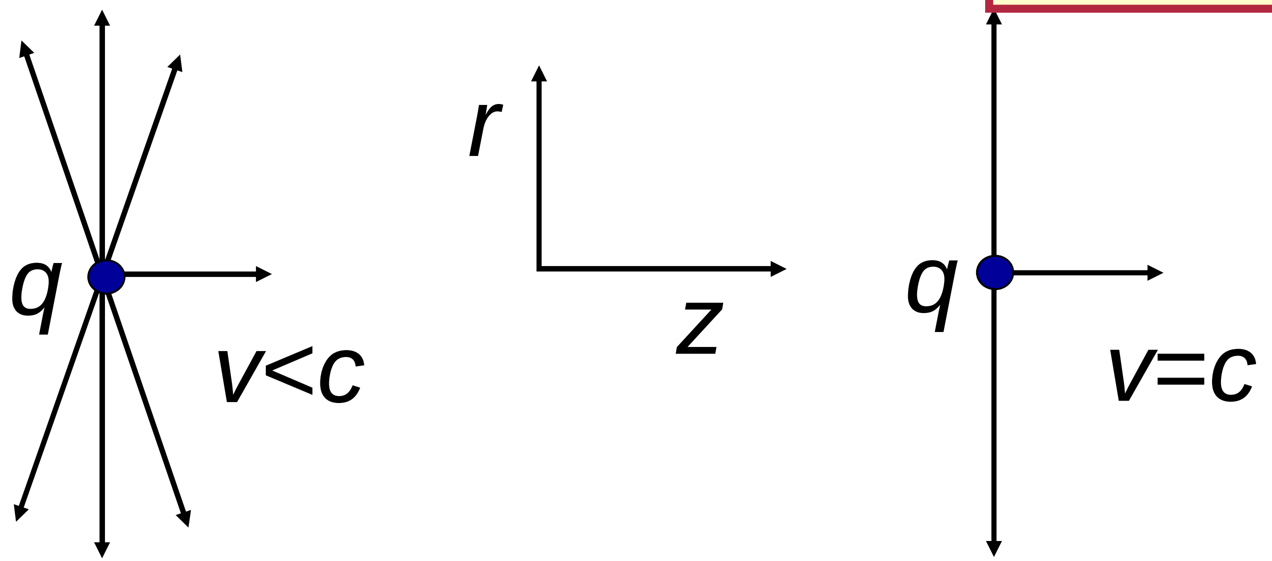
- Wake fields are caused by the interaction of a charged particle beam with the surrounding vacuum chamber in the accelerator
- no fields in front of a relativistic beam
- no wake fields in a smooth circular PEC-pipe
- If the pipe is not smooth and/or not perfectly conducting, a wake is created
- Wake field is a time-domain quantity
- Wake potential is defined as

$$W_{\parallel}(0,s) = -\int_{-\infty}^{\infty} E_z(z,0,\frac{s+z}{c})dz$$

Introduction

- rigid beams assumed in calculations (wake fields do not perturb the bunch)
- no transversal variation of the beam
- beam propagates along the *z*-axis

$$E_r = \frac{q}{2\pi\varepsilon_0 r}\delta(z - ct)$$



Lorentz contracted fields with angular spread

Ultrarelativistic limit: pancake in free space and in smooth pipe

Methods in 2D and in 3D

2D

$$\Phi(z,r,t)=\int_0^r r' E_z(z,r',t)dr'$$

$$\frac{\partial^2\Phi}{c^2\partial t^2}=\frac{\partial^2\Phi}{\partial z^2}+\frac{\partial^2\Phi}{\partial r^2}-\frac{1}{r}\frac{\partial\Phi}{\partial r}$$

$$\varepsilon\frac{\partial E_z}{\partial t}=\frac{1}{r}\frac{\partial}{\partial r}(rH_\theta)$$

$$\varepsilon\frac{\partial E_r}{\partial t}=-\frac{\partial H_\theta}{\partial z}$$

$$\mu\frac{\partial H_\theta}{\partial t}=\frac{\partial E_z}{\partial r}-\frac{\partial E_r}{\partial z}$$

3D

$$\varepsilon\frac{\partial\vec{E}}{\partial t}=\nabla\times\vec{H}$$

$$\mu\frac{\partial\vec{H}}{\partial t}=-\nabla\times\vec{E}$$

Second part of the talk

First part of the talk

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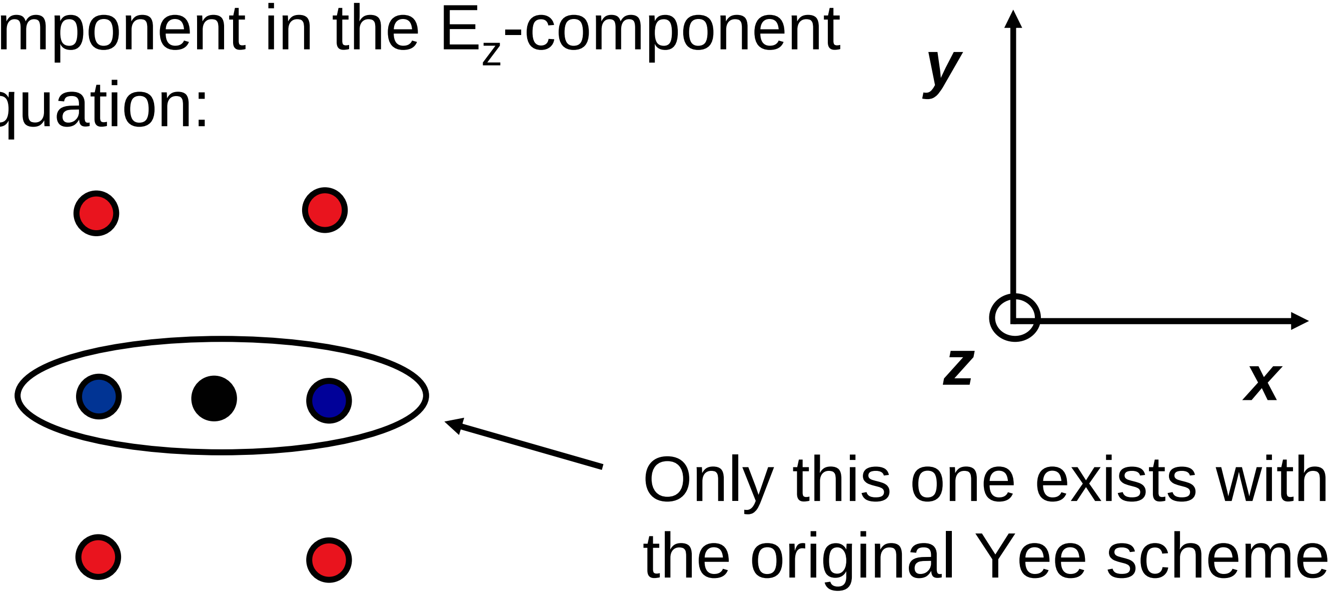
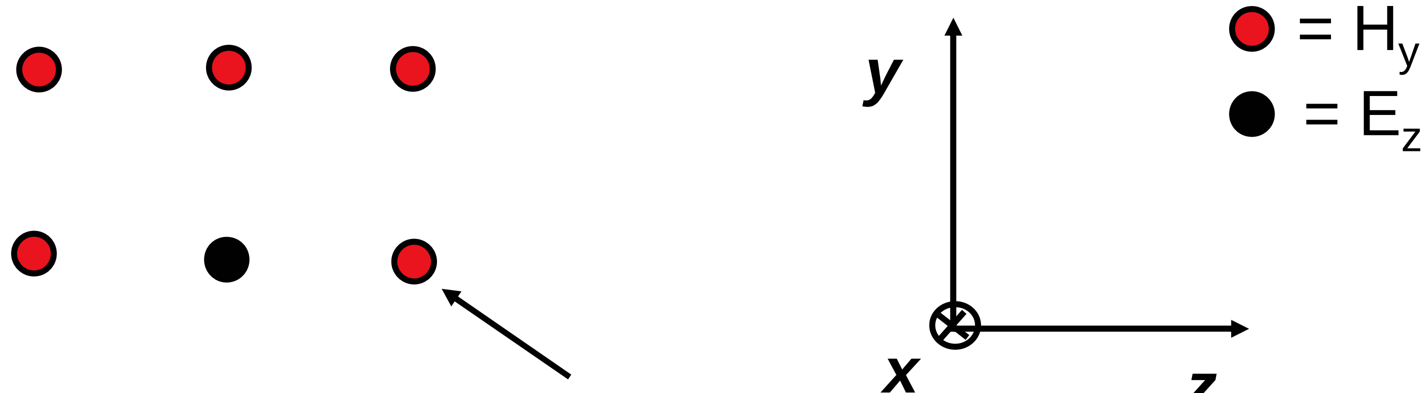
Enlarged Stencil, 3D

Consider computing the *x*-derivative of the H_y-component in the E_z-component update equation:



Only this one exists with the original Yee scheme



More H_y-points used with the new scheme

Update Equations

- Electric field updates modified, magnetic fields updated exactly as in standard FDTD/FIT
- Example: *x*-component of the electric field:

$$
\begin{aligned}
E_x\Big|_{i+1/2,j,k}^{n+1} = {} & E_x\Big|_{i+1/2,j,k}^{n} + \alpha\frac{\Delta t}{\varepsilon\Delta z}\left[H_y\Big|_{i+1/2,j,k+1/2}^{n+1/2} - H_y\Big|_{i+1/2,j,k-1/2}^{n+1/2}\right] + \\
& \beta\frac{\Delta t}{\varepsilon\Delta z}\left[\begin{matrix} H_y\Big|_{i+1/2,j+1,k+1/2}^{n+1/2} - H_y\Big|_{i+1/2,j+1,k-1/2}^{n+1/2} + H_y\Big|_{i+1/2,j-1,k+1/2}^{n+1/2} - H_y\Big|_{i+1/2,j-1,k-1/2}^{n+1/2} + \\ H_y\Big|_{i+3/2,j,k+1/2}^{n+1/2} - H_y\Big|_{i+3/2,j,k-1/2}^{n+1/2} + H_y\Big|_{i-1/2,j,k+1/2}^{n+1/2} - H_y\Big|_{i-1/2,j,k-1/2}^{n+1/2} \end{matrix}\right] + \\
& \gamma\frac{\Delta t}{\varepsilon\Delta z}\left[\begin{matrix} H_y\Big|_{i+3/2,j+1,k+1/2}^{n+1/2} - H_y\Big|_{i+3/2,j+1,k-1/2}^{n+1/2} + H_y\Big|_{i-1/2,j+1,k+1/2}^{n+1/2} - H_y\Big|_{i-1/2,j+1,k-1/2}^{n+1/2} + \\ H_y\Big|_{i+3/2,j-1,k+1/2}^{n+1/2} - H_y\Big|_{i+3/2,j-1,k-1/2}^{n+1/2} + H_y\Big|_{i-1/2,j-1,k+1/2}^{n+1/2} - H_y\Big|_{i-1/2,j-1,k-1/2}^{n+1/2} \end{matrix}\right] + \\
& \alpha\frac{\Delta t}{\varepsilon\Delta y}\left[H_z\Big|_{i+1/2,j+1/2,k}^{n+1/2} - H_z\Big|_{i+1/2,j-1/2,k}^{n+1/2}\right] + \\
& \beta\frac{\Delta t}{\varepsilon\Delta y}\left[\begin{matrix} H_z\Big|_{i+3/2,j+1/2,k}^{n+1/2} - H_z\Big|_{i+3/2,j-1/2,k}^{n+1/2} + H_z\Big|_{i-1/2,j+1/2,k}^{n+1/2} - H_z\Big|_{i-1/2,j-1/2,k}^{n+1/2} + \\ H_z\Big|_{i+1/2,j+1/2,k+1}^{n+1/2} - H_z\Big|_{i+1/2,j-1/2,k+1}^{n+1/2} + H_z\Big|_{i+1/2,j+1/2,k-1}^{n+1/2} - H_z\Big|_{i+1/2,j-1/2,k-1}^{n+1/2} \end{matrix}\right] + \\
& \gamma\frac{\Delta t}{\varepsilon\Delta y}\left[\begin{matrix} H_z\Big|_{i+3/2,j+1/2,k+1}^{n+1/2} - H_z\Big|_{i+3/2,j-1/2,k+1}^{n+1/2} + H_z\Big|_{i-1/2,j+1/2,k+1}^{n+1/2} - H_z\Big|_{i-1/2,j-1/2,k+1}^{n+1/2} + \\ H_z\Big|_{i+3/2,j+1/2,k-1}^{n+1/2} - H_z\Big|_{i+3/2,j-1/2,k-1}^{n+1/2} + H_z\Big|_{i-1/2,j+1/2,k-1}^{n+1/2} - H_z\Big|_{i-1/2,j-1/2,k-1}^{n+1/2} \end{matrix}\right]
\end{aligned}
$$

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Numerical Dispersion & Stability

- Basic observation:
  - The scheme should lead to:

$$\sin^2\left(\frac{\omega\Delta t}{2}\right)=\left(\frac{c\Delta t}{\Delta z}\right)^2\sin^2\left(\frac{k\Delta z}{2}\right)$$

  - With $\alpha+4\beta+4\gamma=1$ and $\Delta t=\Delta z/c$ the update equation leads to numerical phase velocity being equal to c along x,y,z-directions

$$E_x=E_0e^{j(kz-\omega t)},E_y=0,E_z=0$$

$$H_x=0,H_y=\frac{E_0}{\eta}e^{j(kz-\omega t)},H_z=0$$

- The scheme is stable (for example) with $\alpha=7/12$, $\beta=1/12$, $\gamma=1/48$ assuming cubic cells.
- With $\alpha=1$, $\beta=0$, $\gamma=0$ and reduced time step the scheme reduces to the standard FDTD/FIT-scheme

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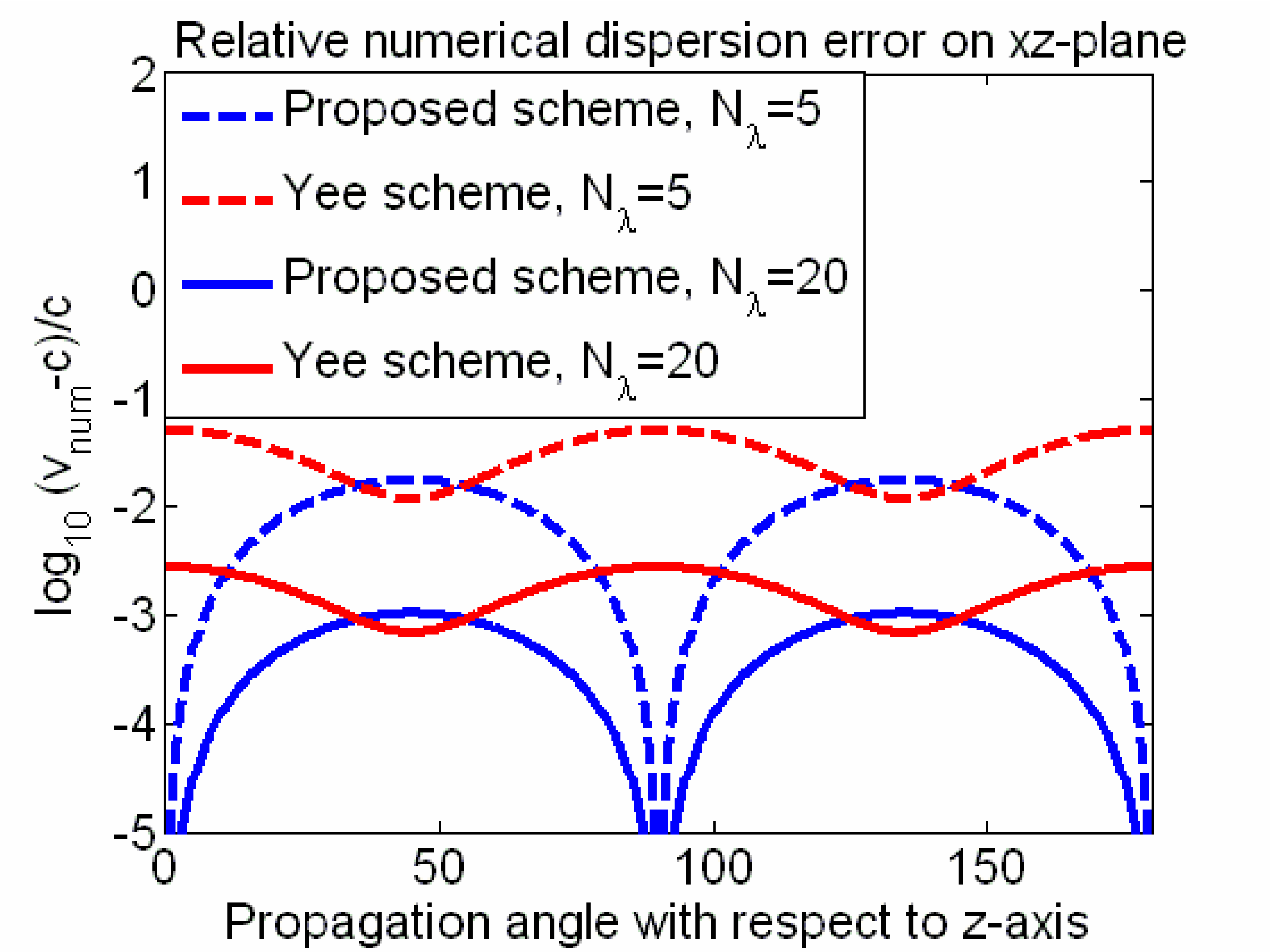
Numerical Dispersion vs. Yee



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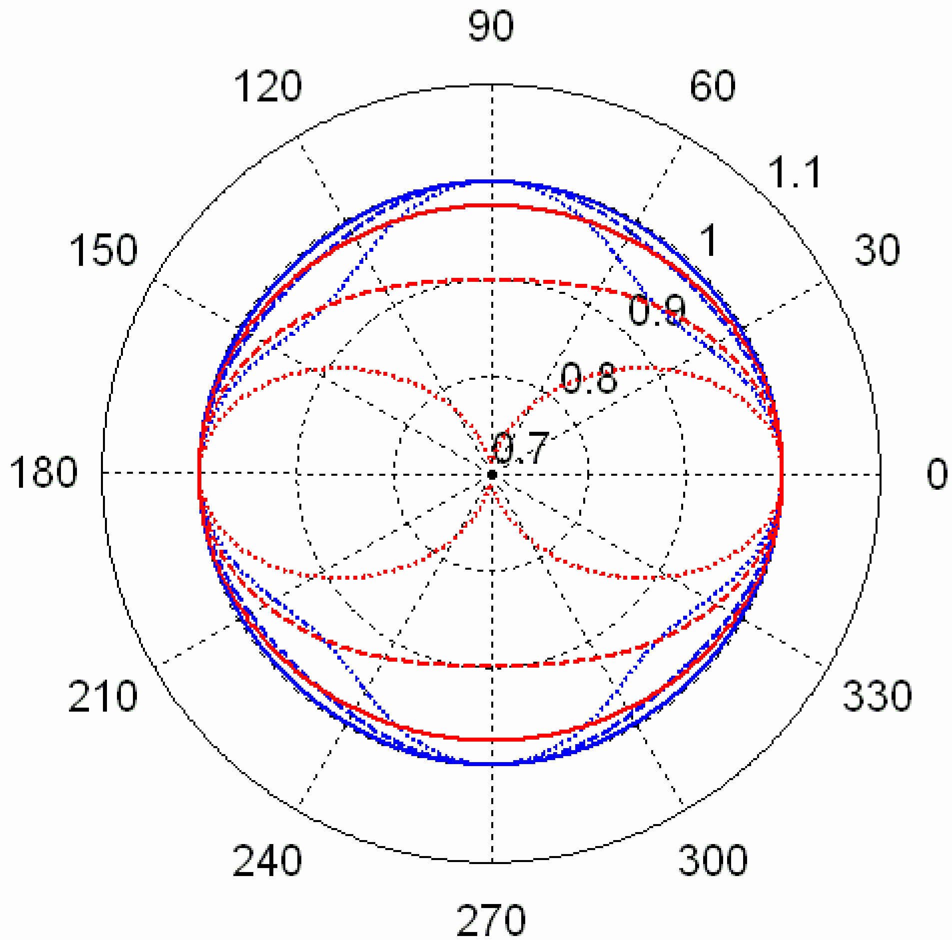
New Scheme vs. LT Splitting

Proposed scheme: Blue curves (3,5 and 10 cells per wavelength). LT splitting: red curves. Phase velocity anisotropy error is lower than with LT splitting (PBCI-code).



Indirect Integration

- The problem
  - How to calculate the wake potential in „infinitely“ long beam pipes ?
- Solution (according to H. Henke)
  - Transformation of the infinite integral into a finite one by using Stokes‘ theorem
  - Requires a solution of a Poisson‘s equation over the arbitrary cross-section of the beam pipe at every time step within the moving window
  - Allows truncating the outgoing beam pipe
- Implementation
  - Direct integration in C-code, post-processing (solutions to Poisson‘s equations) in Matlab

$$W_{\parallel}(0,s) = -\int_{-\infty}^{\infty} E_z(z,0,\frac{s+z}{c})dz$$

$$\nabla^2\Phi = \frac{\partial E_z}{c\partial t} - \frac{\partial E_z}{\partial z}$$

RHS of the Poisson‘s equation is saved within the moving window at the end of the simulation

Details of the scheme can be found in EPAC 2006 paper: H. Henke, W. Bruns, „Calculation of wake potentials in general 3D structures“

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New Discrete Scheme in 2D

- How to discretize the partial differential equation ?

$$\frac{\partial^2 \Phi}{c^2 \partial t^2} = \frac{\partial^2 \Phi}{\partial z^2} + \frac{\partial^2 \Phi}{\partial r^2} - \frac{1}{r}\frac{\partial \Phi}{\partial r}$$

$$\frac{\partial^2 \Phi}{\partial z^2}(i,\ j) = a_z \frac{\Phi(i+1,\ j) - 2\Phi(i,\ j) + \Phi(i-1,\ j)}{\Delta z^2} +$$

$$b_z \frac{\Phi(i+1,\ j+1) - 2\Phi(i,\ j+1) + \Phi(i-1,\ j+1)}{\Delta z^2} +$$

$$b_z \frac{\Phi(i+1,\ j-1) - 2\Phi(i,\ j-1) + \Phi(i-1,\ j-1)}{\Delta z^2}$$

New Discrete Scheme

- How to choose good parameters *a* and *b* ?
  - the scheme must be numerically stable if $\Delta t = \Delta z / c$
  - the scheme must be free of dispersion in the longitudinal direction
  - the scheme must be at least second-order accurate

$$\xi^2 - 2\left[1 + \left(\frac{c\Delta t}{\Delta z}\right)^2 g\right]\xi + 1 = 0$$

Stability equation

$$|\xi| \le 1 \Leftarrow -2 \le g \le 0$$

$$g = \cos(k_z \Delta z) \times \left[1 - 4b_z \sin^2\left(\frac{k_r \Delta r}{2}\right) - \frac{2b_r \Delta z^2}{\Delta r^2}\right] + \frac{\Delta z^2 \cos(k_r \Delta r)}{\Delta r^2} \times \left[1 - 4b_r \sin^2\left(\frac{k_z \Delta z}{2}\right) - \frac{2b_z \Delta r^2}{\Delta z^2}\right] - a_z - \frac{\Delta z^2 a_r}{\Delta r^2}$$

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New Discrete Scheme

The parameter choice

$$a_r = a_z = \frac{\Delta z^2 + 2\Delta r^2}{2\Delta z^2 + 2\Delta r^2}$$

$$b_r = b_z = \frac{\Delta z^2}{4(\Delta z^2 + \Delta r^2)}$$

satisfies the required conditions if *Δr/ Δz>=1*. The first-order term is discretized according to

$$\frac{\partial \Phi}{\partial r}(i,j) = \frac{1}{2}\frac{\Phi(i,j+1) - \Phi(i,j-1)}{2\Delta r} + \frac{1}{4}\frac{\Phi(i+1,j+1) - \Phi(i+1,j-1)}{2\Delta r} + \frac{1}{4}\frac{\Phi(i-1,j+1) - \Phi(i-1,j-1)}{2\Delta r}$$

and the scheme remains stable with the same time-step.

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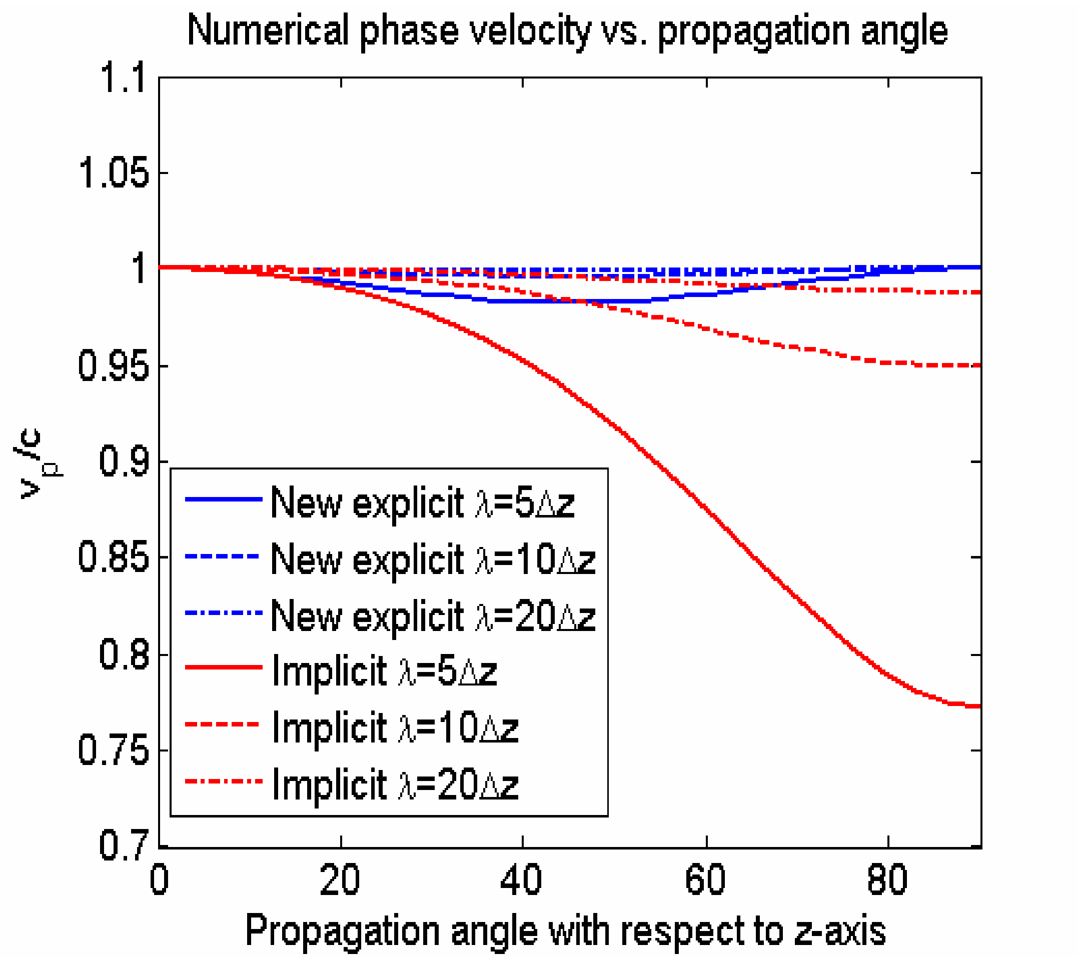
Numerical Phase Velocity

- Comparison of the new explicit scheme and the implicit scheme.



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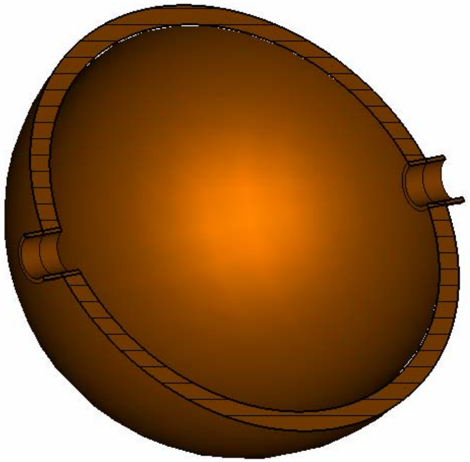
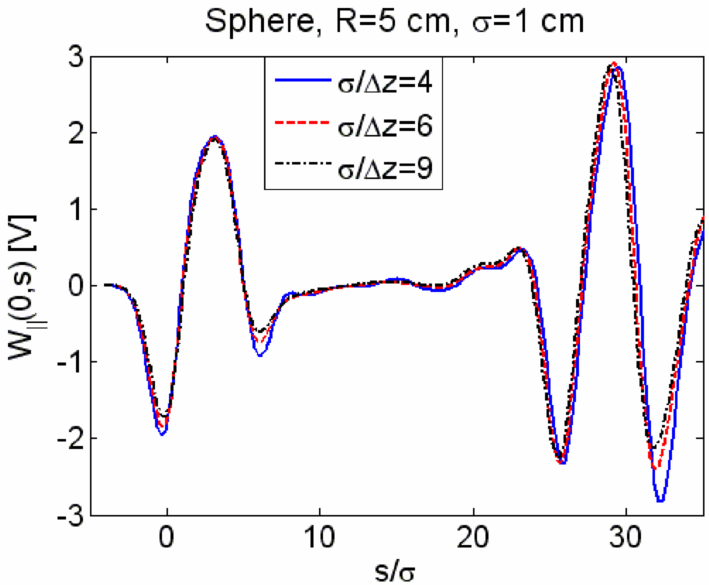
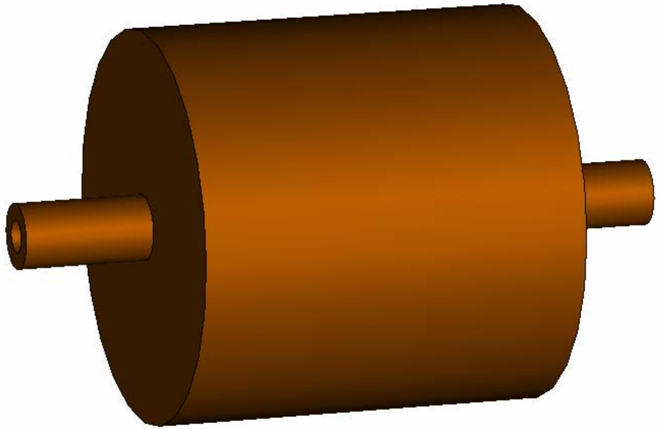
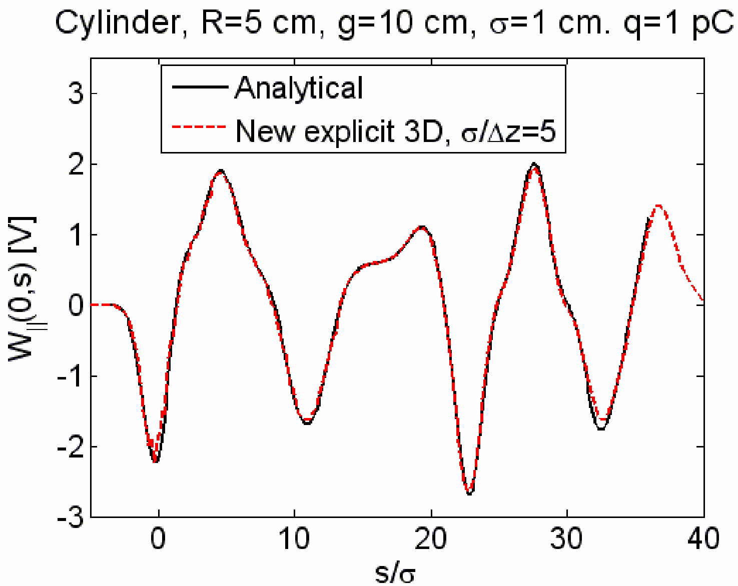
Spherical Cavity & Pillbox









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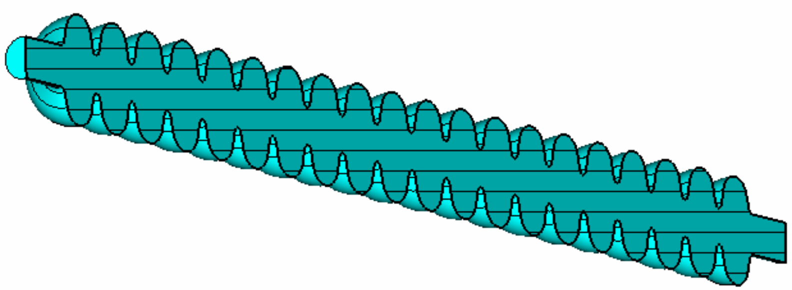
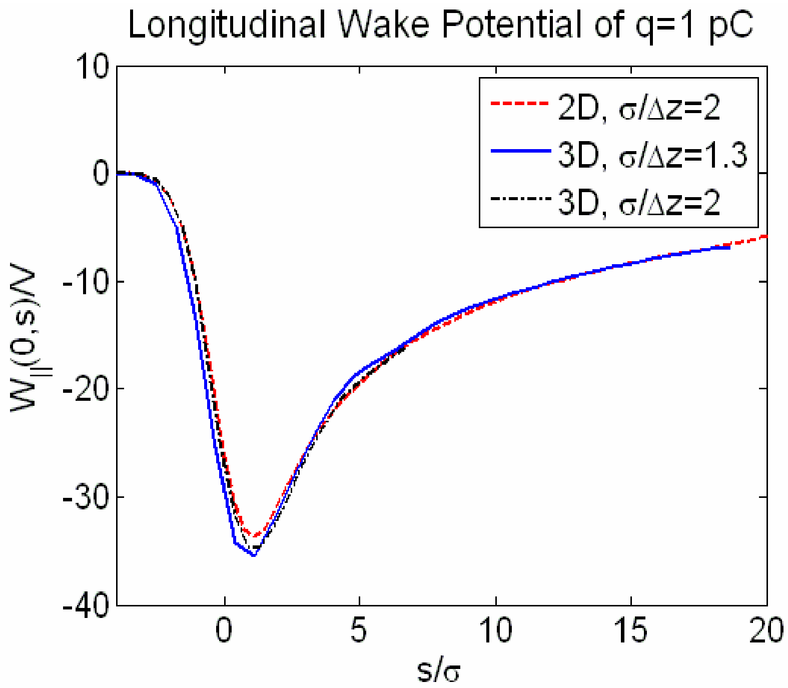
20 Tesla Cells





Simulation on a single processor using $\sigma/\Delta z=2$ takes about 24 hours. Total number of cells in the structure: 1 712 815 142. Number of cells within the moving window: 8 668 824.

Only the fields in the vicinity of the bunch (i.e. inside the „moving window“)are updated

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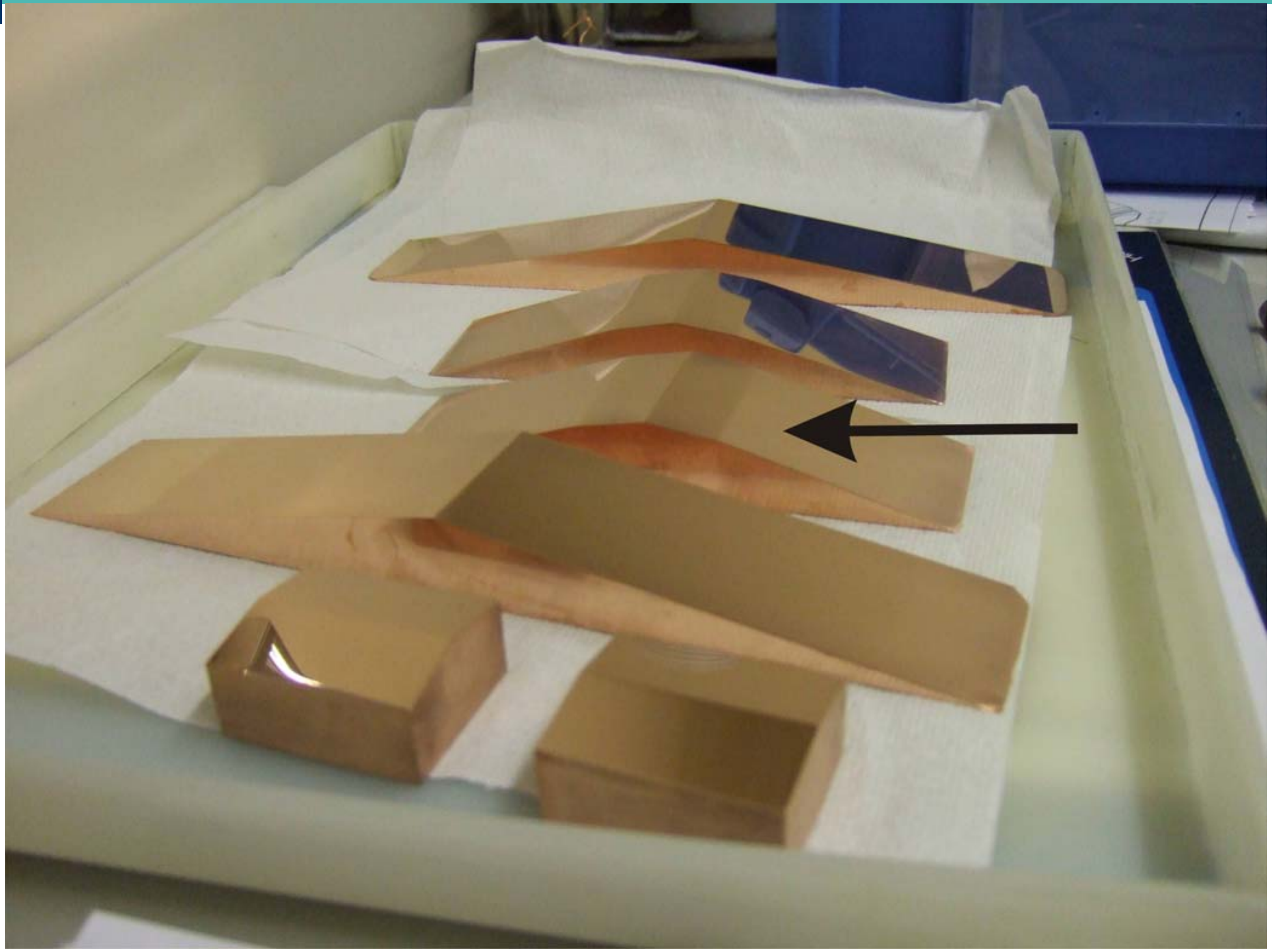
ILC-ESA Collimator Prototype



Collimators were built within the project Spoiler Wakefield and Mechanical Design (SWMD), and were measured with SLAC. Picture from Nigel Watson, CCLRC.

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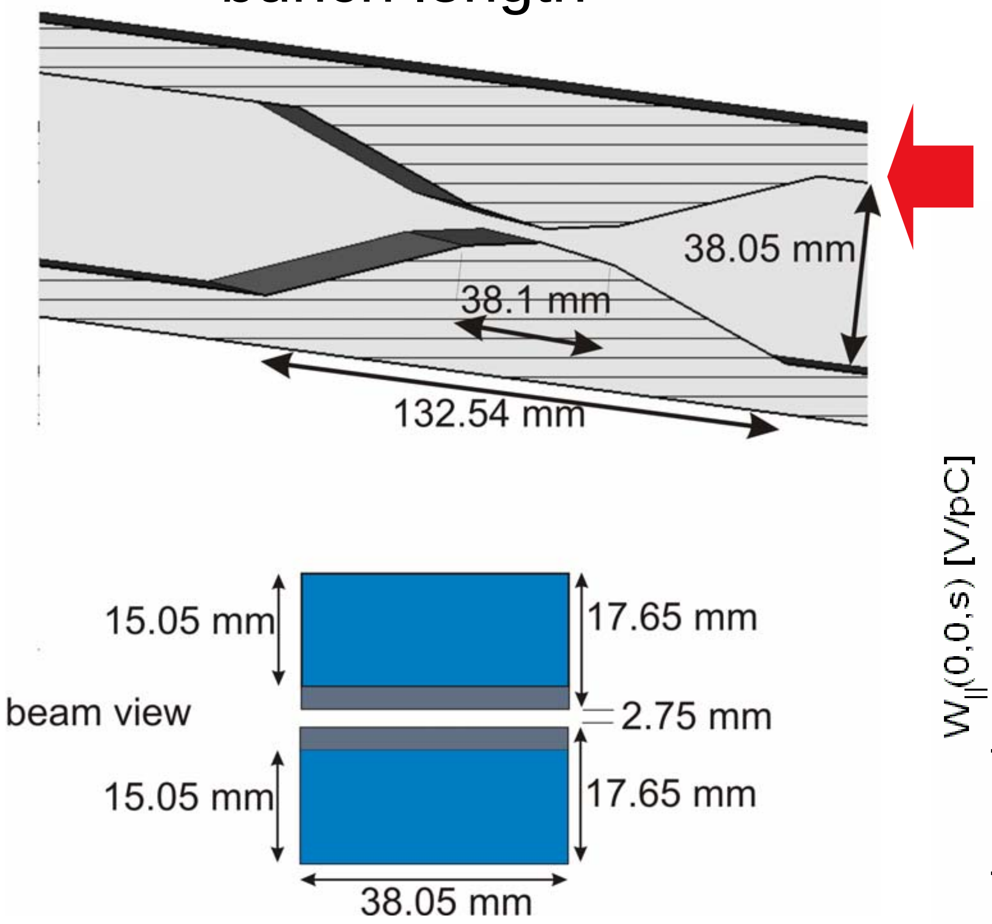
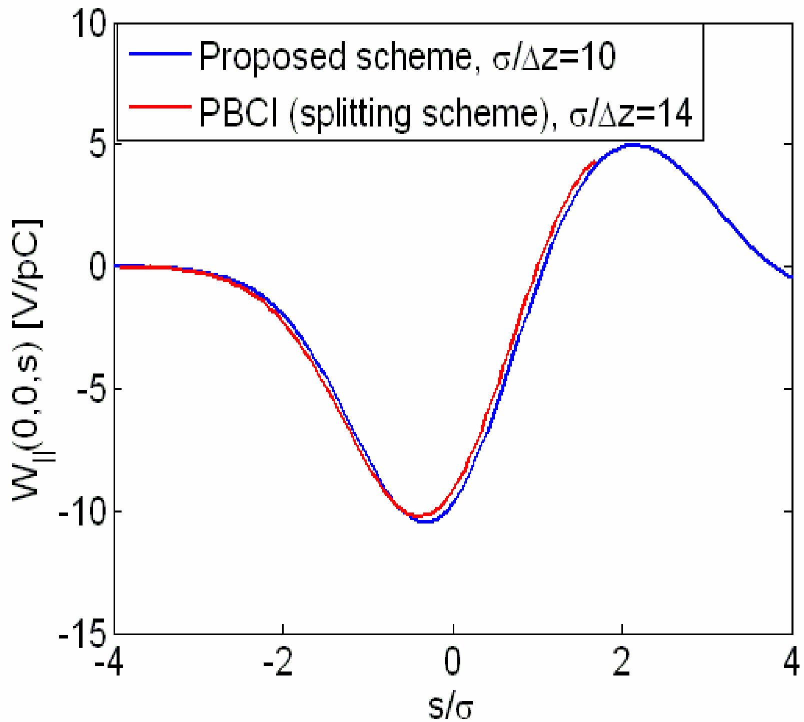
ILC-ESA Collimator Prototype

With a long bunch (1mm) the results are accurate with about 10 cells over the bunch length

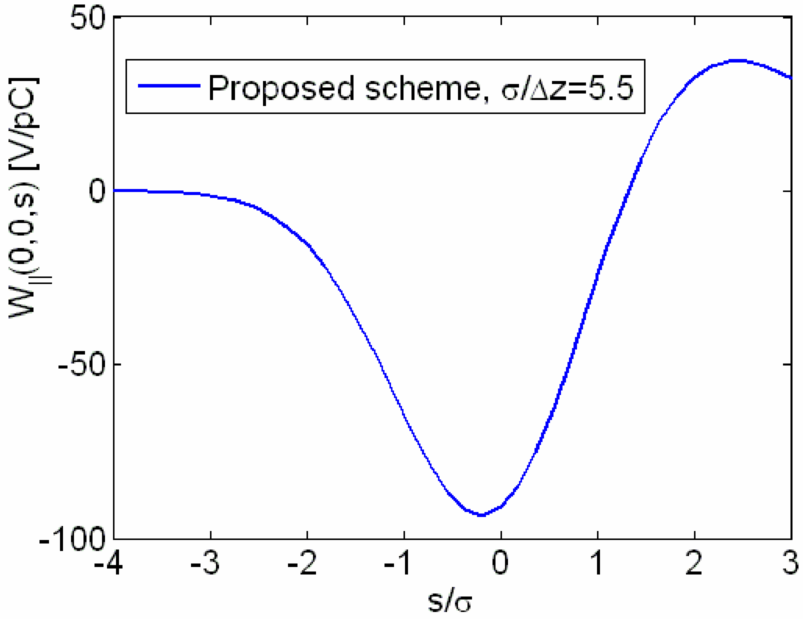


Beam pipe truncated here, indirect wake calculated by solving Poisson‘s equation

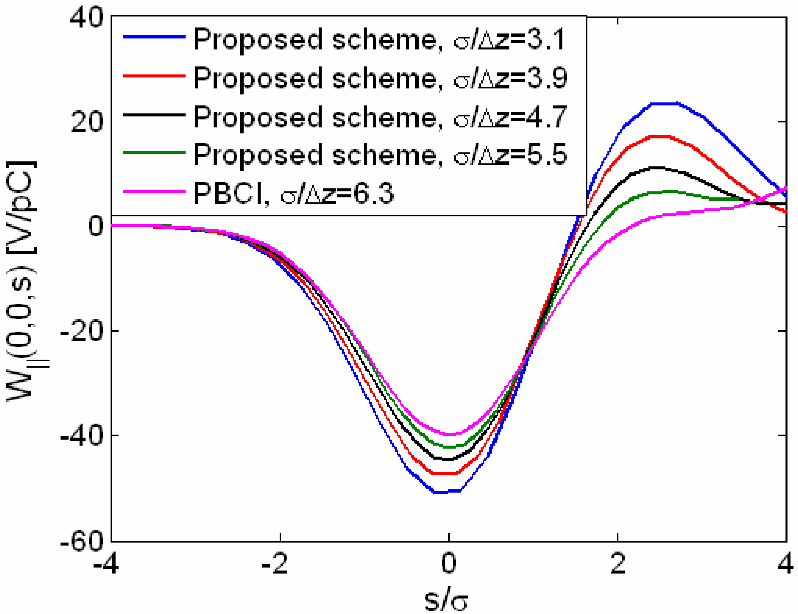


ILC-ESA Collimator Prototype

GdfidL produces fairly similar results



Indirect integration included, i.e. infinite outgoing pipe



Finer mesh or a conformal scheme is needed !

Step Collimators

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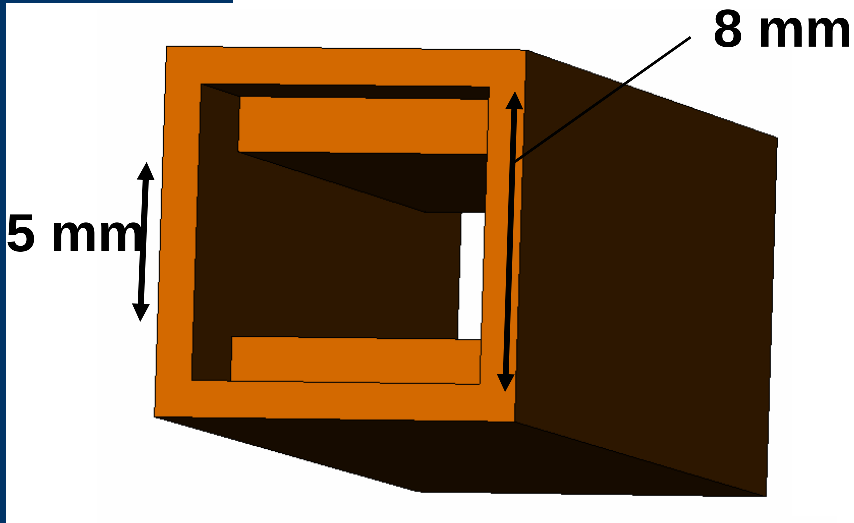
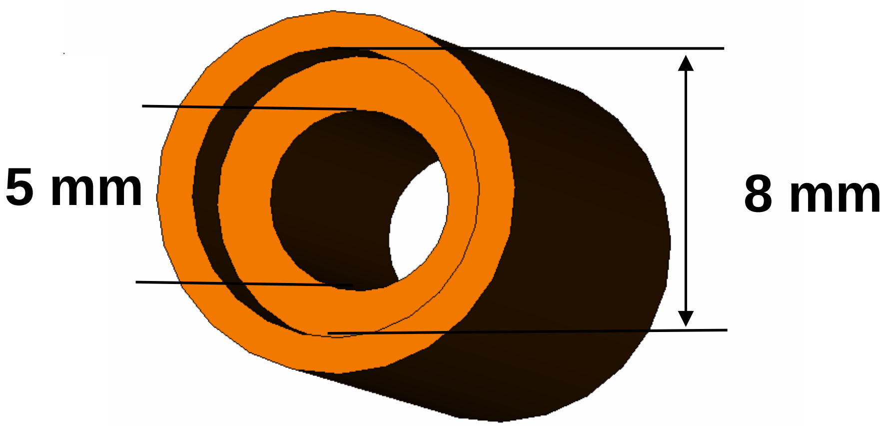
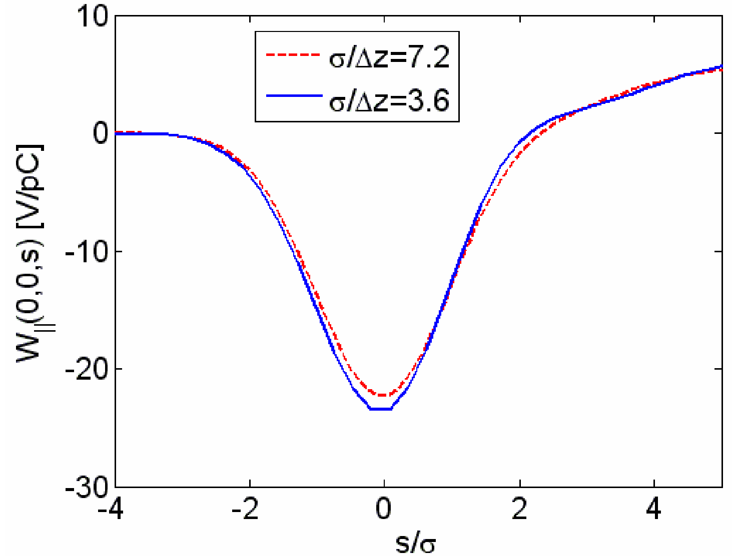
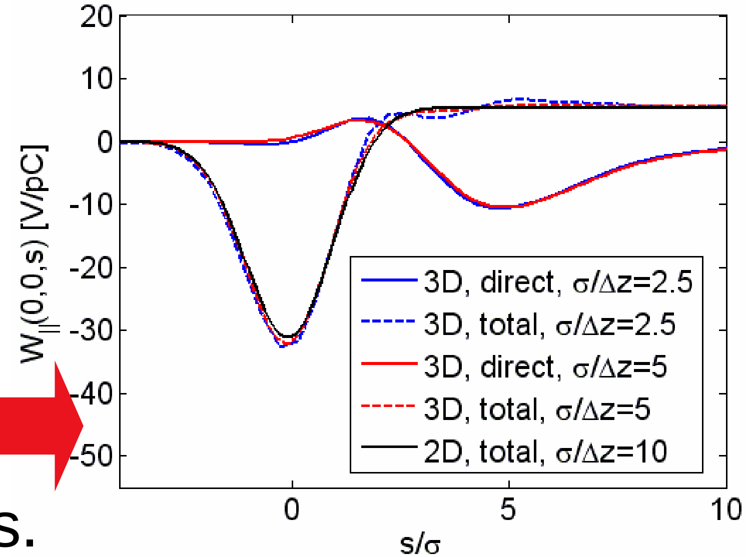


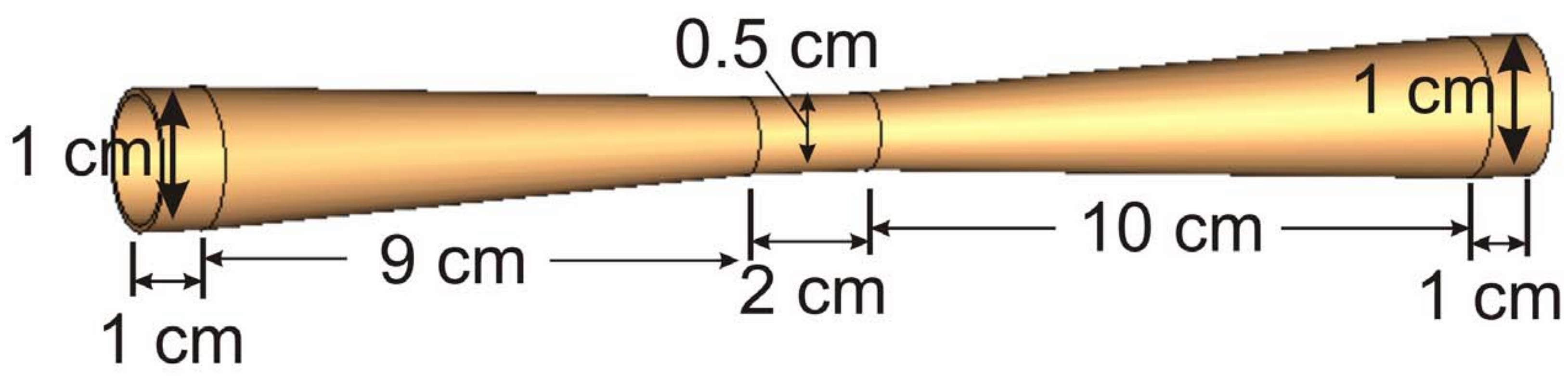


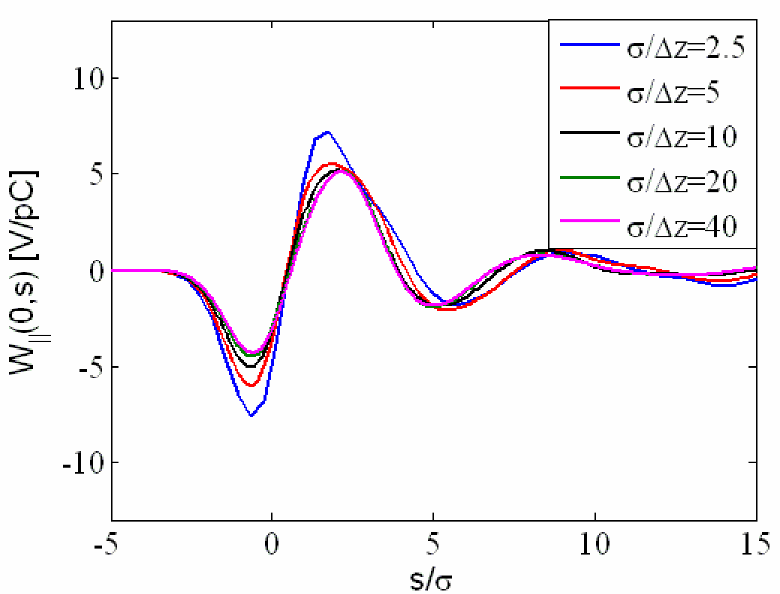




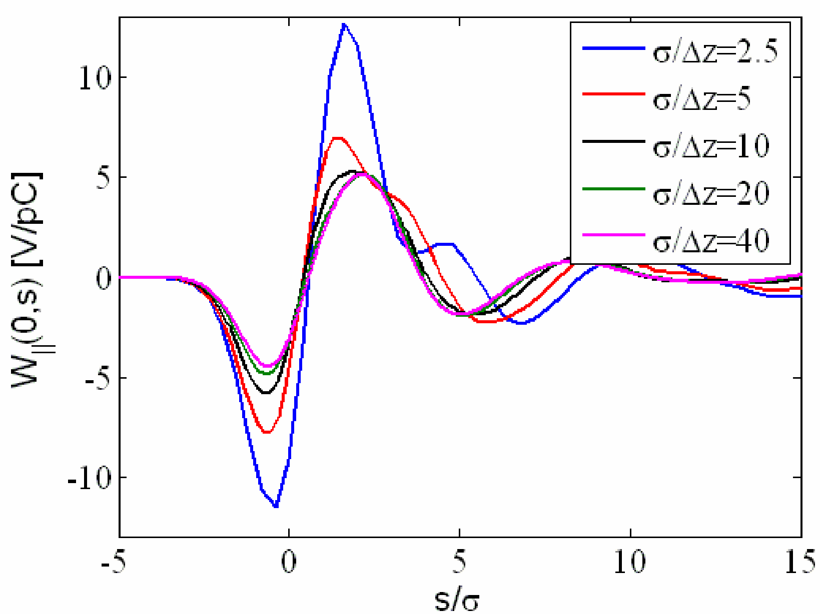
Very good agreement between 2D and 3D. σ=200μm in both cases. Total length is 2.8 cm in both cases.

Staircasing, Collimator





Proposed explicit 2D scheme



ECHO 2D (implicit scheme)

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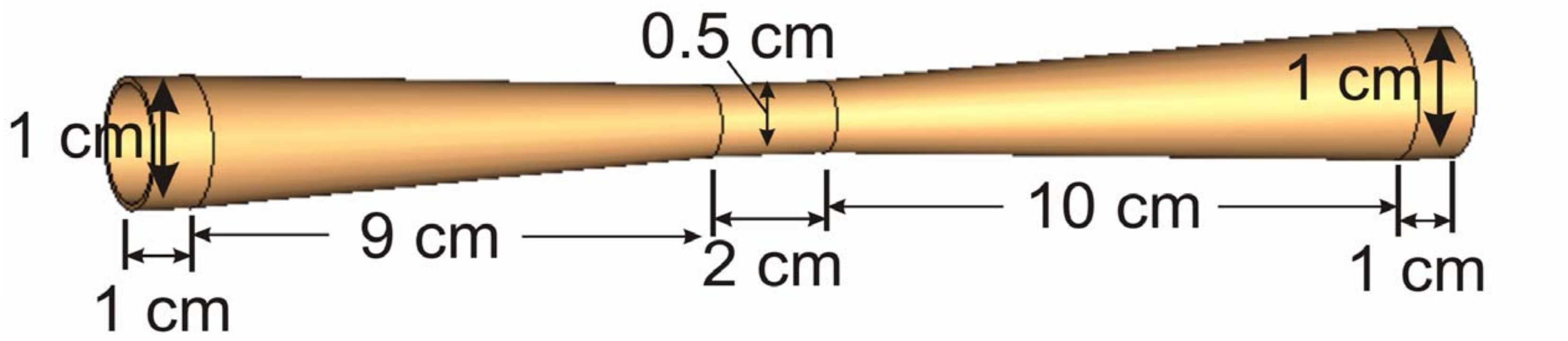
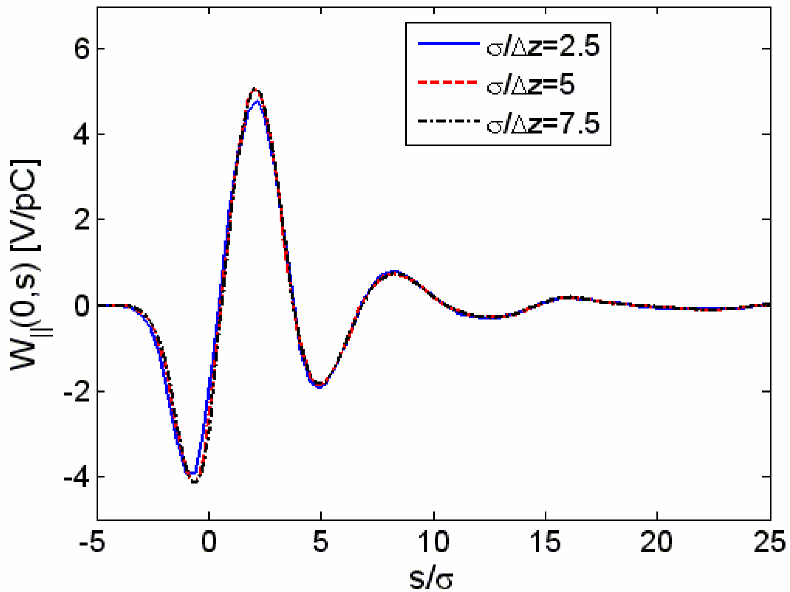
Conformal Scheme, Collimator

- Faster convergence with conformal schemes.





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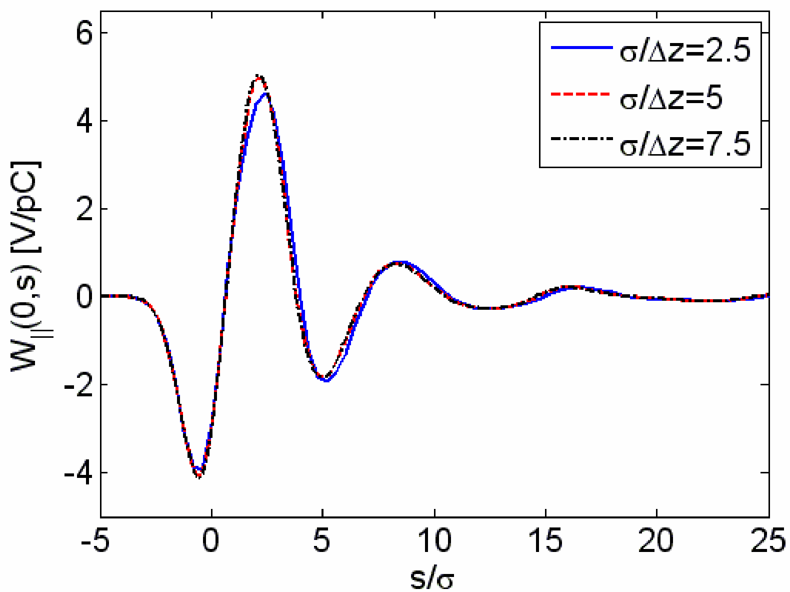

ECHO 2D (implicit scheme)

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Conclusions, Future Plans

- Conclusions:
  - An FDTD scheme for 3D wake field simulations has been introduced and (partially) validated
  - A scheme for cylindrically symmetric problems was introduced
  - Numerical dispersion properties of the schemes are suitable for simulating long accelerator structures
  - Dissatisfactory accuracy for gently tapered structures with stair-casing

- Future plans:
  - Conformal boundary treatment in 3D (PBCI and the proposed scheme)
  - Other extensions

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