Google "Alain Bossavit at TUT" for slides, papers, etc.

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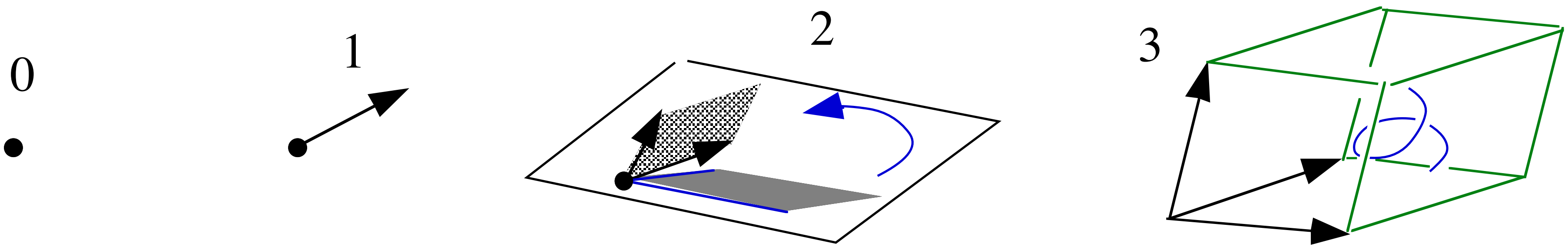
Geometrical methods in computational electromagnetism

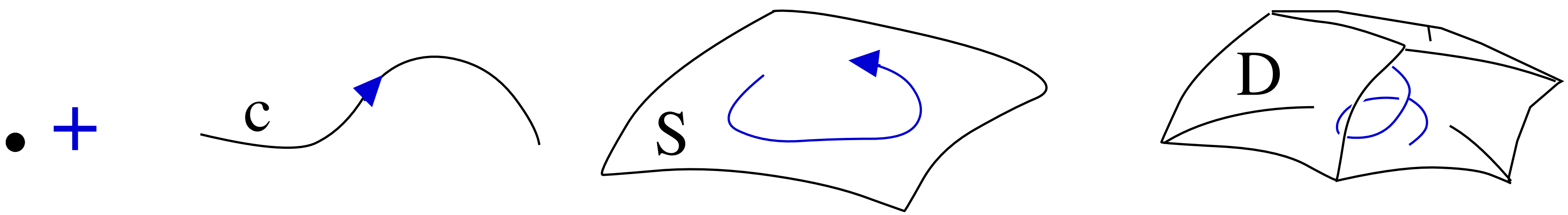
Nuts and bolts of a discretization toolkit

The workshop furniture

- Affine 3D space, with associated vector space, but no orientation, no metric structure (for a while)
- Points, vectors, multivectors (Grassmann algebra)



- Smooth sub-manifolds, with own orientation:



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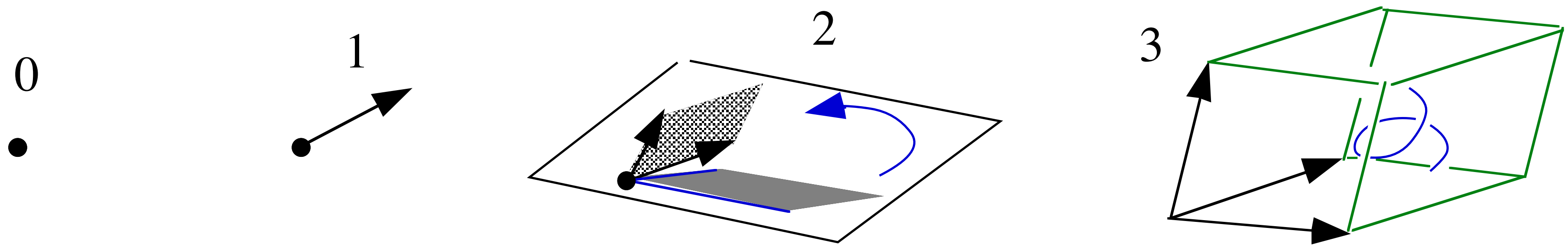

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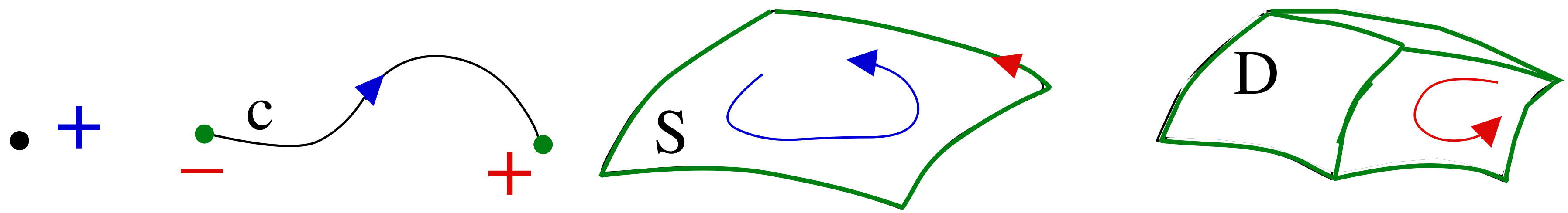

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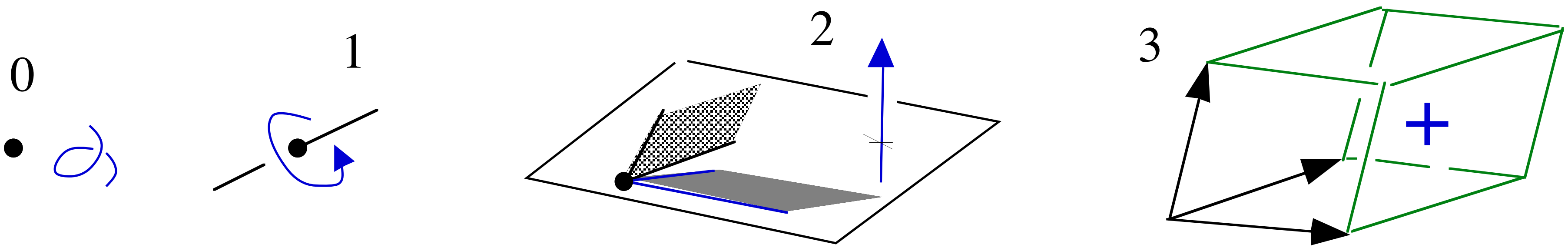

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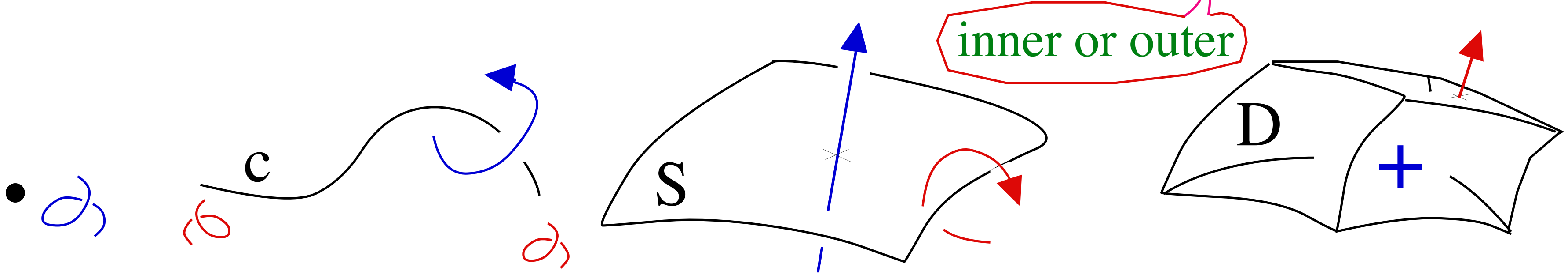
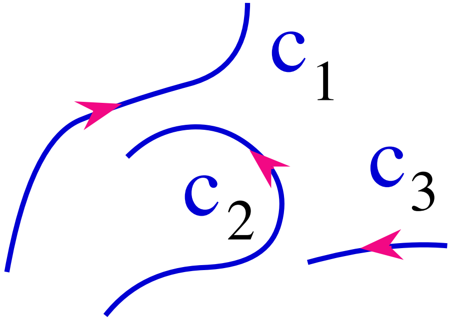


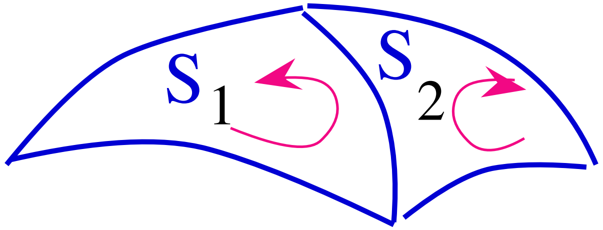
The concept of chain:

1-chains:

Embed set of curves in vector space of *singular* 1-*chains*



2-chains:



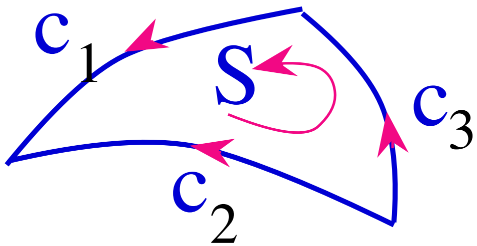
Same with surfaces etc.:

p-*chains*

$$c = r^1 c_1 + r^2 c_2 + r^3 c_3$$

e.g., $S = S_1 - S_2$

Boundary operator ∂:



$$\boxed{\partial S = c_1 - c_2 + c_3}$$

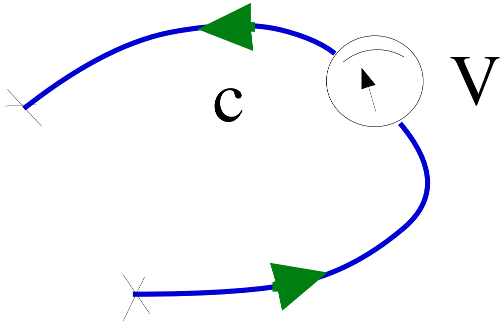
(Linear map: $\partial(S_1 - S_2) = \partial S_1 - \partial S_2$)

What about dual objects (linear functionals), called cochains?

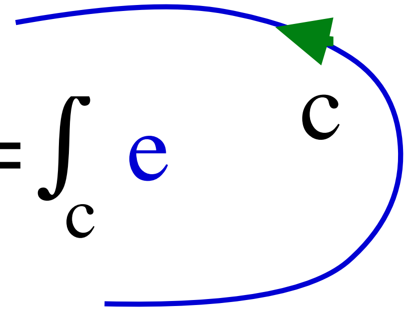
Chains model probes. Cochains model fields.

Voltmeter:

(p = 1)



e.m.f. $V = \int_c e$



Electric field seen as map

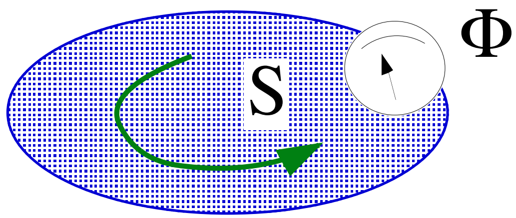
c → <emf along c>,

map here denoted e,

a 1-cochain.

Fluxmeter:

(p = 2)

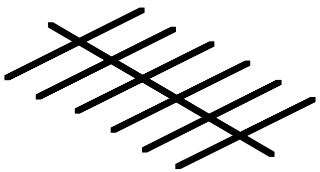


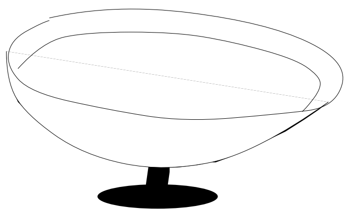
Magnetic induction as map

b, the 2-cochain

S → <flux embraced by S>.

Antennas:





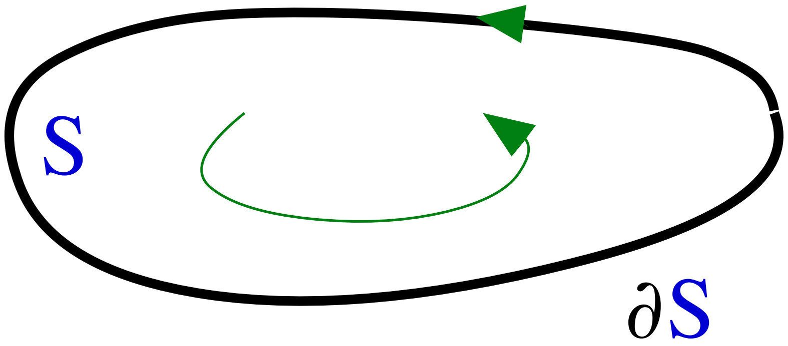
etc.

Small probe <—> p-vector

Local field <—> p-covector

Faraday's law, now straightforward:

$$\underbrace{\int_{\partial S} e}_{\textit{volts}}$$



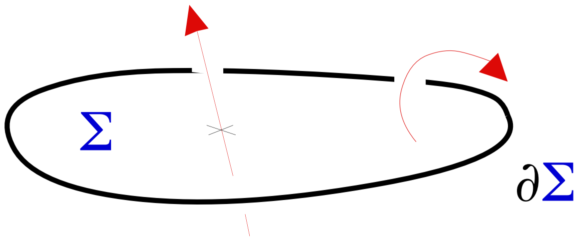
$$\underbrace{\int_{S} b}_{\textit{webers}}$$

$$\boxed{\frac{d}{dt}\int_S b + \int_{\partial S} e = 0}$$

2-cochain (b) 1-cochain (e)

or $\boxed{\partial_t b + de = 0}$, if d **defined by** $\int_S de = \int_{\partial S} e$

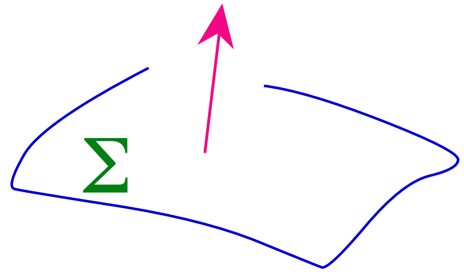
Ampère? almost the same, but not quite: outer oriented surface Σ,



hence "twisted" cochains d , j, and h:

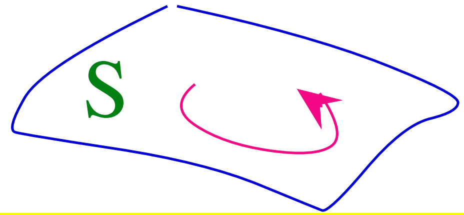
$$\boxed{-\partial_t d + dh = j}$$

(Ampère) $$-\,\partial_t \int_{\Sigma} d + \int_{\partial\Sigma} h = \int_{\Sigma} j \quad \forall$$

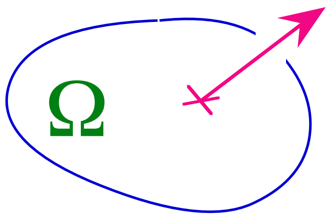


Apart from constitutive laws, yet to come, all equations relate non-metric objects— cochains

(Faraday) $$\partial_t \int_{S} b + \int_{\partial S} e = 0 \quad \forall$$



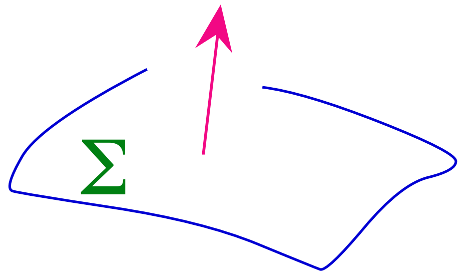
$$\int_{\Omega} q \;\hat{=}\; \int_{\partial\Omega} d$$



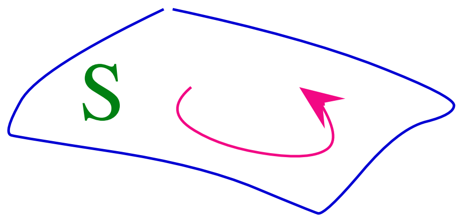
$$\partial_t \int_{\Omega} q + \int_{\partial\Omega} j = 0$$

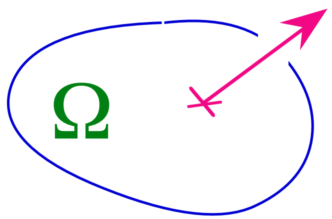
(Defining charge q)

(Charge conservation)

(Ampère) $$-\partial_t \int_\Sigma d + \int_{\partial\Sigma} h = \int_\Sigma j \quad \forall$$ 

$$b = \mu h \qquad\qquad d = \varepsilon e$$

(Faraday) $$\partial_t \int_S b + \int_{\partial S} e = 0 \quad \forall$$ 

$$\int_\Omega q \;\hat{=}\; \int_{\partial\Omega} d$$ 

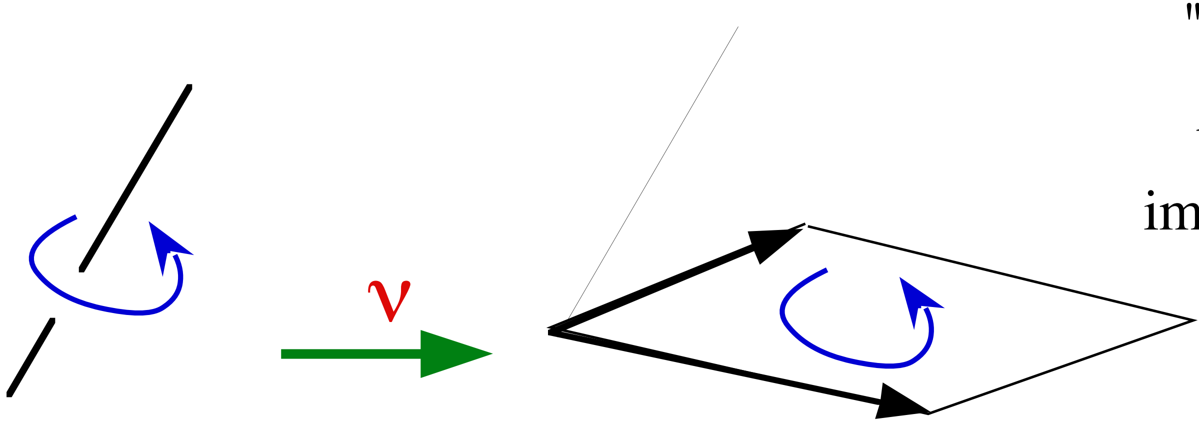
$$\partial_t \int_\Omega q + \int_{\partial\Omega} j = 0$$

Further structuration of space: the Hodge map

p-*VECTOR* $\longrightarrow$ (n – p)-*VECTOR*

twisted or straight — straight or twisted

Equip space with such a map, ν. (Another one, denoted ε, will be needed.)



Only requirement, "non-degeneracy". (Volume $v \wedge \nu v$, built on v and its image, must be $\neq 0$.)

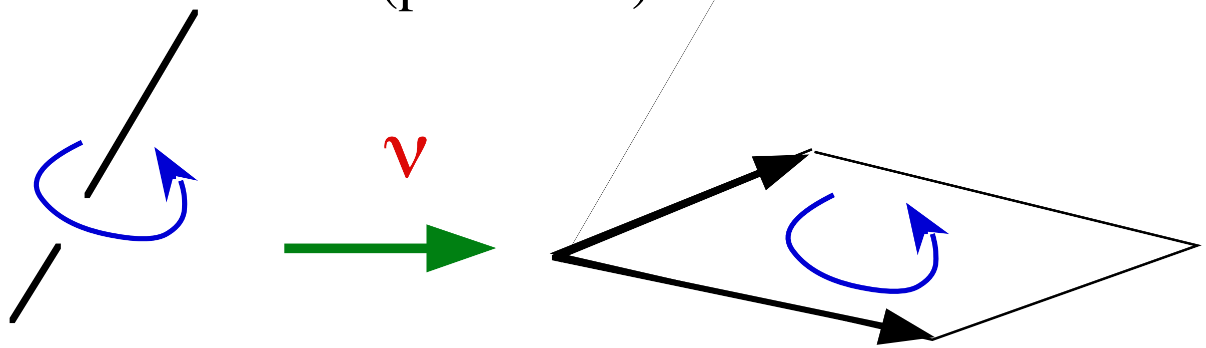
Determines a metric ("ν-adapted")

(Select reference 3-vector Δ and real λ. Then $\lambda^2 v \wedge \nu v = |v|^2_\lambda \Delta$, hence a norm, scaling as λ. Adjust λ for λ-volume of Δ to be λ^2.)

By duality, yields Hodge map on *co*vectors:

$$\text{p-}\mathit{VECTOR} \xrightarrow{\nu} \text{(n – p)-}\mathit{VECTOR}$$

(p = 1 here)



$$\langle \underbrace{\nu v}_{\text{2-vector}} \, ; \overbrace{b}^{\text{2-covector}} \rangle = \langle \underbrace{v}_{\text{1-vector}} \, ; \overbrace{\nu b}^{\text{1-covector}} \rangle$$

$$\text{p-CO}\mathit{VECTOR} \xleftarrow{\nu} \text{(n – p)-CO}\mathit{VECTOR}$$

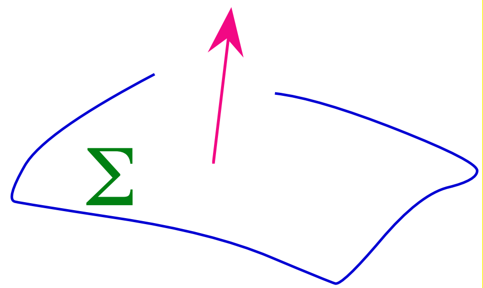
Hence relation $h = \nu b$ (and also $d = \varepsilon e$) between cochains, i.e., *fields*

So space geo-*metry* (in the strong sense of assigning *metric* properties—distances, areas, angles, etc.—to the space we inhabit) amounts to specifying *constitutive laws* in electrodynamics

- Should not sound strange: Don't we use *light rays* to measure the Earth?
- Why *two* metrics ($\nu \equiv \mu^{-1}$ and ε)? Because 3D shadows of Minkowski's 4D metric
- $\varepsilon \neq \varepsilon_0$ and $\mu \neq \mu_0$ when we wish to *ignore* details of microscopic interactions and *geometrize* them wholesale

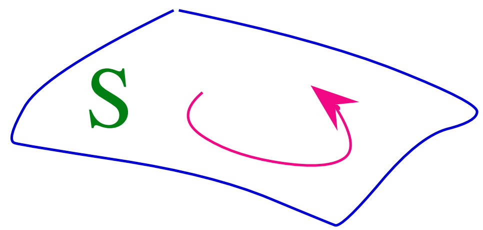
Maxwell, in terms of cochains:

$$-\partial_t \int_\Sigma d + \int_{\partial\Sigma} h = \int_\Sigma j \quad \forall$$



$$d = \varepsilon\, e \qquad\qquad h = \nu\, b$$

$$\partial_t \int_S b + \int_{\partial S} e = 0 \quad \forall$$



$$-\partial_t d + dh = j$$

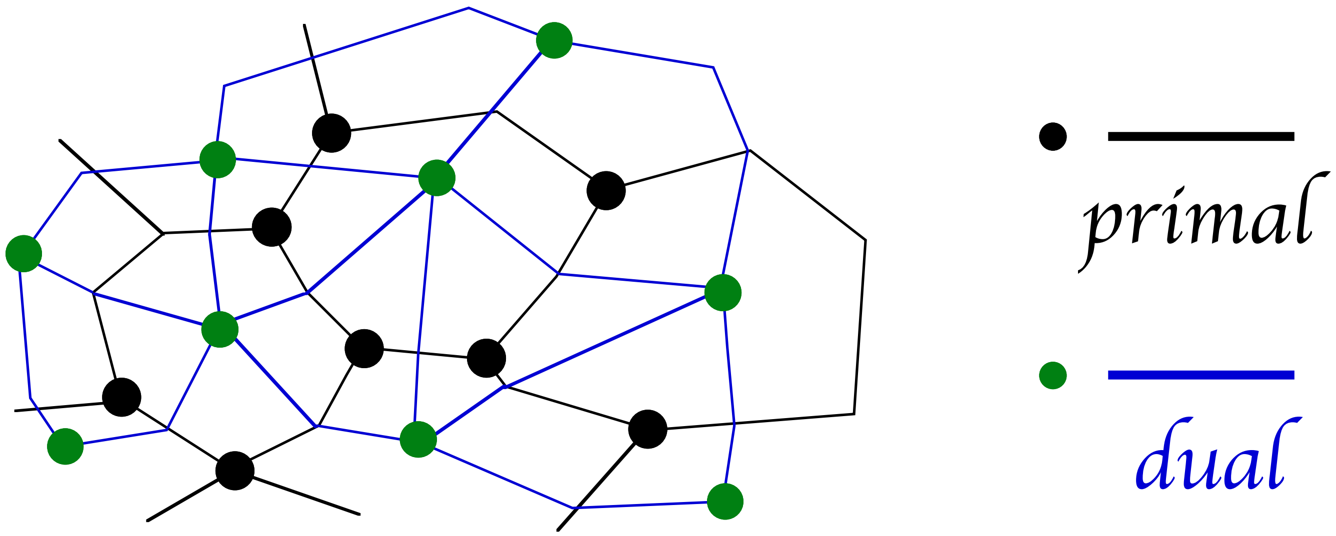
$$d = \varepsilon\, e$$

$$h = \nu\, b$$

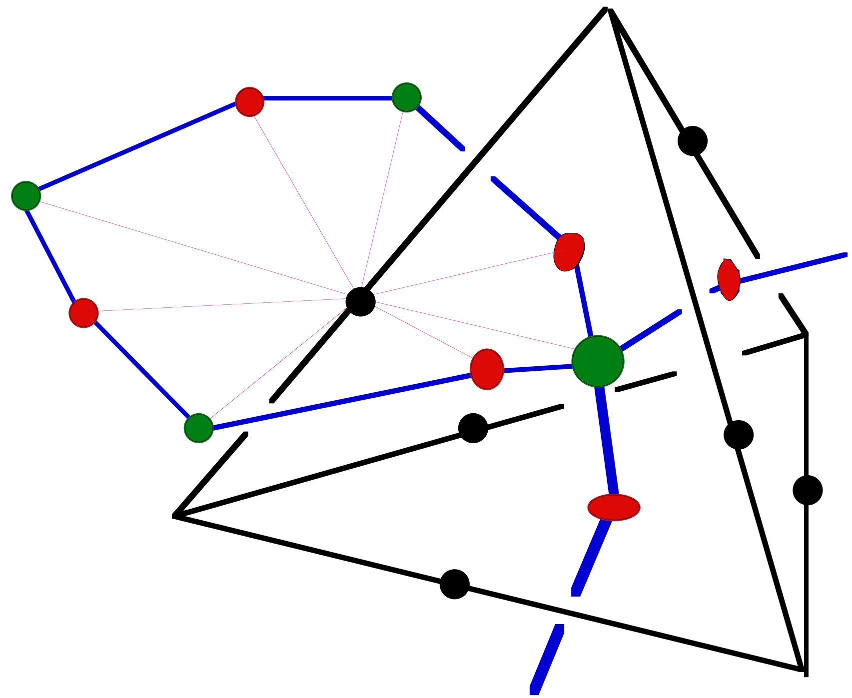
$$\partial_t b + de = 0$$

Discretization strategy: Instead of all S, Σ, enforce this for finite family of surfaces spanned by faces of a mesh

But Hodge requires 1–1 pairing of edges and faces, so...

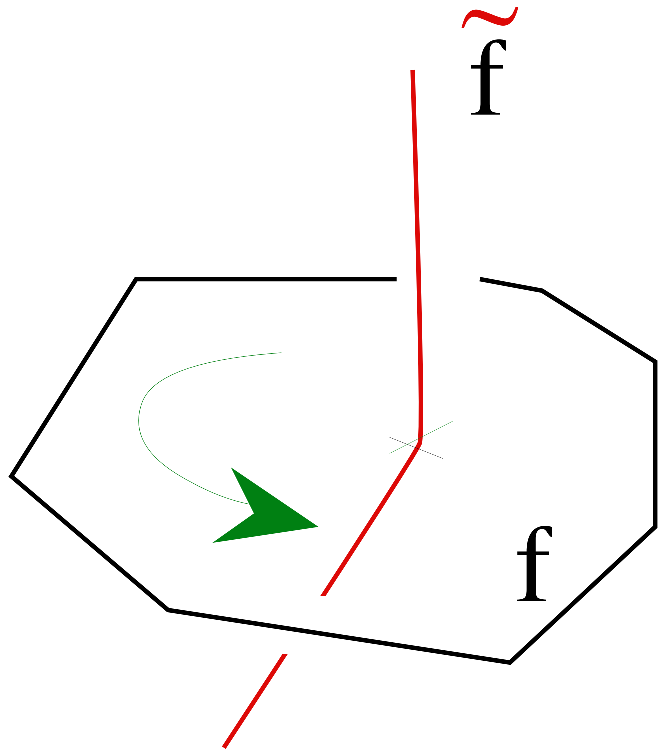


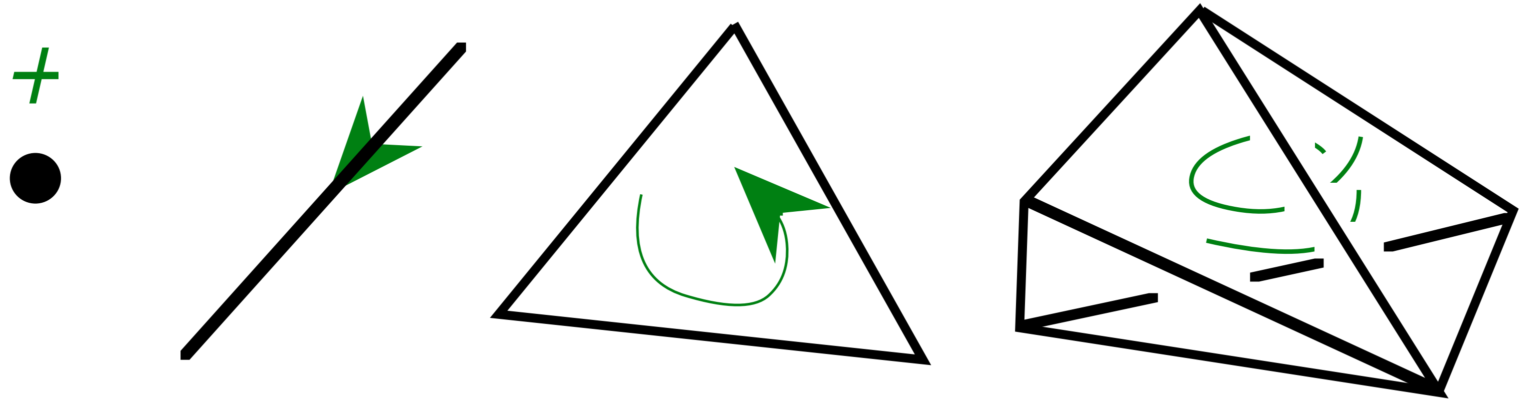
Select centers inside primal simplexes. Join them to make dual.

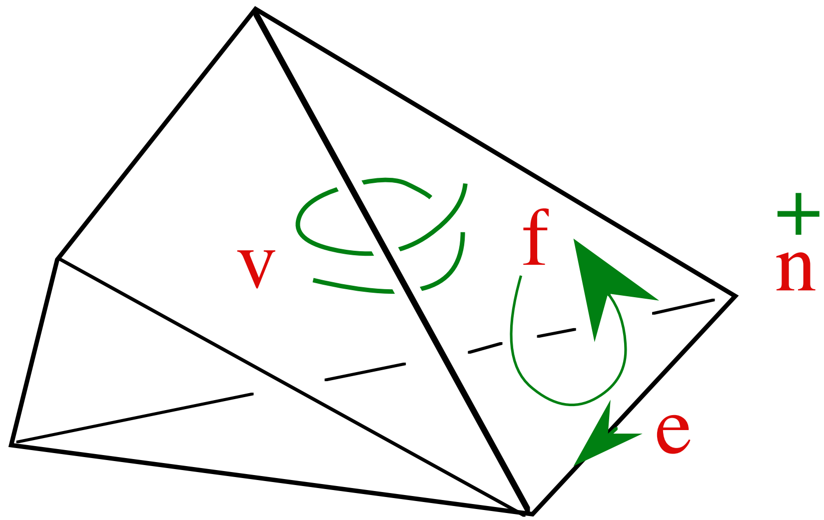


To each cell its own orientation,

inner (for m) or outer (for $\tilde{m}$)







$$G_{en} = -1$$
$$R_{fe} = -1$$
$$D_{vf} = 1$$

$$DR = 0, \quad RG = 0$$

(in analogy with grad, rot, div)

Incidence matrices for dual are transposes $\mathbf{D}^t$, $\mathbf{R}^t$, $\mathbf{G}^t$

Discretization: General Principles

Replace generic **surfaces** S, Σ,

as well as abstractions of voltmeter, fluxmeter, etc.,

by **cell-chains** (i.e., based on mesh cells)

- Primal cells when emf's, or magnetic "fluxes" are concerned (integrals of fields e, a, b, ...)
- Dual cells when genuine, *substantial*, fluxes are concerned (energy, charge, momentum...) (integrals of fields h, d, j, ..., and more involved things like —to appear later— $e \wedge h$, $i\ b \wedge j$, $i\ b \wedge h$, $i\ d \wedge e$, ...)

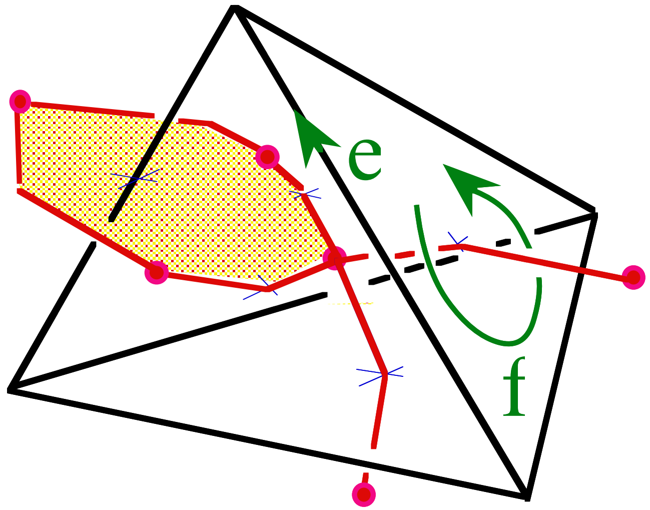
Replace fields by **cell-cochains**

thus, field modelled by "DoF array" of scalar values, one per cell of right kind

$$\mathcal{N} \xrightarrow[\text{grad}]{G} \mathcal{E} \xrightarrow[\text{rot}]{R} \mathcal{F} \xrightarrow[\text{div}]{D} \mathcal{V}$$

b *at faces*

e, a *at edges*



here, $R_{fe} = -1$

h *at dual edges (i.e., faces)*

d, j *at dual faces*

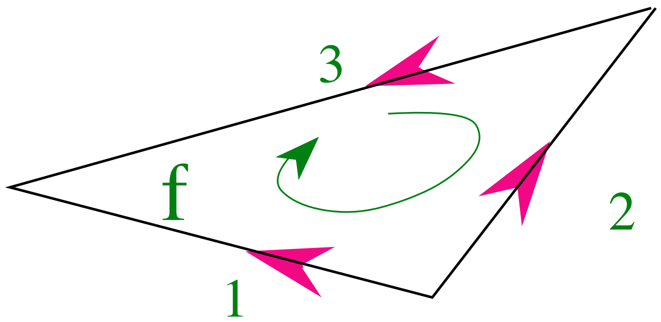
$$\mathbf{b} = \{b_f : f \in \mathcal{F}\} \xrightarrow{\nu} \mathbf{h} = \{h_f : f \in \mathcal{F}\}$$

$$\mathbf{e} = \{e_e : e \in \mathcal{E}\} \xrightarrow{\varepsilon} \mathbf{d} = \{d_e : e \in \mathcal{E}\}$$

Enforce Faraday's law, $\partial_t \int_S b + \int_{\partial S} e = 0$

not for all surfaces S, but for all those made of primal faces. This requires (when $S = f$, a primal face),

$$\partial_t b_f + e_1 - e_2 - e_3 = 0$$



i.e.,

$$\partial_t \mathbf{b} + \mathbf{R}\mathbf{e} = 0$$

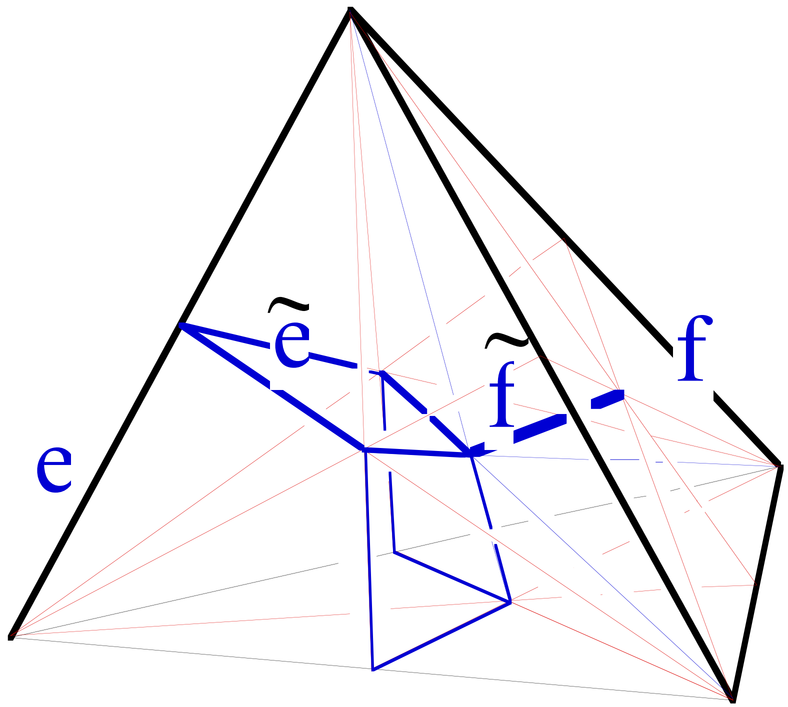
Same thing on dual network:

$$-\partial_t \mathbf{d} + \mathbf{R}^t \mathbf{h} = \mathbf{j}$$

Which leaves $\boldsymbol{\varepsilon}$ and $\boldsymbol{\mu}$ to be built

Note *automaticity* of process: Equations forced on us by implementation decisions taken so far.

Metric structure: to any twisted vector v, associate a bivector νv.



But according to "general principles", only available twisted vectors, in discrete framework, are the $\tilde{f}$'s; and only bivectors, the f's. Therefore, define:

$$\nu \tilde{f} = \sum_{f' \in \mathcal{F}} \nu^{ff'} f'$$

By duality, the $\nu^{ff'}$ make the needed matrix ν

Compatibility with earlier map ν (given reluctivity "tensor") provides convergence criterion for numerical analysis.

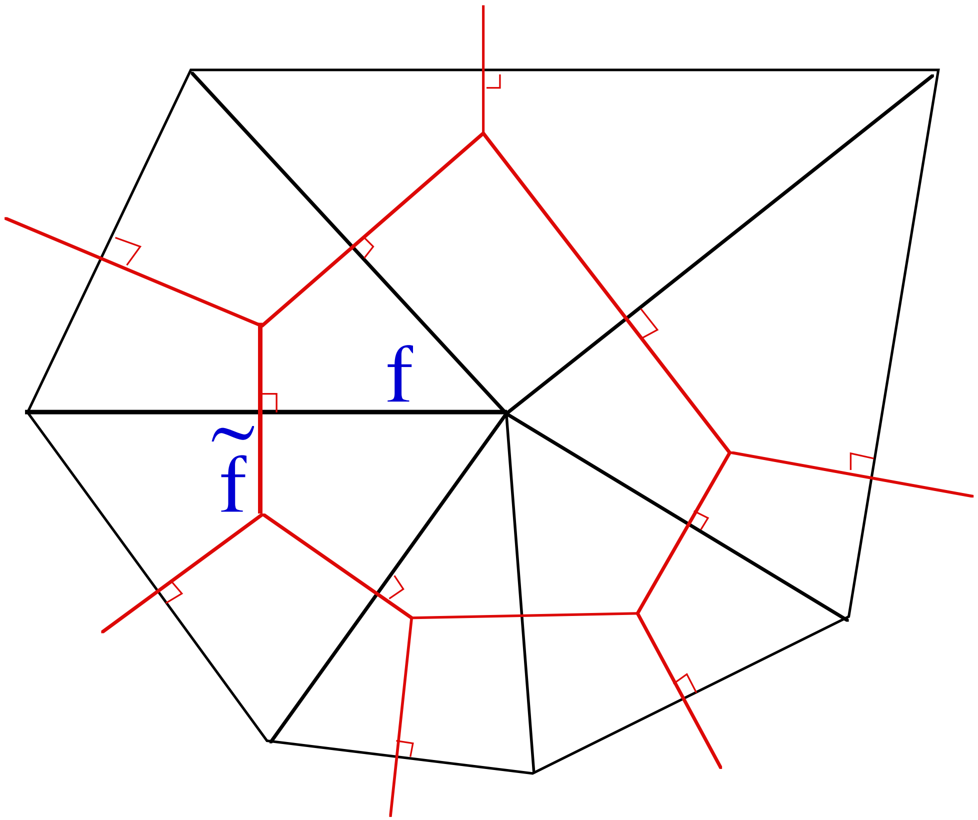
If dual mesh barycentric, use the "Galerkin Hodge", defined as

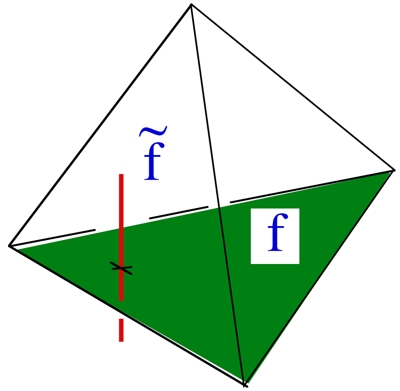
$$\boldsymbol{\nu}^{ff'} = \int \nu \; \mathbf{w}^{f} \cdot \mathbf{w}^{f'}$$

$$\boldsymbol{\varepsilon}^{ee'} = \int \varepsilon \; \mathbf{w}^{e} \cdot \mathbf{w}^{e'}$$

where w^s is Whitney form of simplex s

If mutual orthogonality of primal and dual meshes can be achieved,





Then,

$$\nu^{ff} = \nu \frac{\text{length}(\tilde{f})}{\text{area}(f)}$$

i.e., ν diagonal

Then, automatic spatial discretization of Maxwell's equations:

$$\partial_t \mathbf{b} + R\mathbf{e} = 0 \qquad -\partial_t \mathbf{d} + R^t \mathbf{h} = \mathbf{j}$$

$$\mathbf{h} = \boldsymbol{\nu}\,\mathbf{b} \qquad \mathbf{d} = \boldsymbol{\varepsilon}\,\mathbf{e}$$

hence (using leap-frog) a Yee-like scheme:

$$(\boldsymbol{\nu} = \boldsymbol{\mu}^{-1})$$

$$\frac{\mathbf{b}^{k+1/2} - \mathbf{b}^{k-1/2}}{\delta t} + R\mathbf{e}^{k} = 0$$

$$-\varepsilon\,\frac{\mathbf{e}^{k+1} - \mathbf{e}^{k}}{\delta t} + R^t \boldsymbol{\nu}\,\mathbf{b}^{k+1/2} = \mathbf{j}^{k+1/2}$$

Whitney forms

Once obtained the cochains **b**, **e** (or **a**), **h**, **d**, what about the fields themselves?

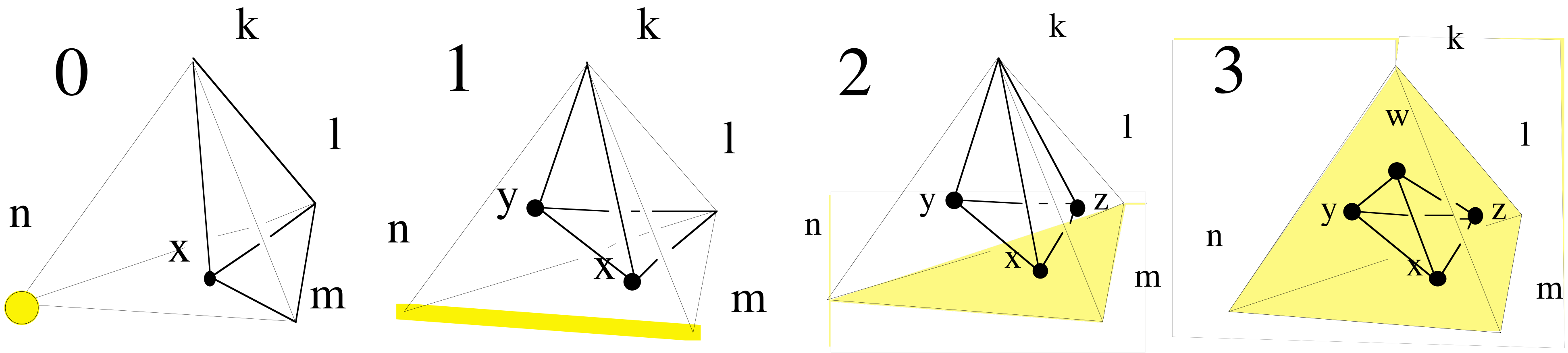
or else:

Are there objects that would be to differential forms what finite elements are to functions, i.e., to 0-forms?

$\mathbf{a} \rightarrow \sum_{e \in \mathcal{E}} \mathbf{a}_e \ w^e$ $\qquad\qquad$ $\mathbf{b} \rightarrow \sum_{f \in \mathcal{F}} \mathbf{b}_f \ w^f$

(Vector potential.)

Whitney forms



w^{n} $\quad w^{\{m,\, n\}}$ $\quad w^{\{l,\, m,\, n\}}$ $\quad w^{\{k,\, l,\, m,\, n\}}$

$$\lambda^{n}$$

$$\lambda^{n}\, d\lambda^{m} - \lambda^{m} d\lambda^{n}$$

$$2[\, \lambda^{l}\, d\lambda^{m} \wedge d\lambda^{n} + \dots + \dots]$$

$$6\, d\lambda^{k} \wedge d\lambda^{l} \wedge d\lambda^{m}$$

The tools in the box:

Surfaces, curves, etc. $\longrightarrow$ Cell chains

Fields b, h, ... $\longrightarrow$ Cell cochains (DoF arrays) $\mathbf{b}$, $\mathbf{h}$, ...

Constitutive laws $\longrightarrow$ "Discrete hodges", $\boldsymbol{\varepsilon}$, $\boldsymbol{\nu}$, $\boldsymbol{\sigma}$ ...

grad, rot, div $\longrightarrow$ G, R, D (primal side), $-\mathrm{D}^t$, R^t, $-\mathrm{G}^t$ (dual side)

products, $E \times H$, $J \cdot E$ $\longrightarrow$ "wedge" product, $e \wedge h$, $j \wedge e$

$$-\partial_t D + \mathrm{rot}\, H = J, \quad D = \varepsilon E \qquad -\partial_t \mathbf{d} + \mathrm{R}^t \mathbf{h} = \mathbf{j}, \quad \mathbf{d} = \boldsymbol{\varepsilon} \mathbf{e}$$

$$\partial_t B + \mathrm{rot}\, E = 0, \quad H = \nu B \qquad \partial_t \mathbf{b} + \mathrm{R}\, \mathbf{e} = 0, \quad \mathbf{h} = \boldsymbol{\nu} \mathbf{b}$$

$$\mathrm{div}\, D = Q, \quad \mathrm{div}\, B = 0 \qquad -\mathrm{G}^t \mathbf{d} = \mathbf{q}, \quad \mathrm{D}\mathbf{b} = 0$$

$$E = -\,\mathrm{grad}\, \varphi - \partial_t A \qquad \mathbf{d} = -\mathrm{G}\, \boldsymbol{\varphi} - \partial_t \mathbf{a}$$

etc.

Good, but not enough:

What about "force related" entities, like

- $E \times H$ (Poynting) ?
- $Q(E + v \times B)$ (Lorentz) ?
- $J \times B$ (Laplace) ?
- $B \otimes H$ (Maxwell) ?

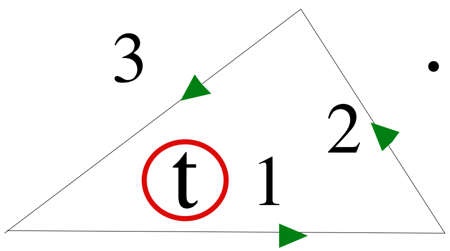
Heuristic hint: force is a covector, cf. $v \to \langle v \, ; f \rangle$

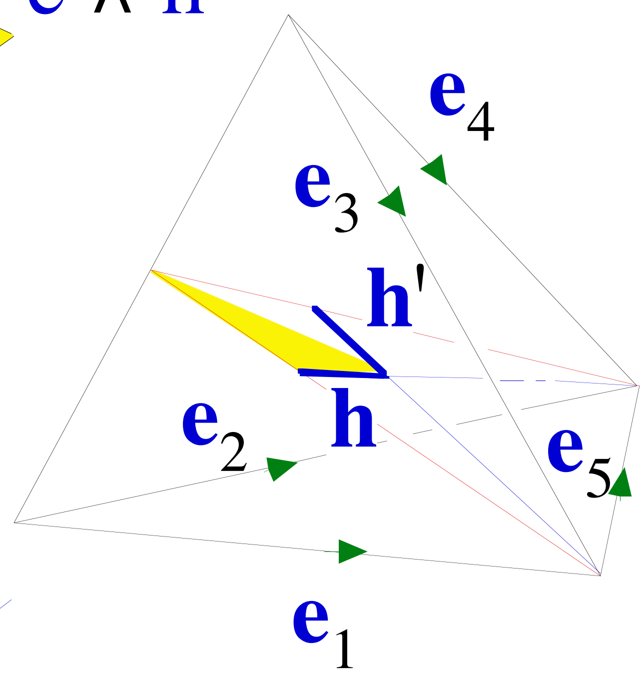
Flux of Poynting "vector"

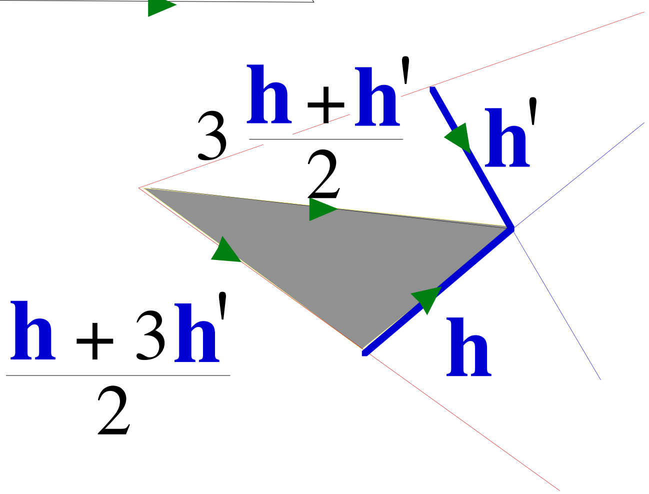
Actually, $\int_\Sigma \mathrm{e} \wedge \mathrm{h}$, with Σ a dual 2-chain

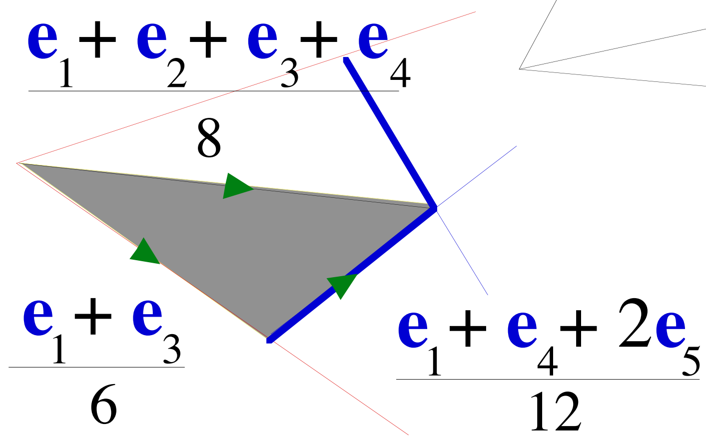
Knowing DoF-arrays $\mathbf{e}$, $\mathbf{b}$, compute $\int \mathrm{e} \wedge \mathrm{h}$

$$\int_t \mathrm{e} \wedge \mathrm{h} = \frac{1}{6}\,[\mathbf{e}_1\mathbf{h}_2 + \mathbf{e}_2\mathbf{h}_3 + \mathbf{e}_3\mathbf{h}_2 \;\cdots\; - \mathbf{h}_1\mathbf{e}_2 - \mathbf{h}_2\mathbf{e}_3 - \mathbf{h}_3\mathbf{e}_1]$$







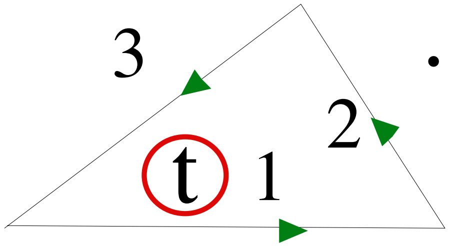


Flux of Poynting "vector"

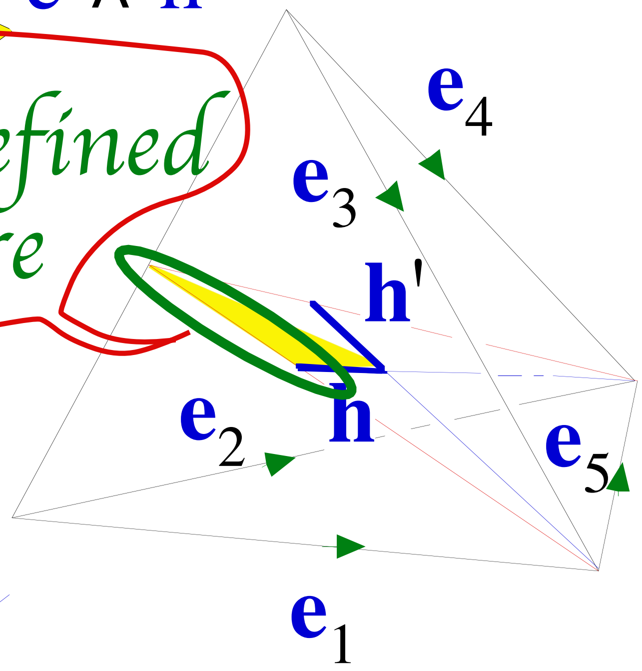
Actually, $\int_\Sigma \mathrm{e} \wedge \mathrm{h}$, with Σ a dual 2-chain

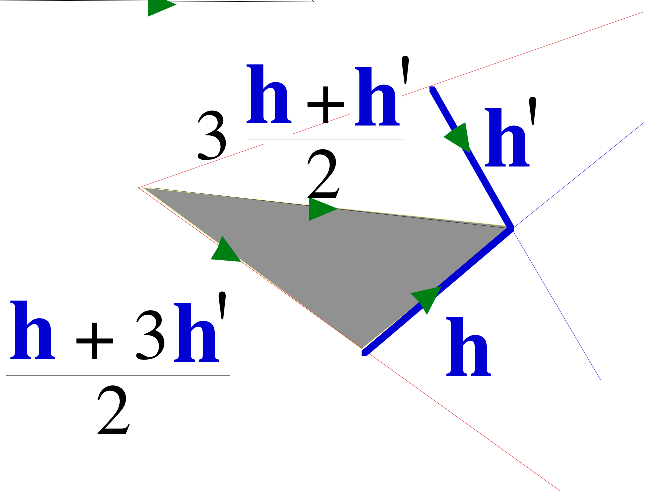
Knowing DoF-arrays **e**, **b**, compute $\int \mathrm{e} \wedge \mathrm{h}$

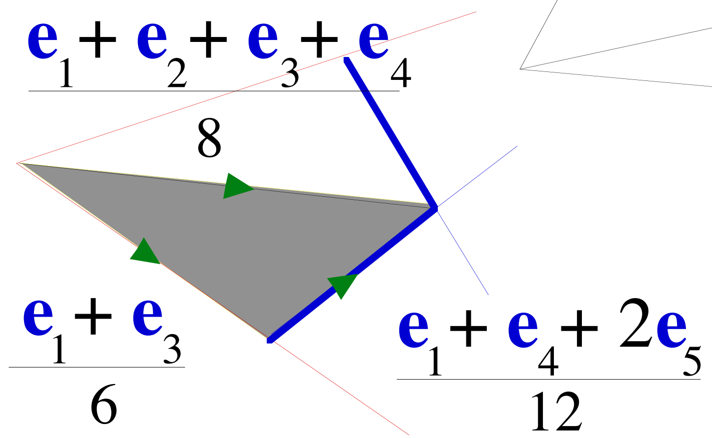
$$\int_t \mathrm{e} \wedge \mathrm{h} = \frac{1}{6}\,[\mathbf{e}_1\mathbf{h}_2 + \mathbf{e}_2\mathbf{h}_3 + \mathbf{e}_3\mathbf{h}_2 \ \dots \ - \mathbf{h}_1\mathbf{e}_2 - \mathbf{h}_2\mathbf{e}_3 - \mathbf{h}_3\mathbf{e}_1]$$



h *ill-defined there*







Flux of Poynting "vector"

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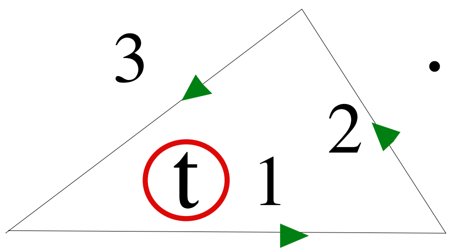
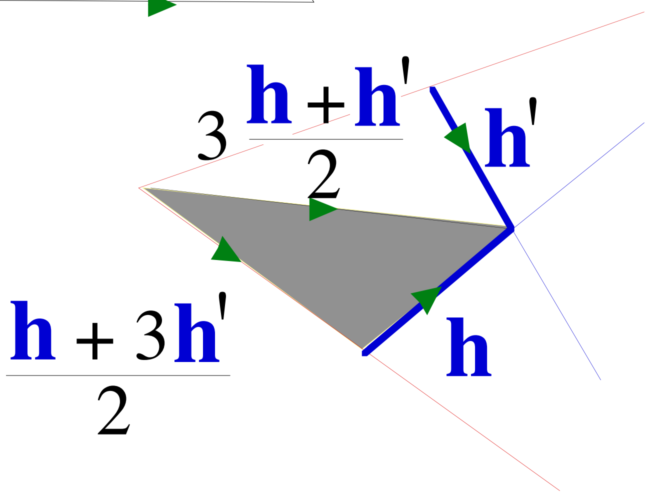
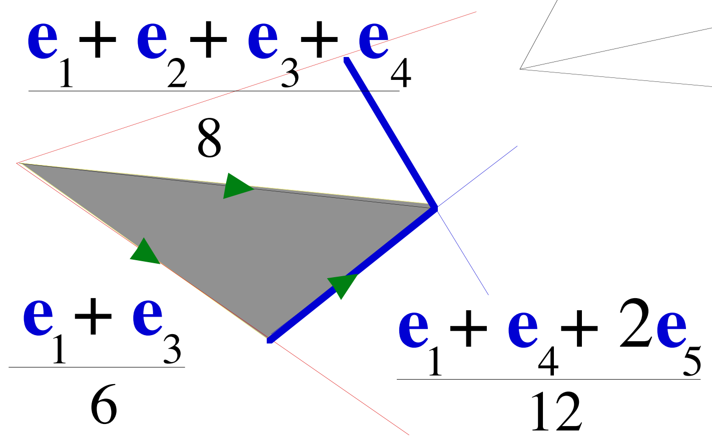
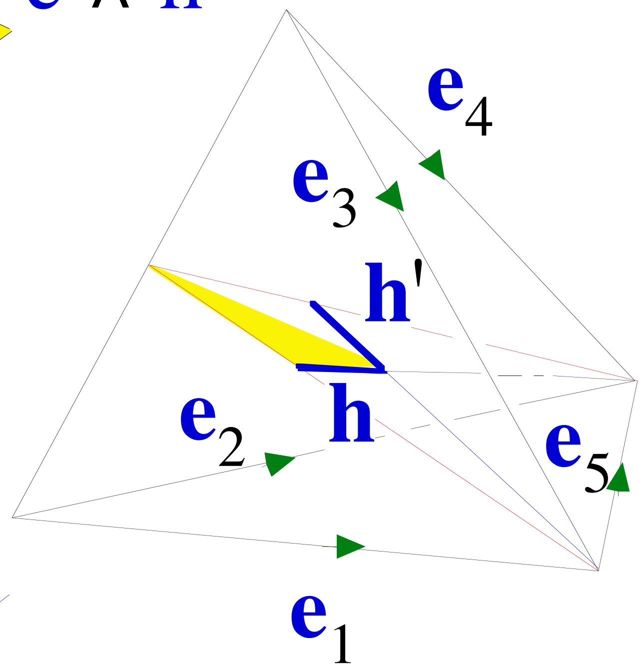


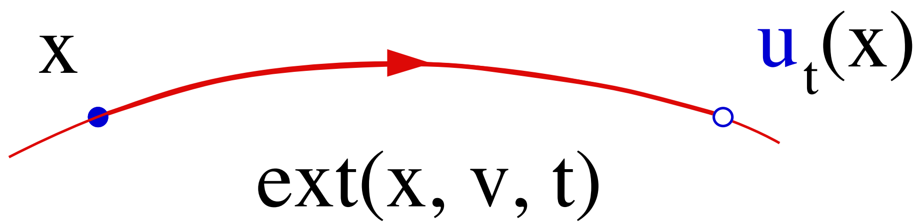






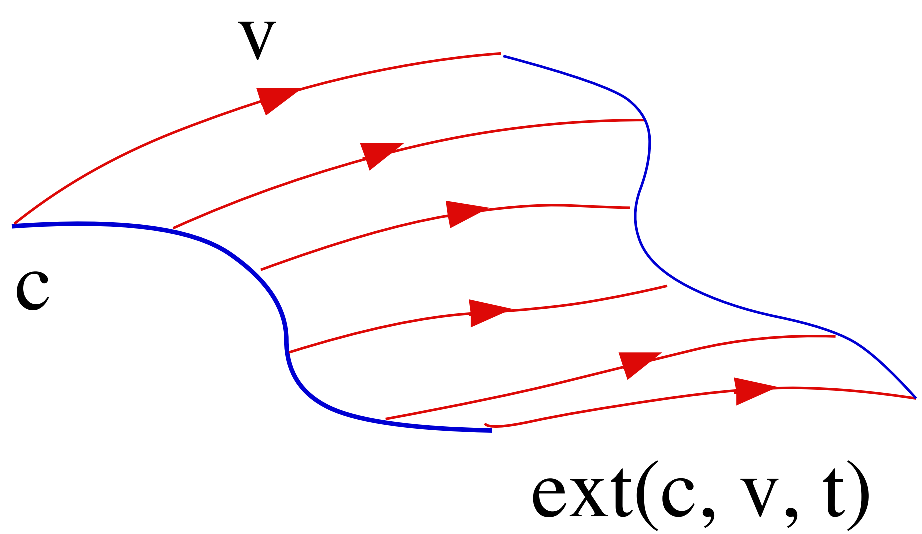
Extrusion *(by the flow of a vector field* v):

- of a point:



$$\begin{cases} d_t u_t(x) = v(u_t(x)) \\ u_0(x) = x \end{cases}$$

- of a p-manifold:



Inner product:

$$\int_c i_v b = \lim_{t \to 0} \frac{1}{t} \int_{ext(c,\, v,\, t)} b$$

The Lorentz force

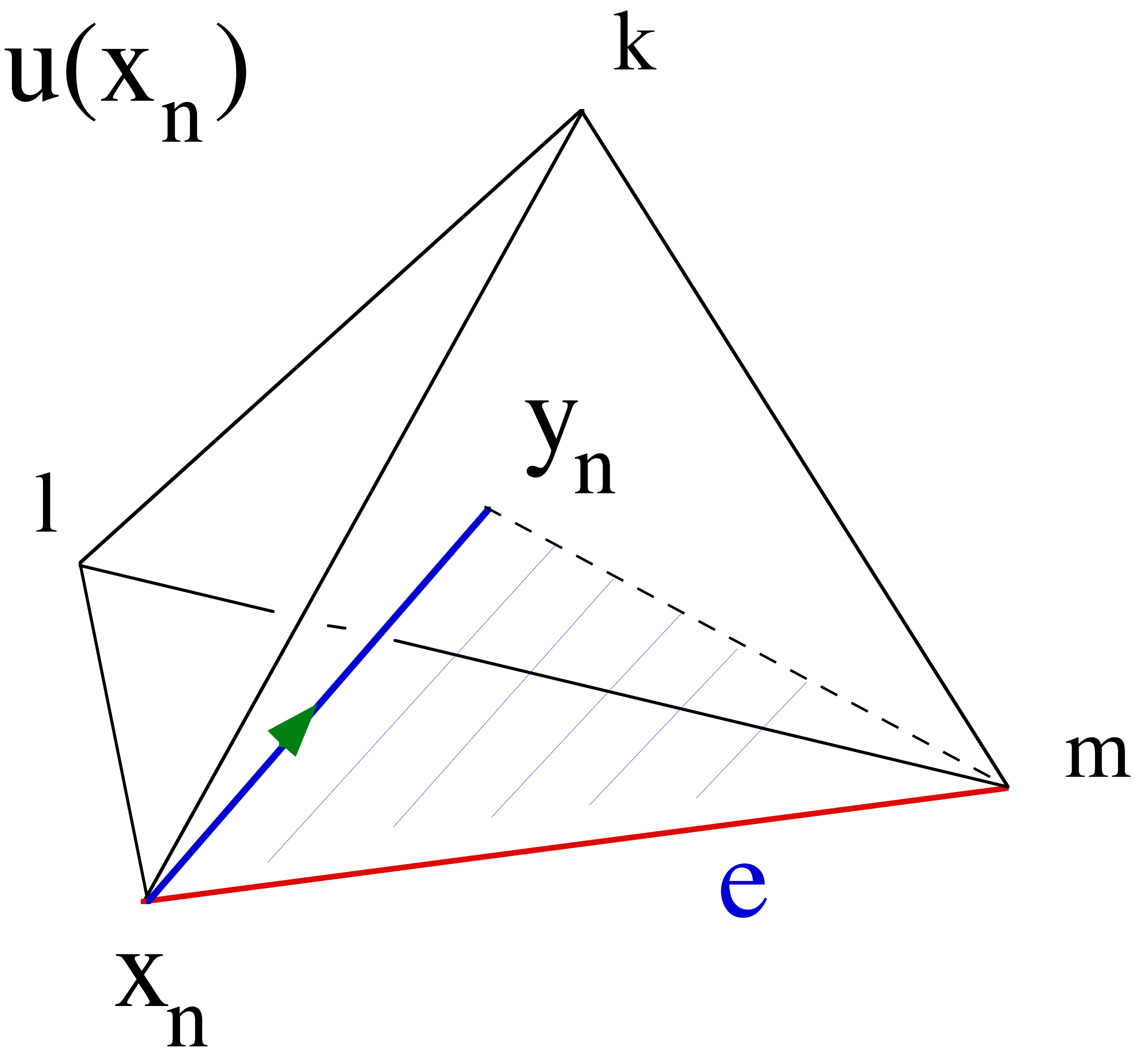
$v \times B$ proxy for $-i_v b$

(vector fields) (1-cochain)

$$\int_e i_v b \sim \int_{ext(e,\, v)} b$$

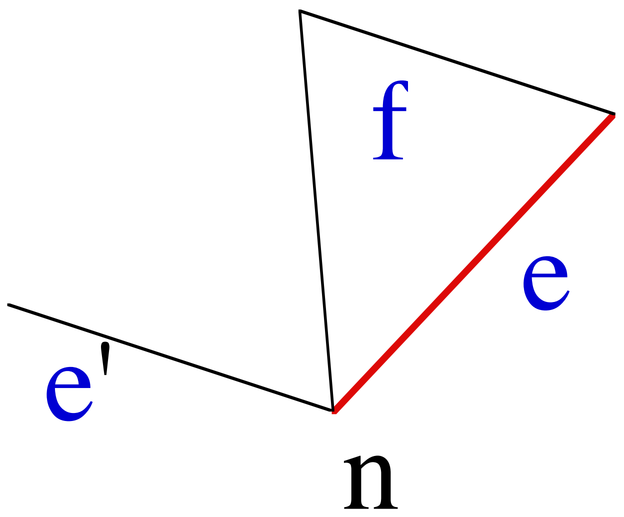
Extrusion of an edge, as a chain of facets?

$$y_n = x_n + u(x_n)$$



$$\mathrm{ext}(e, u) \approx \lambda^k(y_n)\ nmk + \lambda^l(y_n)\ nml$$

$I(e, e', f)$ = weight of facet f in extrusion of edge e by the field $\lambda^n e'$



$$v \approx \sum_n \lambda^n(x)\, \mathbf{v}_n = \sum_{n,\, e'} \lambda^n(x)\, v_n^{e'}\, e'$$

$$b = \sum_f \mathbf{b}_f\, w^f$$

$$(i_v b)_e = \sum_{e',\, f} I(e, e', f)\, \mathbf{b}_f\, v_n^{e'}$$

The Laplace force

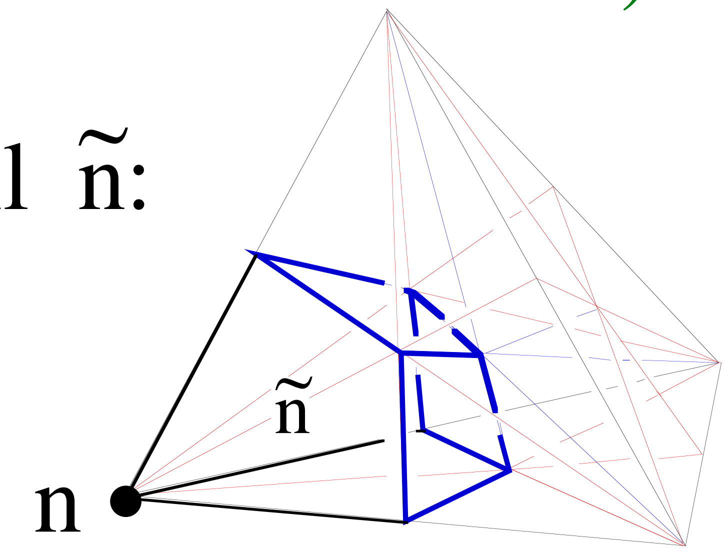
$\mathbf{J} \times \mathbf{B}$ proxy for $v \to i_v b \wedge j$

(vector field) (covector-valued twisted 3-form)

To be integrated over dual 3-cell $\tilde{n}$:

Similar to $\int e \wedge h$, but now

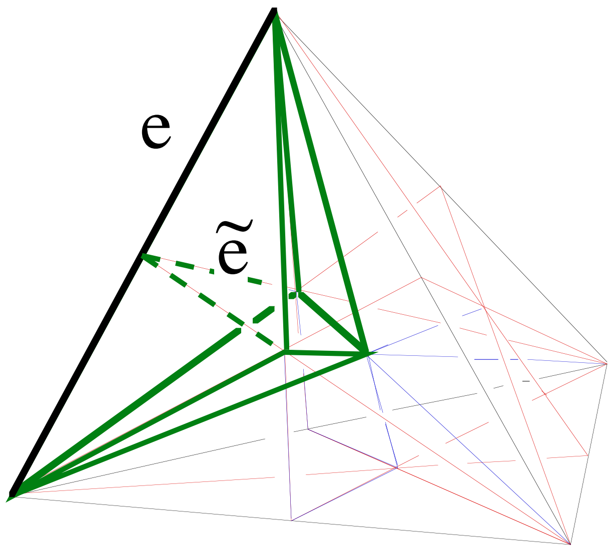
$1 \wedge \tilde{2}$ instead of $1 \wedge \tilde{1}$

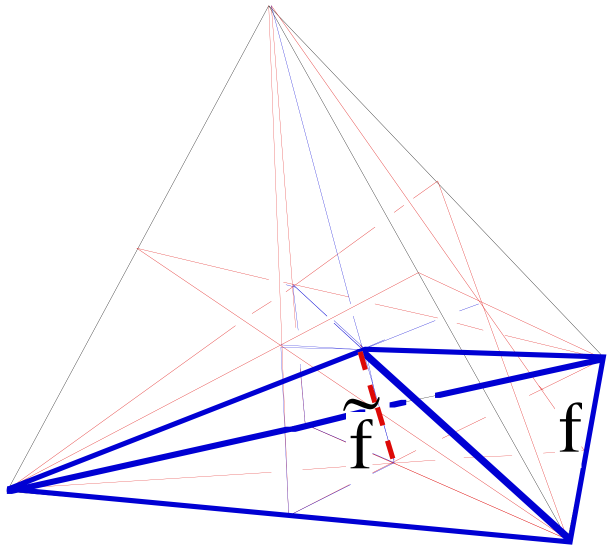


Then, covector $v \to \int_{\tilde{n}} i_v b \wedge j$ is force exerted on $\tilde{n}$

Electric energy, $\int_{\tilde{n}} e \wedge d$, treated like $\int_{\tilde{n}} i_v b \wedge j$

Energy





$$\sum_{e \in \mathcal{E}} e_e \, d_e$$

(electric)

$$\sum_{f \in \mathcal{F}} h_f \, b_f$$

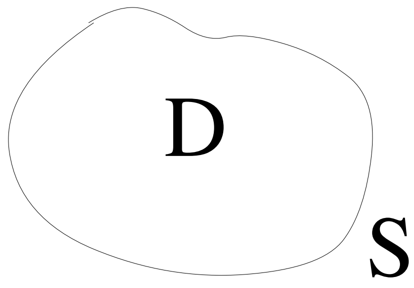
(magnetic)

The Maxwell "tensor"

Start from   wedge multiply by

$$-\partial_t \mathbf{d} + d\mathbf{h} = \mathbf{j} \qquad \wedge\ i_v \mathbf{b}$$

$$\partial_t \mathbf{b} + d\mathbf{e} = 0 \qquad \wedge\ i_v \mathbf{d}$$



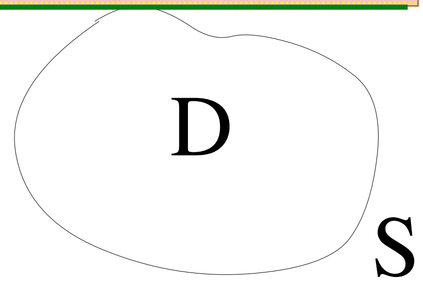
add, integrate over D, use $\mathbf{q} = d\mathbf{d}$,

set $\boxed{\mathbf{f} = i_v \mathbf{q} \wedge \mathbf{e} + i_v \mathbf{b} \wedge \mathbf{j}}$ (force *density*, covector-valued twisted 3-form)

find eventually that $\int_D \mathbf{f}$ is equal to

$$\underbrace{\partial_t [\int_D i_v \mathbf{d} \wedge \mathbf{b}]}_{\text{momentum}} + \int_S \underbrace{[i_v \mathbf{h} \wedge \mathbf{b} + i_v \mathbf{e} \wedge \mathbf{d} - \frac{1}{2} i_v(\mathbf{h} \wedge \mathbf{b} + \mathbf{e} \wedge \mathbf{d})]}_{\text{Maxwell (covector-valued, twisted) 2-form}}$$

The Maxwell "tensor"



$$\int_D f =$$

$$\partial_t \Big[\int_D i_v d \wedge b\Big] + \int_S \Big[i_v h \wedge b + i_v e \wedge d - \frac{1}{2}\, i_v (h \wedge b + e \wedge d)\Big]$$

momentum Maxwell (covector-valued, twisted) 2-form

$$\int_S \Big[i_v h \wedge b - \frac{1}{2}\, i_v (h \wedge b)\Big] = \int_S \Big[i_v b \wedge h + \frac{1}{2}\, i_v (h \wedge b)\Big]$$

treat like $e \wedge h$

extrude dual faces by v, use result about $h \wedge b$

Conclusion

- Object-oriented programming agenda
- Specific difficulty: infinite dimensional entities (fields) vs finite data structures
- Candidates to "object" status (mesh-related things) have been identified,
- and procedures that apply to them, described
- Discrete avatars of *geometrical* objects, for which traditional vector fields are only *proxies*