A Framework for Maxwell's Equations in Noninertial Frames Based on Differential Forms

Stefan Kurz
ETAS GmbH
stefan.kurz@gmx.de

Bernd Flemisch, Barbara Wohlmuth
Universität Stuttgart
flemisch@ians.uni-stuttgart.de

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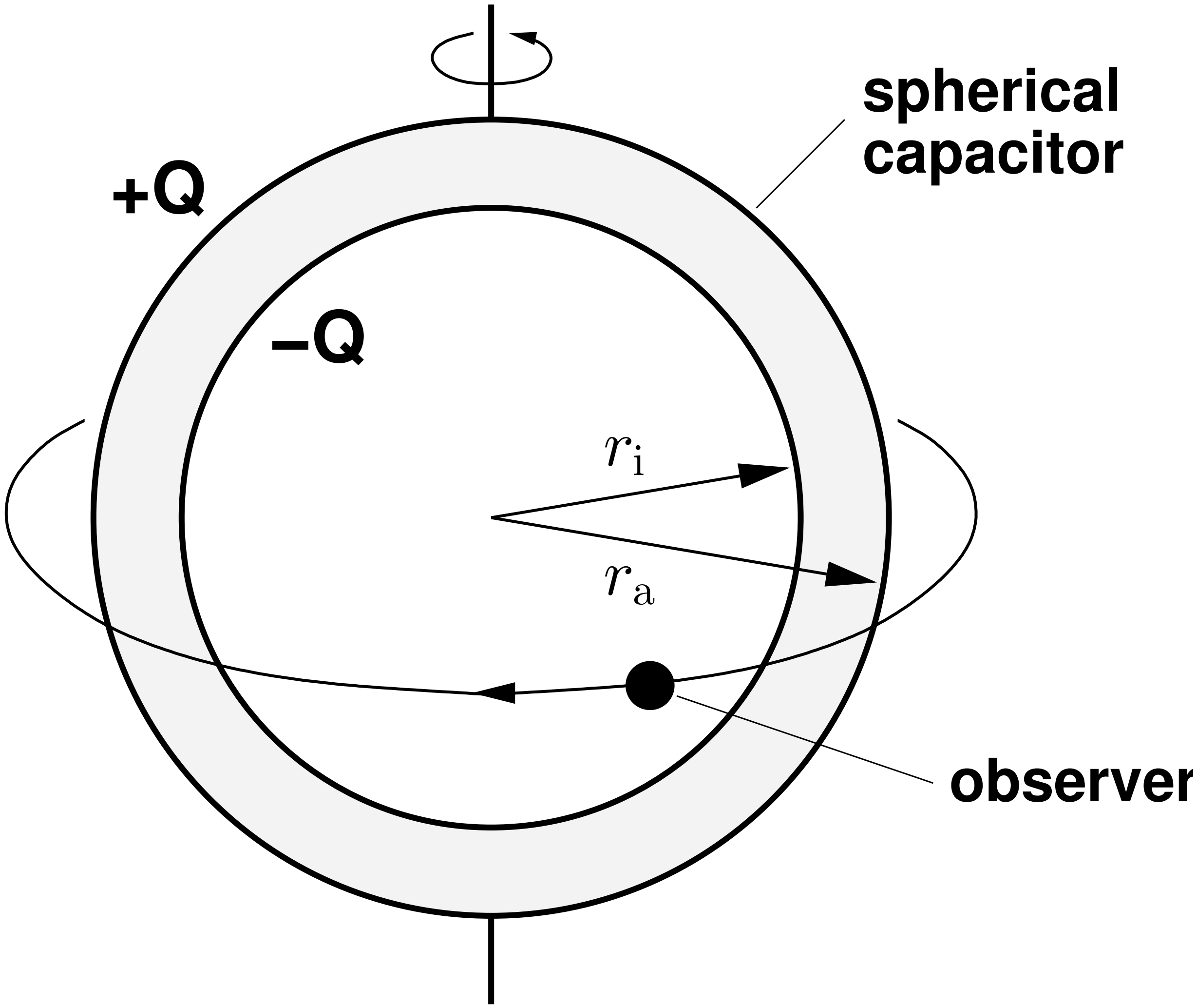
Outline

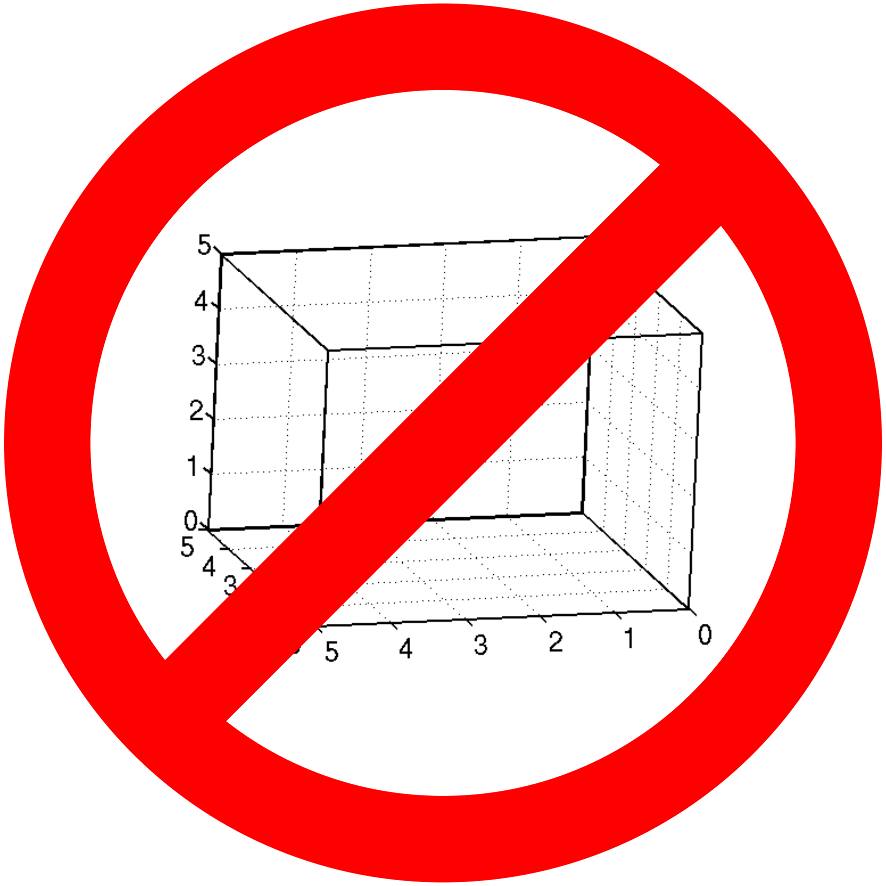
Schiff's Paradox

Leonard Isaac Schiff (1915 – 1971)



L.I. Schiff. A question in general relativity. *Proc. Nat. Acad. Sci. USA* **25**: 391-395, 1939.

“We may not enter”



NO coordinates

NO reference frames

Maxwell's Equations in 4D

$$\boxed{\mathrm{d}\,\underline{F} = 0, \qquad \mathrm{d}\,\underline{G} = \underline{\mathcal{J}}}$$

David van Dantzig

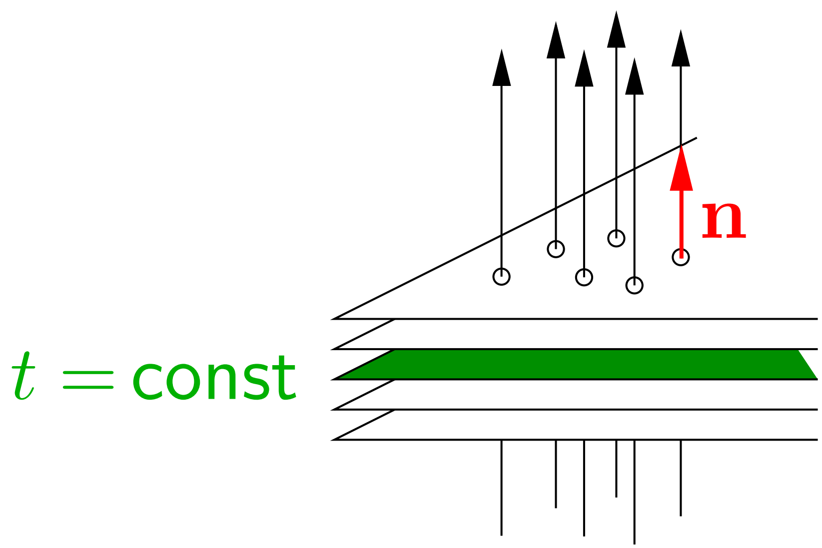
Differential form		Name	Unit
$\underline{F}$	2-form	Electromagnetic field	Vs
$\underline{G}$	2-form	Electromagnetic excitation	As
$\underline{\mathcal{J}}$	3-form	Four current density	As

- David van Dantzig: "The fundamental equations of electromagnetism, independent of metrical geometry", *Proc. Cambridge Phil. Soc. 30 (1934), 421*.
- E. Jan Post: "Formal Structure of Electromagnetics", *North-Holland 1962*.

Observer

- Relative space: Fibration of space-time manifold M by a three-parameter vector field $\mathbf{n}$.
- Relative time: Foliation of space-time manifold M by a one-parameter family of hypersurfaces $t =$ const., $\boldsymbol{\sigma} = \mathrm{d}\,t$.
- After scaling:

$$(\mathbf{n}, \boldsymbol{\sigma}), \quad \boldsymbol{\sigma} \big| \mathbf{n} = 1$$



The Decomposition Mechanism

- Given: observer $(\mathbf{n}, \boldsymbol{\sigma})$.
- Decomposition P only depends on $\mathbf{n}$ and $\boldsymbol{\sigma}$.
- $\mathcal{F}^p = \mathcal{F}^p(M)$ smooth p-forms on M,

$$\mathcal{F}^p_{\mathbf{n}} = \{\boldsymbol{\omega} \in \mathcal{F}^p : \mathbf{i}_{\mathbf{n}}\boldsymbol{\omega} = 0\}\,.$$

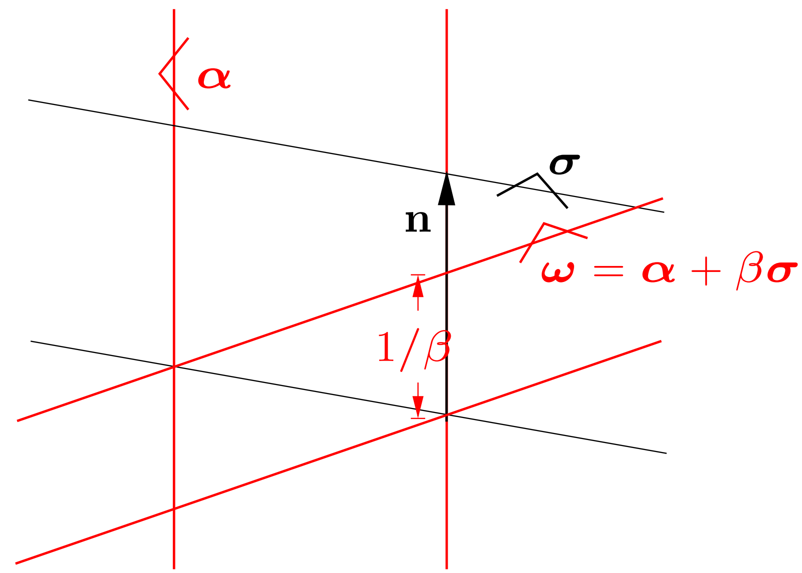
- Decomposition of a p-form by

$$P : \mathcal{F}^p \to \mathcal{F}^p_{\mathbf{n}} \times \mathcal{F}^{p-1}_{\mathbf{n}}, \quad \boldsymbol{\omega} \mapsto (\boldsymbol{\alpha}, \boldsymbol{\beta}),$$

$$\boldsymbol{\beta} = \mathbf{i}_{\mathbf{n}}\boldsymbol{\omega}, \quad \boldsymbol{\alpha} = \mathbf{i}_{\mathbf{n}}(\boldsymbol{\sigma} \wedge \boldsymbol{\omega}) = \boldsymbol{\omega} - \boldsymbol{\sigma} \wedge \boldsymbol{\beta},$$

$\boldsymbol{\sigma} \wedge \boldsymbol{\beta}$: longitudinal part, “time” piece
$\boldsymbol{\alpha}$: transversal part, “space” piece

Projection of a covector $\boldsymbol{\omega}$



(F. Hehl, Y. Obukhov. *Foundations of Classical Electromagnetism*. Birkhäuser, Boston, 2003.)

Projection of the Exterior Derivative d

- Temporal derivative $^{\bullet} = \mathcal{L}_{\mathbf{n}} = \mathbf{i}_{\mathbf{n}} \circ \mathrm{d} + \mathrm{d} \circ \mathbf{i}_{\mathbf{n}}$.
- Define spatial derivative d_3 by $P \circ \mathrm{d} = \begin{pmatrix} \bullet & \mathrm{d}_3 \\ & -\mathrm{d} \circ \mathbf{i}_{\mathbf{n}} \end{pmatrix}$.
- For the composition $P \circ \mathrm{d} \circ P^{-1}$, it holds that

$$P \circ \mathrm{d} \circ P^{-1} \begin{pmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \end{pmatrix} = P \circ \mathrm{d}(\boldsymbol{\alpha} + \boldsymbol{\sigma} \wedge \boldsymbol{\beta}) = \begin{pmatrix} \bullet & \mathrm{d}_3 \\ & -\mathrm{d} \circ \mathbf{i}_{\mathbf{n}} \end{pmatrix} (\boldsymbol{\alpha} + \boldsymbol{\sigma} \wedge \boldsymbol{\beta})$$

$$= \ldots = \begin{pmatrix} \mathrm{d}_3 & \mathrm{d}_3 \boldsymbol{\sigma} \wedge \\ \bullet & -\mathrm{d}_3 + \dot{\boldsymbol{\sigma}} \wedge \end{pmatrix} \begin{pmatrix} \boldsymbol{\alpha} \\ \boldsymbol{\beta} \end{pmatrix}.$$

- Here: $\mathrm{d}\boldsymbol{\sigma} = \mathrm{d}\,\mathrm{d}t = 0 \;\Rightarrow\; \mathrm{d}_3 \boldsymbol{\sigma} = \dot{\boldsymbol{\sigma}} = 0$,

$$\boxed{P \circ \mathrm{d} \circ P^{-1} = \begin{pmatrix} \mathrm{d}_3 & 0 \\ \bullet & -\mathrm{d}_3 \end{pmatrix}}$$

Decomposition of Maxwell's Equations

- Decompose 4D quantities and define 3D components:

$$\begin{pmatrix} \underline{D} \\ \underline{H}/c_0 \end{pmatrix} = P\underline{G}, \qquad \begin{pmatrix} \underline{B} \\ -\underline{E}/c_0 \end{pmatrix} = P\underline{F}, \qquad \begin{pmatrix} \underline{\rho} \\ -\underline{J}/c_0 \end{pmatrix} = P\underline{\mathcal{J}}.$$

- Project 4D equations:

$$P\,\mathrm{d}\,P^{-1}P\underline{F} = 0, \qquad P\,\mathrm{d}\,P^{-1}P\underline{G} = P\underline{\mathcal{J}}.$$

- General (3+1) Maxwell's equations

$$\mathrm{d}_3\underline{H} = \underline{J} + \dot{\underline{D}} + \dot{\boldsymbol{\sigma}} \wedge \underline{H}, \qquad \mathrm{d}_3\underline{B} = 0 + \mathrm{d}_3\boldsymbol{\sigma} \wedge \underline{E},$$

$$\mathrm{d}_3\underline{E} = -\dot{\underline{B}} \quad + \dot{\boldsymbol{\sigma}} \wedge \underline{E}, \qquad \mathrm{d}_3\underline{D} = \underline{\rho} - \mathrm{d}_3\boldsymbol{\sigma} \wedge \underline{H}.$$

- Here: $\mathrm{d}\boldsymbol{\sigma} = \mathrm{d}\,\mathrm{d}t = 0$ (holonomic observer) $\Rightarrow$ red terms vanish!

The Constitutive Relations

- To get a particular electromagnetic theory out of Maxwell's equations, one must adopt constitutive relations that determine $\underline{G}$ in terms of $\underline{F}$.
- It is here that the metric structure of space-time enters the story.
- 4-metric induces an isomorphism $g : \mathcal{X}^p \to \mathcal{F}^p$,

$$\|\boldsymbol{\omega}\|^2 = \left|\, \boldsymbol{\omega} \middle| g^{-1}(\boldsymbol{\omega}) \,\right|, \quad \text{“extent of } \boldsymbol{\omega}\text{”}.$$

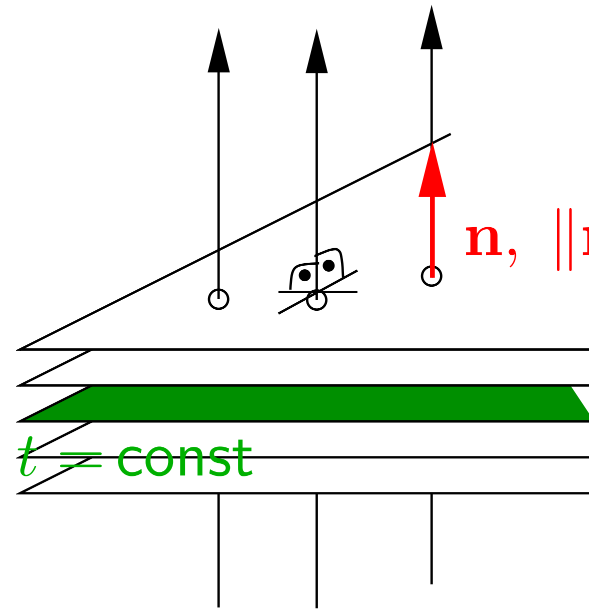
- Volume form $\boldsymbol{\Omega} \in \mathcal{F}^4$, $\|\boldsymbol{\Omega}\| = 1$. Hodge $*$: $\mathcal{F}^p \to \mathcal{F}^{4-p}$,

$$\boxed{*\,\boldsymbol{\omega} = -\boldsymbol{\Omega} \big| g^{-1}(\boldsymbol{\omega})\ .}$$

- Set $Z_0 = \sqrt{\mu_0/\varepsilon_0}$. In vacuum: $\boxed{\underline{G} = Z_0^{-1} * \underline{F}}$

Inertial Observer

- Intertial observer: Choice of a
  - fibration consisting of parallel time-like lines with
  - orthogonal space-like hyperplanes as foliation.
- Let $\|\mathbf{n}\| = 1$: $\mathbf{n}$ can be seen as four-velocity.



- Consequences:
  - $\boldsymbol{\sigma}\big|\mathbf{n} = 1$, $\boldsymbol{\sigma} = \lambda g(\mathbf{n})$, $\|\mathbf{n}\| = 1 \ \Rightarrow\ \lambda = 1$, $\boldsymbol{\sigma} = g(\mathbf{n})$.
  - Inertial observer defined by constant four-velocity $\mathbf{u}$.
  - Locally inertial observer defined by (smooth) four-velocity $\mathbf{u}$ defines measurable field quantities according to the hypothesis of locality.

B. Mashhoon: The hypothesis of locality in relativistic physics. *Physics Letters A* **145**: 147–153, 1990.

Decomposition of the Hodge $*$

- Let g_3^{-1} be the transversal part of the isomorphism g^{-1} (up to sign).
- For the 3-metric g_3, the induced Hodge $*_3$ can be shown to be

$$*_3 = \|\mathbf{n}\|^{-1}\mathbf{i}_{\mathbf{n}}* \ .$$

- Define constitutive parameters by $(\mathbf{w}, \lambda) = Pg^{-1}(\boldsymbol{\sigma})$.
- The Hodge $*$ decomposes to

$$P * P^{-1}\begin{pmatrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{pmatrix} = \|\mathbf{n}\|\begin{pmatrix} - *_3 \mathbf{i}_{\mathbf{w}} s & \lambda *_3 s - \mathbf{i}_{\mathbf{w}} *_3 \mathbf{i}_{\mathbf{w}} \\ *_3 & -\mathbf{i}_{\mathbf{w}} *_3 s \end{pmatrix}\begin{pmatrix}\boldsymbol{\alpha}\\ \boldsymbol{\beta}\end{pmatrix}.$$

- Locally inertial observer: $\mathbf{w} = 0,\ \lambda = \|\mathbf{n}\| = 1,$

$$\boxed{P * P^{-1} = \begin{pmatrix} 0 & *_3 s \\ *_3 & 0 \end{pmatrix}.}$$

Maxwell's Equations and Material Laws in (3+1)D

- For a locally inertial observer in vacuum,

  we obtain from $\underline{G} = Z_0^{-1} * \underline{F}$

$$P\underline{G} = Z_0^{-1} P * P^{-1} P \underline{F}$$

$$\Rightarrow \begin{pmatrix} \underline{D} \\ \underline{H}/c_0 \end{pmatrix} = \frac{1}{Z_0} \begin{pmatrix} 0 & *_3 s \\ *_3 & 0 \end{pmatrix} \begin{pmatrix} \underline{B} \\ -\underline{E}/c_0 \end{pmatrix}.$$

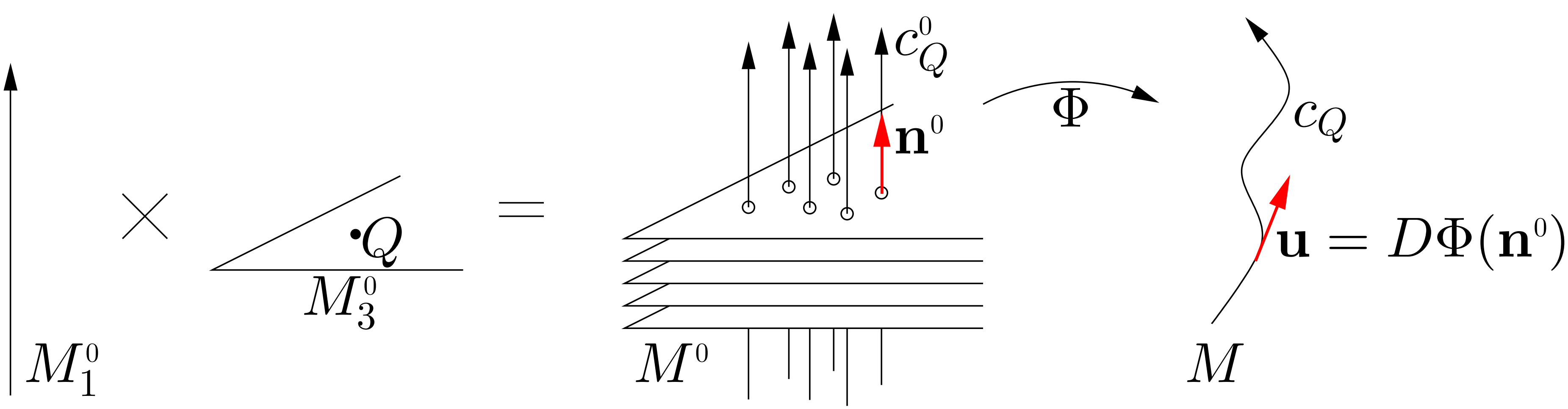
- Canonical form

$$\underline{D} = \varepsilon_0 *_3 \underline{E}$$

$$\underline{H} = \mu_0^{-1} *_3 \underline{B}.$$

The Reference Space M^0

- Lagrangian observer describes the events from a reference space $M^0 = M_1^0 \times M_3^0$,
  - M_1^0: one-dimensional oriented affine space,
  - configuration space M_3^0: three-dimensional oriented affine space.
- Consider a point $Q \in M_3^0$: $\rightarrow$ curve $c_Q^0 = M_1^0 \times Q \subset M^0$.
- Placement mapping: diffeomorphism $\Phi : M^0 \rightarrow M$ such that $c_Q = \Phi(c_Q^0) \subset M$ is the worldline of Q.
- Parametrize the curves c_Q^0 by arc-length w.r.t. the pulled-back metric g^0.



4D Lagrangian Electrodynamics

- Reformulation of Maxwell's equations and material laws in M^0.
- Pull-back via Φ of the involved field quantities and operators:

$$\underline{F}^0 = \Phi^* \underline{F}, \quad \underline{G}^0 = \Phi^* \underline{G}, \quad \underline{\mathcal{J}}^0 = \Phi^* \underline{\mathcal{J}}.$$

- 4D Lagrangian description:

$$\boxed{\mathrm{d}^0 \underline{F}^0 = 0, \qquad \mathrm{d}^0 \underline{G}^0 = \underline{\mathcal{J}}^0}$$

- In vacuum:

$$\boxed{\underline{G}^0 = Z_0^{-1} *^0 \underline{F}^0}$$

- Hodge $*^0$ taken with respect to pulled-back metric g^0.

Introduction of two observers

- Foliation: Observer $(\mathbf{n}^0, \boldsymbol{\sigma}^0)$, $\boldsymbol{\sigma}^0 = \mathrm{d}\,t^0$.
- Holonomic by construction, $\mathrm{d}\,\boldsymbol{\sigma} = 0$: Maxwell's equations retain simple form.
- In general not locally inertial: Constitutive relations become involved.

- Metric: Observer $(\mathbf{n}', \boldsymbol{\sigma}') = (\mathbf{n}^0, g(\mathbf{n}^0))$.
- In general anholonomic: Maxwell's equations become involved.
- Locally inertial by construction: Constitut. relations retain simple form.
- Push-forward by $D\Phi$ yields locally inertial frame in M, in which the considered material element is instantaneously at rest.
- Most simple form of constitutive relations also for more complex media.

- Idea: Always use *simple* relations by employing *both* observers simultaneously!

Q: How can we connect the two observers?

- Answer:

$$\boldsymbol{\omega} = (P^0)^{-1} \begin{pmatrix} \boldsymbol{\alpha}^0 \\ \boldsymbol{\beta}^0 \end{pmatrix} = (P')^{-1} \begin{pmatrix} \boldsymbol{\alpha}' \\ \boldsymbol{\beta}' \end{pmatrix}$$

- This yields

$$\begin{pmatrix} \boldsymbol{\alpha}^0 \\ \boldsymbol{\beta}^0 \end{pmatrix} = P^0 (P')^{-1} \begin{pmatrix} \boldsymbol{\alpha}' \\ \boldsymbol{\beta}' \end{pmatrix}$$

$$= \ldots$$

$$= \underbrace{\begin{pmatrix} 1 & (\boldsymbol{\sigma}' - \boldsymbol{\sigma}^0)\wedge \\ 0 & 1 \end{pmatrix}}_{\Psi} \begin{pmatrix} \boldsymbol{\alpha}' \\ \boldsymbol{\beta}' \end{pmatrix}$$

Consistent Lagrangian (3+1) Framework

Maxwell's Equations	Transformation Ψ	Constitutive Laws
$\mathrm{d}^0 \underline{E}^0 = -\dot{\underline{B}}^0$ $\mathrm{d}^0 \underline{B}^0 = 0$	$\underline{B}^0 = \underline{B}' - (\boldsymbol{\sigma}' - \boldsymbol{\sigma}^0) \wedge \underline{E}'/c_0$ $\underline{E}^0 = \underline{E}'$	$\underline{D}' = *_3^0 \varepsilon \underline{E}'$
$\mathrm{d}^0 \underline{H}^0 = \underline{J}^0 + \dot{\underline{D}}^0$ $\mathrm{d}^0 \underline{D}^0 = \underline{\rho}^0$	$\underline{D}^0 = \underline{D}' + (\boldsymbol{\sigma}' - \boldsymbol{\sigma}^0) \wedge \underline{H}'/c_0$ $\underline{H}^0 = \underline{H}'$	$\underline{B}' = *_3^0 \mu \underline{H}'$

1. Possibility: Elimination of the primed quantities.
   J. Van Bladel. *Relativity and Engineering.* Springer-Verlag, Berlin, 1984.

2. Possibility: Elimination of the quantities with index ^{0}.
   T.C. Mo. Theory of electrodynamics in media in noninertial frames and applications. *Journal of Mathematical Physics,* **11**(8): 2589-2610, August 1970.

Schiff's Paradox: Simplified Setup

- **Case 1:** $\Omega = 0, \quad \omega = \omega_0.$

  Convection current $\longrightarrow$ electromagnetic field in the interior of the cylinder,

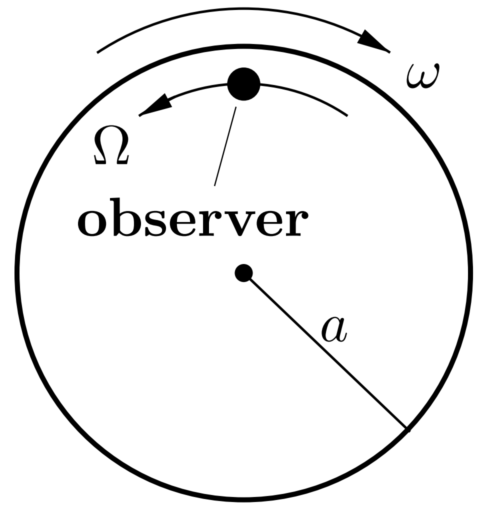
$$D_r = 0, \quad H_z = -\frac{\omega q}{2\pi}.$$

- **Case 2:** $\Omega = \omega_0, \quad \omega = 0.$

  Interior free of an electromagnetic field, valid for any observer. Observer should find

$$D'_r = H'_z = 0.$$

- Observer:
  Both cases are kinematically identical!

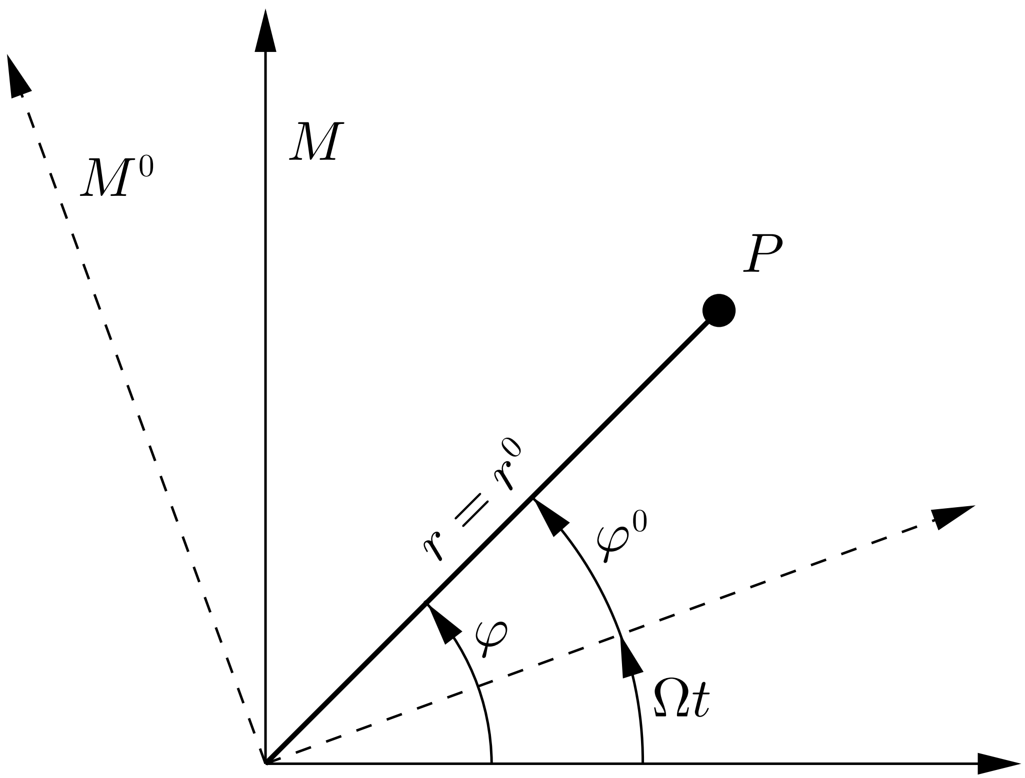


Homogeneously charged cylinder, radius a, charge per unit length q.

$$\rho = \frac{q}{2\pi a}\,\delta(r-a)$$

$$j_\varphi = -\frac{\omega q}{2\pi}\delta(r-a)$$

Schiff's Paradox: rotating coordinates



Placement mapping

$$\Phi : \quad \begin{pmatrix} t \\ r \\ \varphi \\ z \end{pmatrix} = \begin{pmatrix} \gamma t^0 \\ r^0 \\ \varphi^0 + \gamma \Omega t^0 \\ z^0 \end{pmatrix},$$

$$\gamma = (1 - (\Omega r^0 / c_0)^2)^{-1/2}.$$

Natural coordinate frames for M (analogously for M^0):

$$\left(\frac{1}{c_0}\partial t,\ \partial r,\ \partial \varphi,\ \partial z\right),$$

$$(c_0 \,\mathrm{d}t,\ \mathrm{d}r,\ \mathrm{d}\varphi,\ \mathrm{d}z).$$

Frames related by $D\Phi$, i.e.,

$$\begin{aligned} \mathrm{d}t &= \gamma \,\mathrm{d}t^0, \\ \mathrm{d}\varphi &= \mathrm{d}\varphi^0 + \gamma\Omega \,\mathrm{d}t^0, \\ \partial t &= \gamma^{-1}\partial t^0 - \Omega \partial \varphi^0. \end{aligned}$$

Schiff's Paradox: Maxwell's Equations in M^0

- Differential forms for the sources ρ, j_φ

$$\underline{\rho} = \frac{q}{2\pi}\delta(r-a)\,\mathrm{d}\,r \wedge \mathrm{d}\,\varphi \wedge \mathrm{d}\,z, \qquad \underline{J} = 0, \quad \text{w.r.t. } (c_0^{-1}\partial t,\, c_0\,\mathrm{d}\,t).$$

- $\underline{\mathcal{J}} = P^{-1}(\underline{\rho}, -\underline{J}/c_0) = \underline{\rho}$. Pull-back

$$\underline{\mathcal{J}}^0 = \frac{q}{2\pi}\delta(r^0 - a)\,\mathrm{d}\,r^0 \wedge (\mathrm{d}\,\varphi^0 + \gamma\Omega\,\mathrm{d}\,t^0) \wedge \mathrm{d}\,z^0.$$

- Decomposition w.r.t. P^0:

$$\begin{pmatrix} \underline{\rho}^0 \\ -\underline{J}^0/c_0 \end{pmatrix} = P^0\underline{\mathcal{J}}^0 = \frac{q}{2\pi}\delta(r^0-a)\begin{pmatrix} \mathrm{d}\,r^0 \wedge \mathrm{d}\,\varphi^0 \wedge \mathrm{d}\,z^0 \\ \gamma\Omega c_0^{-1}\,\mathrm{d}\,z^0 \wedge \mathrm{d}\,r^0 \end{pmatrix}.$$

- Maxwell's equations reduce to

$$\frac{\mathrm{d}}{\mathrm{d}r^0}D_r^0 = \rho^0 = \frac{q}{2\pi}\quad\delta(r^0-a) \quad\rightarrow\quad D_r^0 = \frac{q}{2\pi}\begin{cases} c_1 & r^0 < a, \\ c_1 + 1 & r^0 > a. \end{cases}$$

$$\frac{\mathrm{d}}{\mathrm{d}r^0}H_z^0 = -j_\varphi^0 = \frac{\gamma\Omega q}{2\pi}\,\delta(r^0-a) \quad\rightarrow\quad H_z^0 = \frac{q}{2\pi}\begin{cases} c_2 & r^0 < a, \\ c_2 + \gamma\Omega & r^0 > a. \end{cases}$$

Schiff's Paradox: Transformation to Inertial Observer

- Transformation of D_r^0 to the observable quantity D_r' by means of Ψ

$$D_r' \;=\; D_r^0 + \gamma\frac{\Omega (r^0)^2}{c_0^2} H_z^0 \;=\; \frac{q}{2\pi}\begin{cases} c_1 + \dfrac{c_2 \gamma \Omega (r^0)^2}{c_0^2} & r^0 < a, \\ c_1 + \dfrac{c_2 \gamma \Omega (r^0)^2}{c_0^2} + \gamma^2 & r^0 > a. \end{cases}$$

- Integration constants:

  D_r' bounded for $r^0 \to \infty \;\to\; c_2 = 0.$

  $E_r' = D_r'/(\varepsilon \gamma r^0)$ bounded for $r^0 \to 0 \;\to\; c_1 = 0.$

  $\to$ Interior is free of an electromagnetic field, the paradox is resolved!

- Solution

$$\underline{D}' = \frac{\gamma^2 q}{2\pi}\,\mathrm{d}\,\varphi^0 \wedge \mathrm{d}\,z^0 \begin{cases} 0 & r^0 < a \\ 1 & r^0 > a \end{cases}, \qquad \underline{H}' = H_z^0 = \frac{\gamma q}{2\pi}\,\mathrm{d}\,z^0 \begin{cases} 0 & r^0 < a \\ \Omega & r^0 > a \end{cases}.$$

Other Applications

- Constitutive laws for moving bodies (experiments by Wilson and Röntgen/Eichwald).
- Lorentz boost $B(\mathbf{u}, \mathbf{u}')$ for differential forms: Identify physically equivalent fields.
- Standard Lorentz transformation for differential forms by $P \circ B(\mathbf{u}', \mathbf{u}) \circ P^{-1}$.
- Extended Galilean relativity for eddy current problems: all rigid observers are equivalent.
- Apply decomposition repeatedly: (1+(1+2))-decomposition of fields in waveguides.

Summary

- Language of differential forms, no coordinates, no frames.
- Maxwell's equations and constitutive relations in 4D.
- Observers $(\mathbf{n}, \boldsymbol{\sigma})$ and related decompositions.
  - Holonomic observers $(\mathrm{d}\,\boldsymbol{\sigma} = 0)$ yield simple Maxwell's equations.
  - Locally inertial observers $(\boldsymbol{\sigma} = g(\mathbf{n}))$ yield simple constitutive relations.
- Lagrangian framework encompasses both kinds of observers. Simplified Schiff's paradox as example.