# Geometry of Electromagnetism and its Implications in Field and Wave Analysis

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## Background

What does *geometry of electromagnetism* mean precisely?

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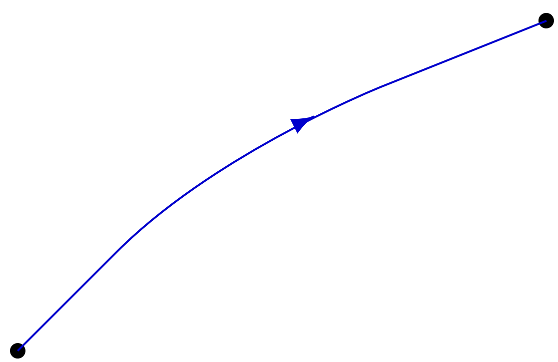
*If the geometry of electromagnetism is an answer, what is the question?*

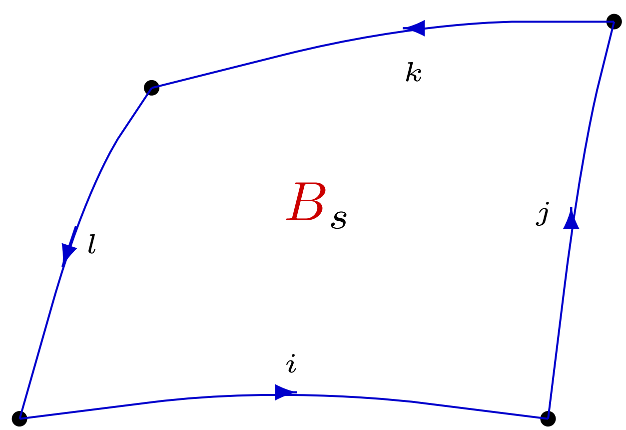
It seems the term is commonly employed to point to two ideas:

i) The degrees of freedoms are related to edges, faces and volumes.

ii) Stokes's theorem gives a meaning to "discrete differentiation".



$$E_i = \int_{\text{edge}_i} e$$




$$-\partial_t B_s = E_i + E_j + E_k + E_l$$

$$\mathbf{D}E = -\partial_t B$$

As a result one ends up with a primal-dual mesh pair, and a "discrete countepart" of electromagnetic equations:

$$\begin{aligned} \mathbf{D}E &= -\partial_t B\,, \\ \mathbf{D}H &= -\partial_t D + J\,, \\ \mathbf{D}D &= Q\,, \\ \mathbf{D}B &= 0\,, \end{aligned}$$

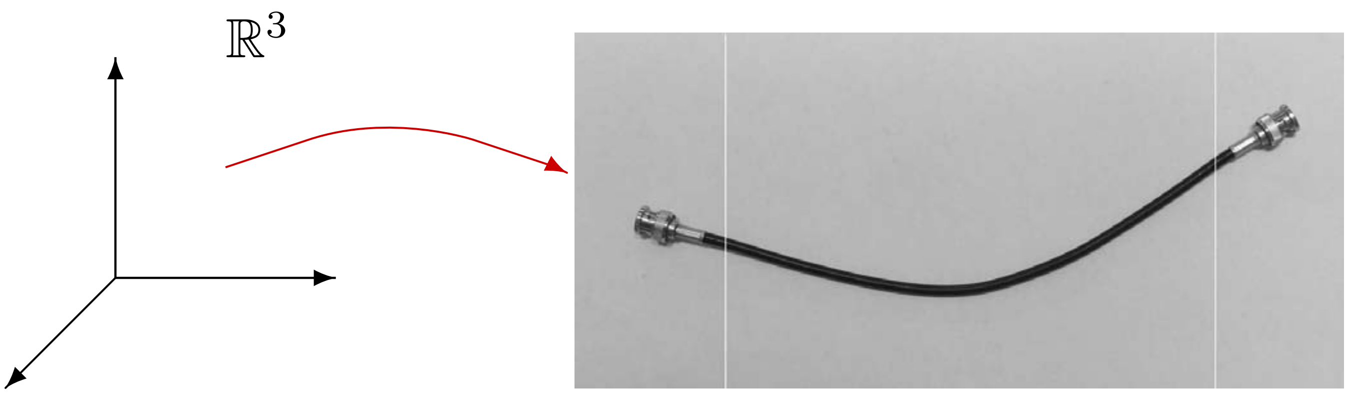
$$\begin{aligned} D &= \mathbf{M}_\epsilon E \\ J &= \mathbf{M}_\sigma E + J_s \\ B &= \mathbf{M}_\mu H \end{aligned}$$

# Manifolds

A *manifold* is

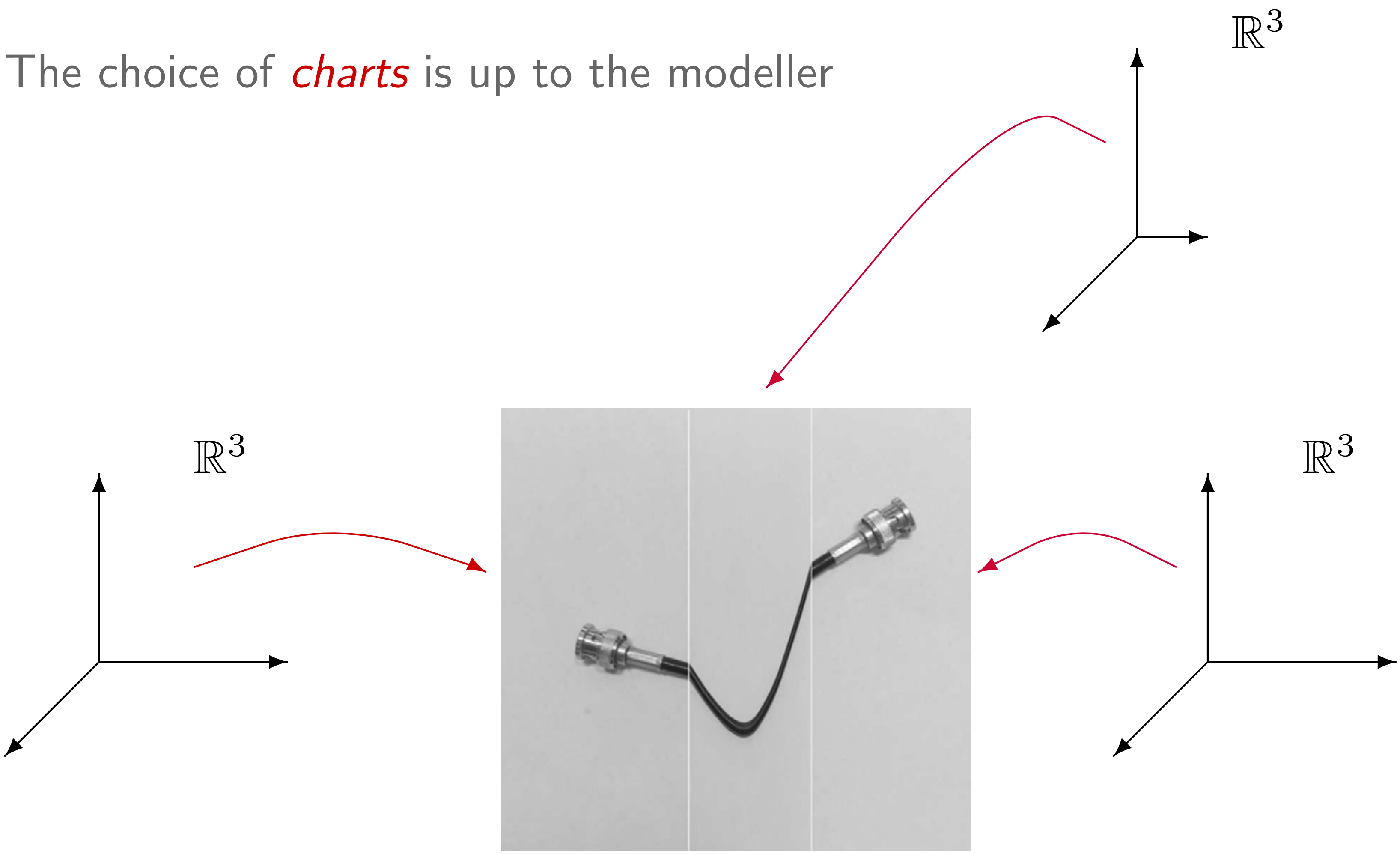
i) a set $X$ and

ii) a collection of mappings from $\mathbb{R}^n$ to $X$.




point set $X$

The choice of *charts* is up to the modeller



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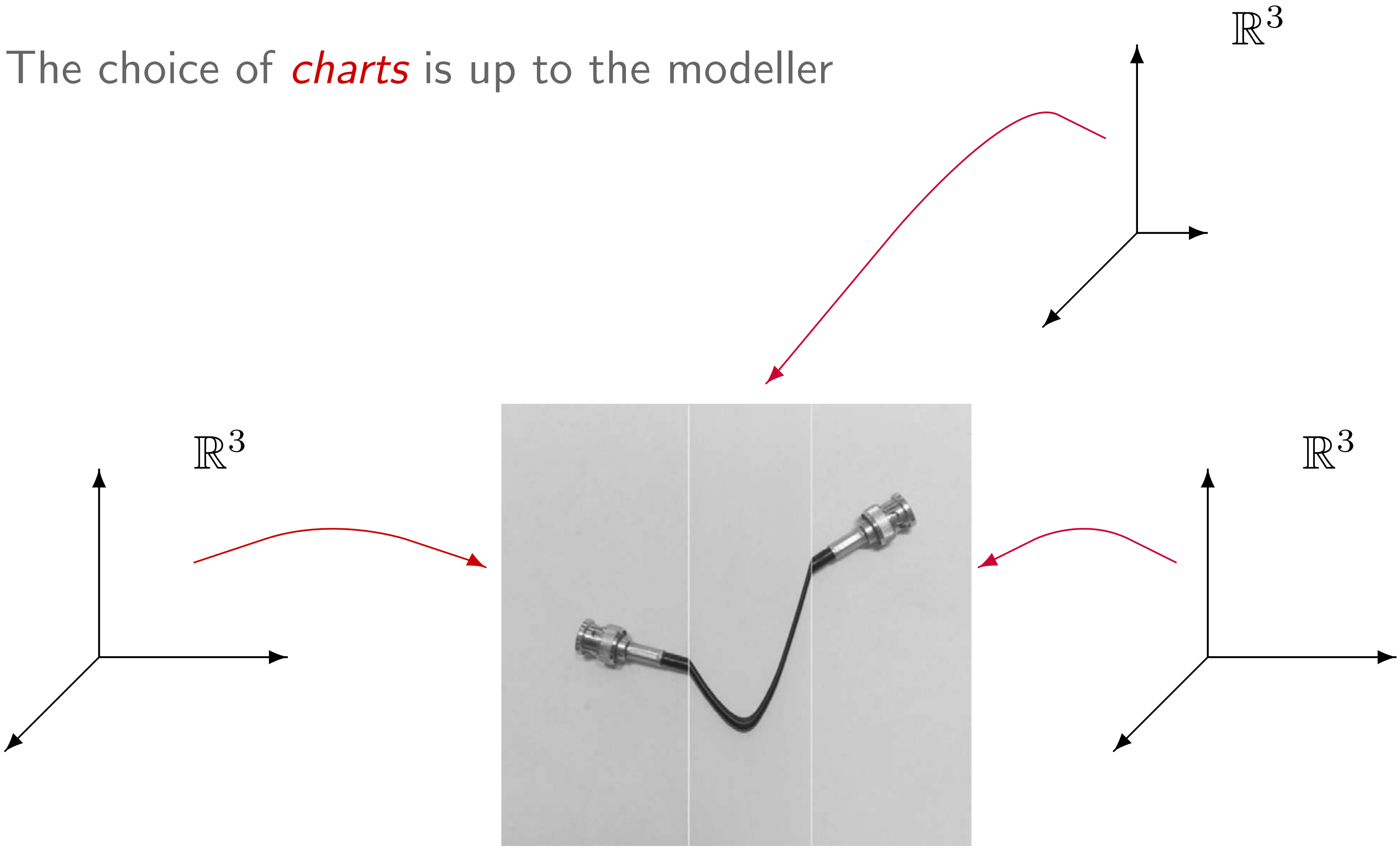
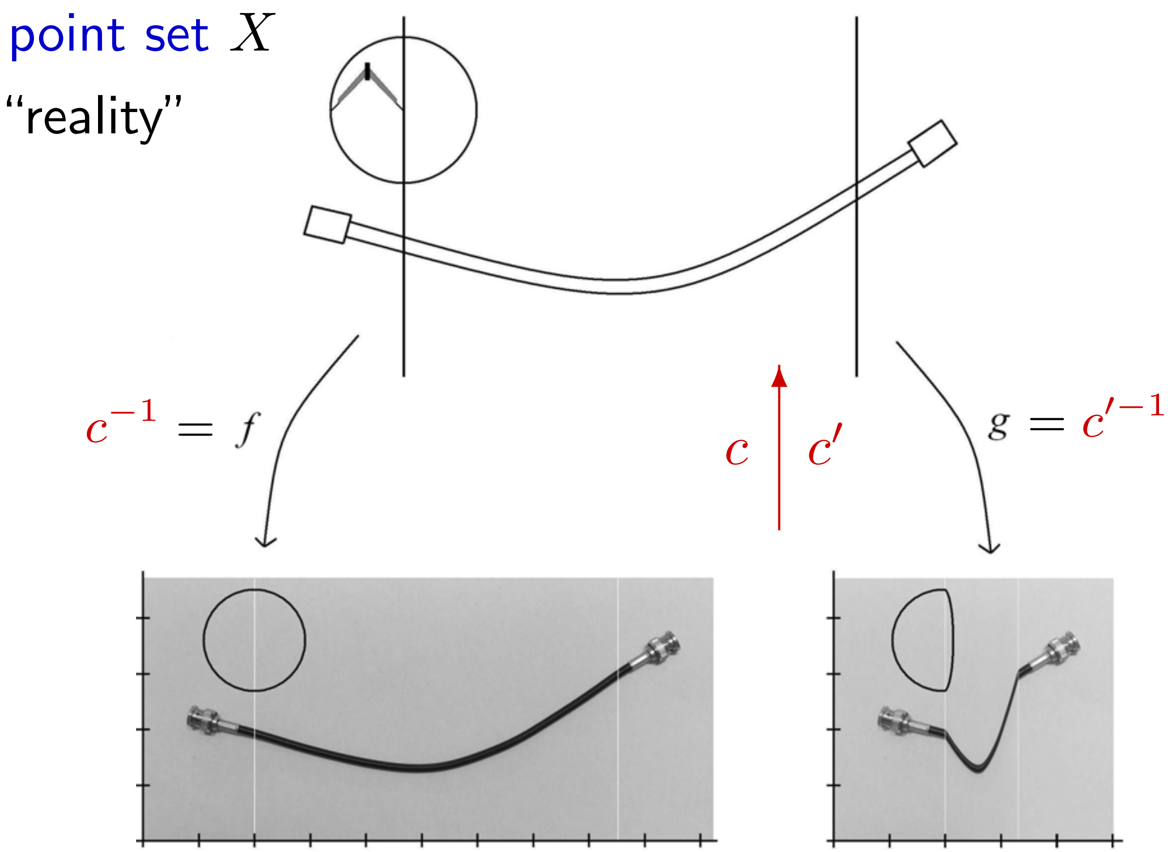



*It's still a collection of charts to the* same *point set* $X$

So, let's put it this way:

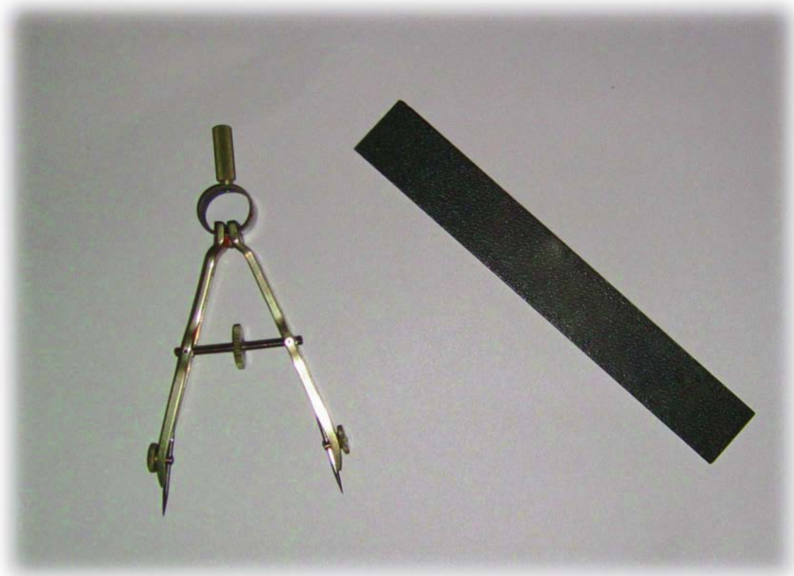
The two images point to the ***same object*** in reality. They are just two different parametrizations of one and the same point set.



# Euclidean geometry

For us here, the main idea behind Euclidean geometry is, that any domain of $\mathbb{R}^3$ can be parametrized by

i) a ruler (no grading is needed), and
ii, a pair of dividers



# 1 Euclidean geometry

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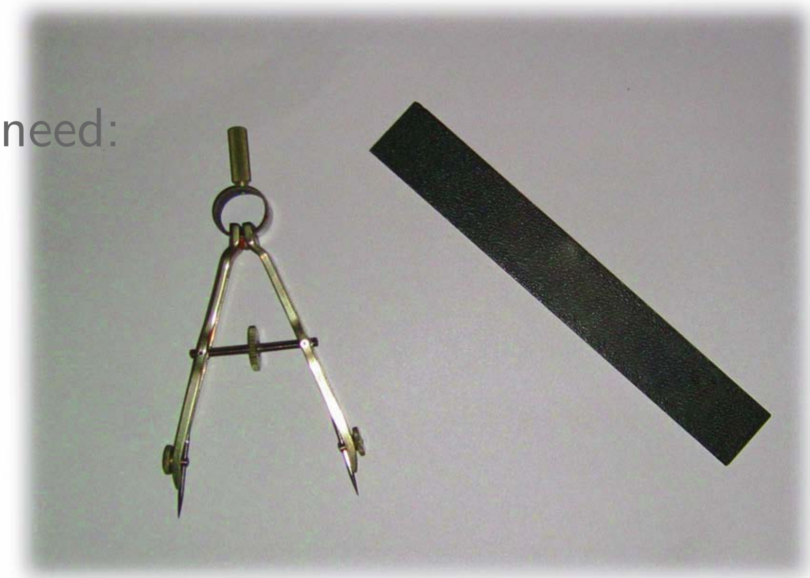
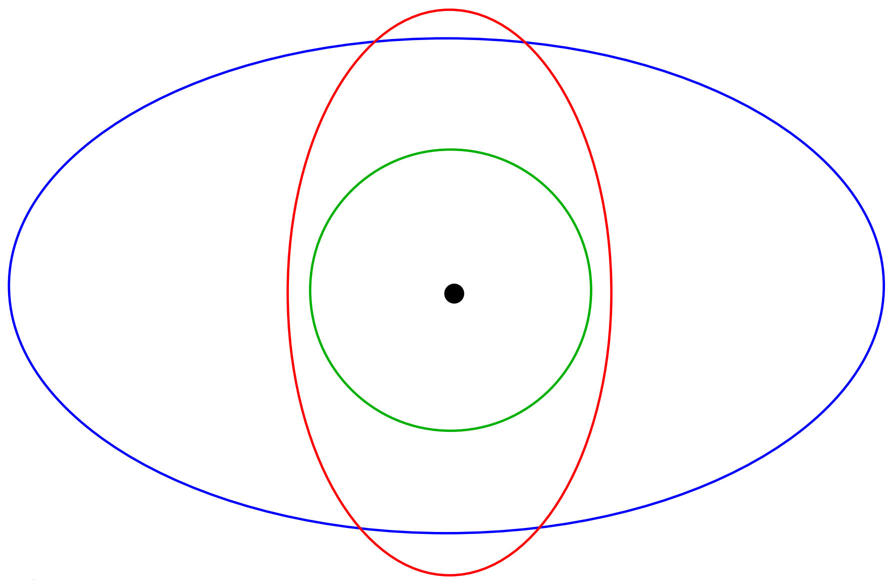
Compare, the models of physics need:
i) the notion of straight lines
ii) the abstraction of rigid body



Second, the pair of dividers enable us to recognize the most "symmetrical" barrels which generate norms that do not priviledge any direction.



Examples of barrels
with *maximal isotropy group*

A key point:

There is no absolute choice of the most symmetrical barrel.

The symmetry depends on what is called a pair of dividers, i.e., on what we decide to call a rigid body.

- This sounds rather preposterous;
  *One is not able to recognize a sphere from a set of ellipsoids?*

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  *One is not able to recognize a sphere from a set of ellipsoids?*

  *Spheres can be recognized, but there is no absolute sphere.*

Recognition of a sphere is up to what we've decided to call a rigid body.

Another key point:

Euclidean norms stem for a scalar product

All Euclidean structures are the same up to a linear transformation L:

$$u \cdot v = \mathrm{L}u' \cdot \mathrm{L}v' .$$

## Geometric manifold

Let say we have some point set $X$ modelling a laboratory and some device in it.

Consider now a geometric chart and the collection $\mathcal{G}$ of all charts compatible to the first one.

We call pair $\{\mathcal{G}, X\}$ by name **geometric manifold**.

The emphasis is now on the fact that $X$ is parametrizable, and not in choosing some specific chart.

## 2 Geometry of electromagnetism

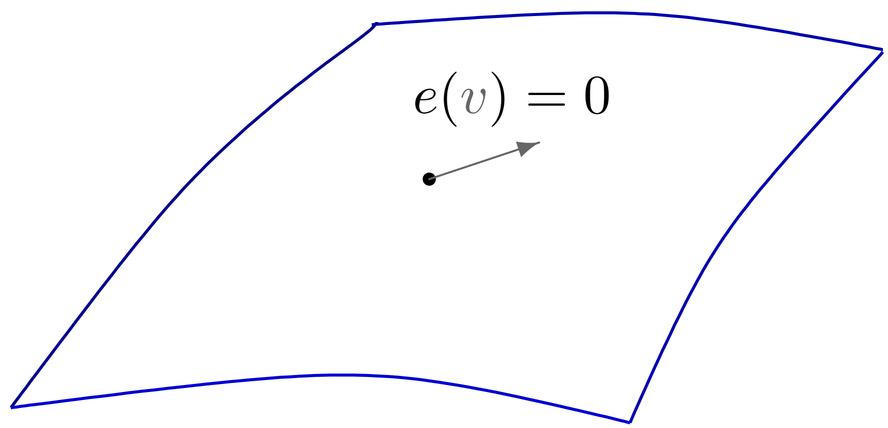
Now, that we know $X$ is a manifold the basic equations of electrostatics are fully meaningful:

$$\begin{aligned} \mathrm{d}e &= 0\,, \\ \mathrm{d}d &= q\,, \\ d &= \star_\epsilon e \end{aligned}$$

That is, the pair $\{e, d\}$ is well specified. But, without choosing some chart from $\mathcal{G}$ we are not able to express the solutions by numbers.

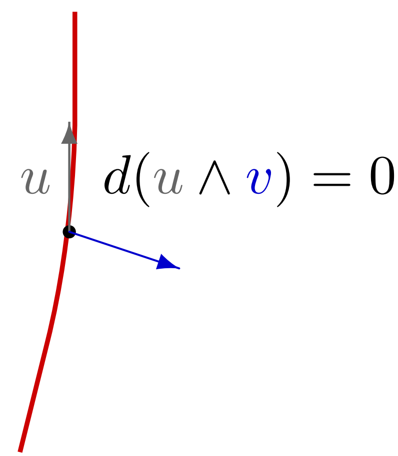
Equipotential layers are also meaningful:

**Definition 1:** A 2-dimensional connceted submanifold $S$ of $M = \{\mathcal{G}, X\}$ is an *equipotential layer*, if for all $x \in S$ and $v \in T_x S$ (i.e, vector $v$ is in the tangent space of $S$ at point $x$) implies $e_x(v) = 0$.



Furthermore, field lines are meaningful:

**Definition 2:** A 1-dimensional submanifold $F$ of $M$ is a *field line*, if for all $x \in F$ properties $u \in T_x F$ and $v \in T_x X$ imply $d_x(u \wedge v) = 0$.



Field lines correspond to straight lines

Equipotential layers correspond to straight lines

This implies ($c$ is a charged conductor),

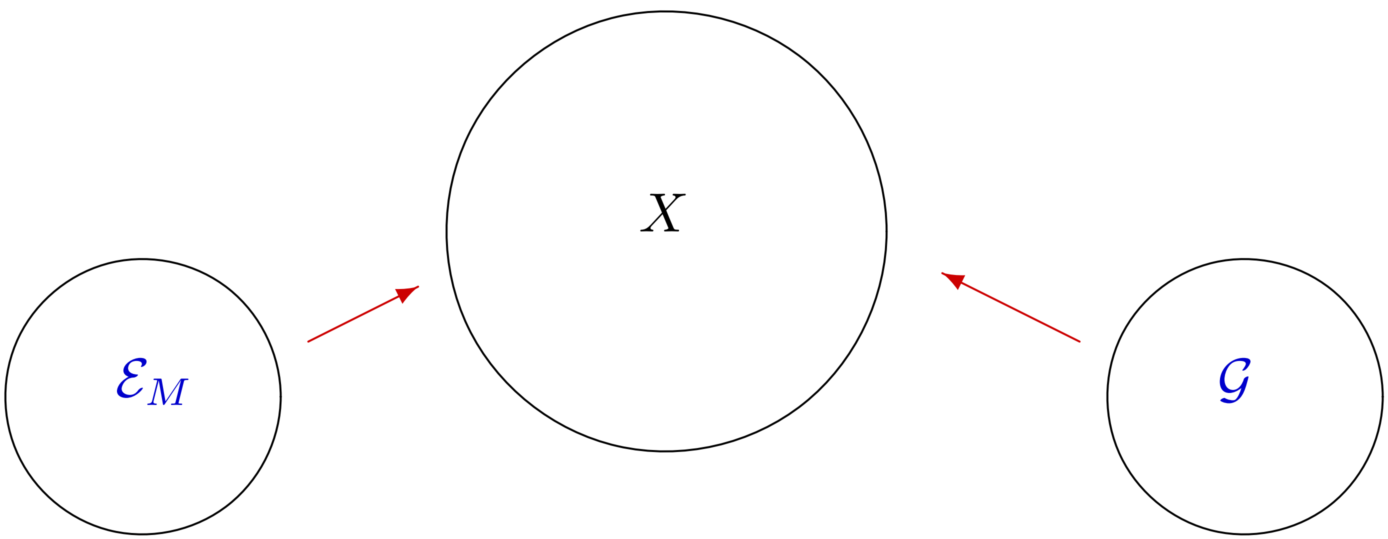
| Set $X \backslash c$ can be parametrized by the electrostatic fields |
|---|

Consider now the collection $\mathcal{E}$ of all those mutually compatible charts mapping to $X \backslash c$ one can get from electrostatics

We call pair $\{\mathcal{E}, X \backslash c\}$ by name **geometry of electrostatics**.

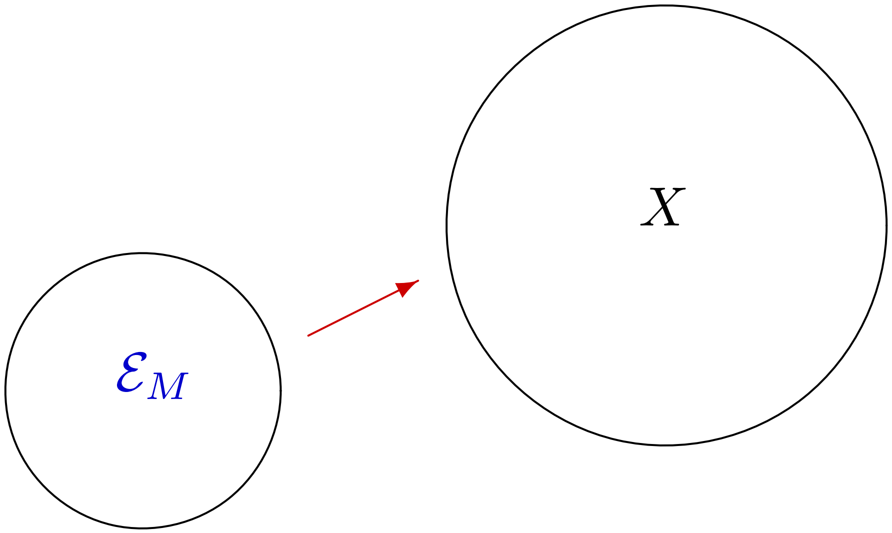
Consider next the collection $\mathcal{E}_M$ of all those mutually compatible charts mapping to $X$ one can get from electromagnetism

We call pair $\{\mathcal{E}_M, X\}$ by name **geometry of electromagnetism**.




Manifold $\{\mathcal{A}, X\}$, where $\mathcal{E}_M, \mathcal{G} \in \mathcal{A}$

The electromagnetic manifold $\{\mathcal{E}_M, X\}$ is meaningfull without $\mathcal{G}$.

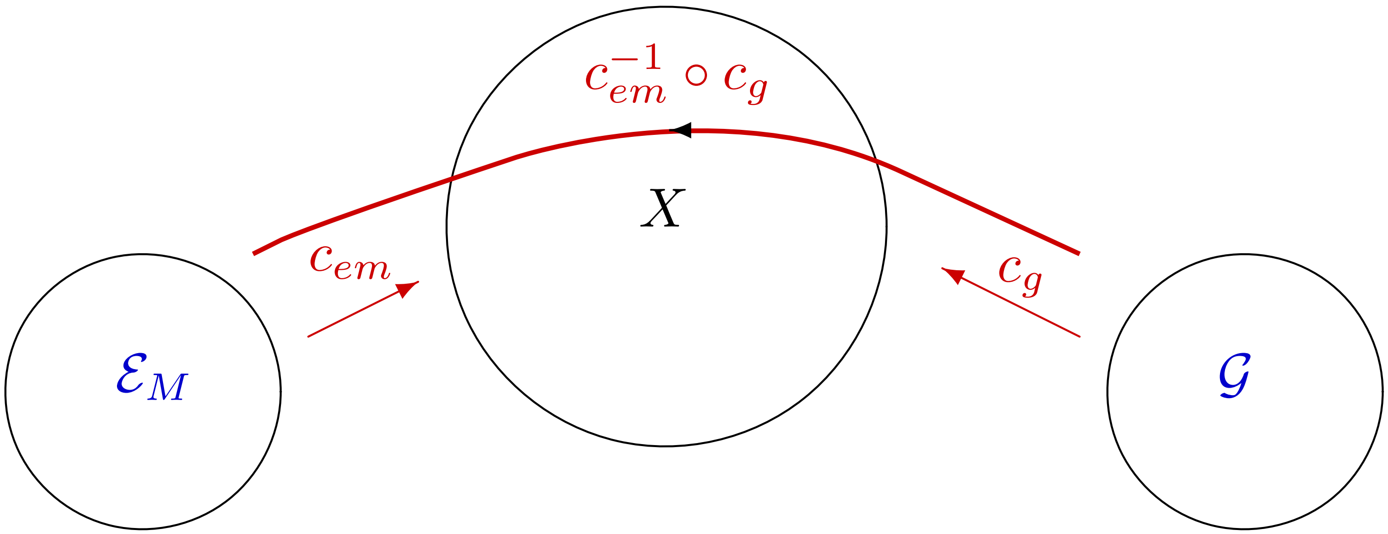



But, it is of rather abstract nature. There is no arithmetic to express points of $X$ with numbers.

$$\boxed{\text{parametrizable} \sim \text{parameters}}$$

# 3 Software systems

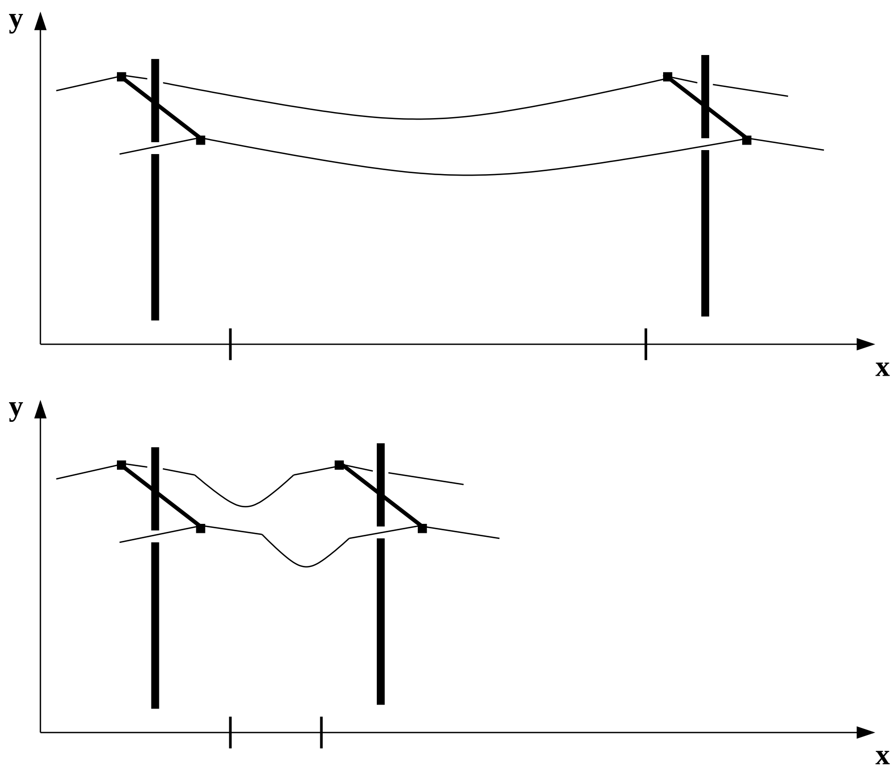
Software systems are **transition maps** from $\mathcal{G}$ to $\mathcal{E}_M$.



# 4 Practical implications

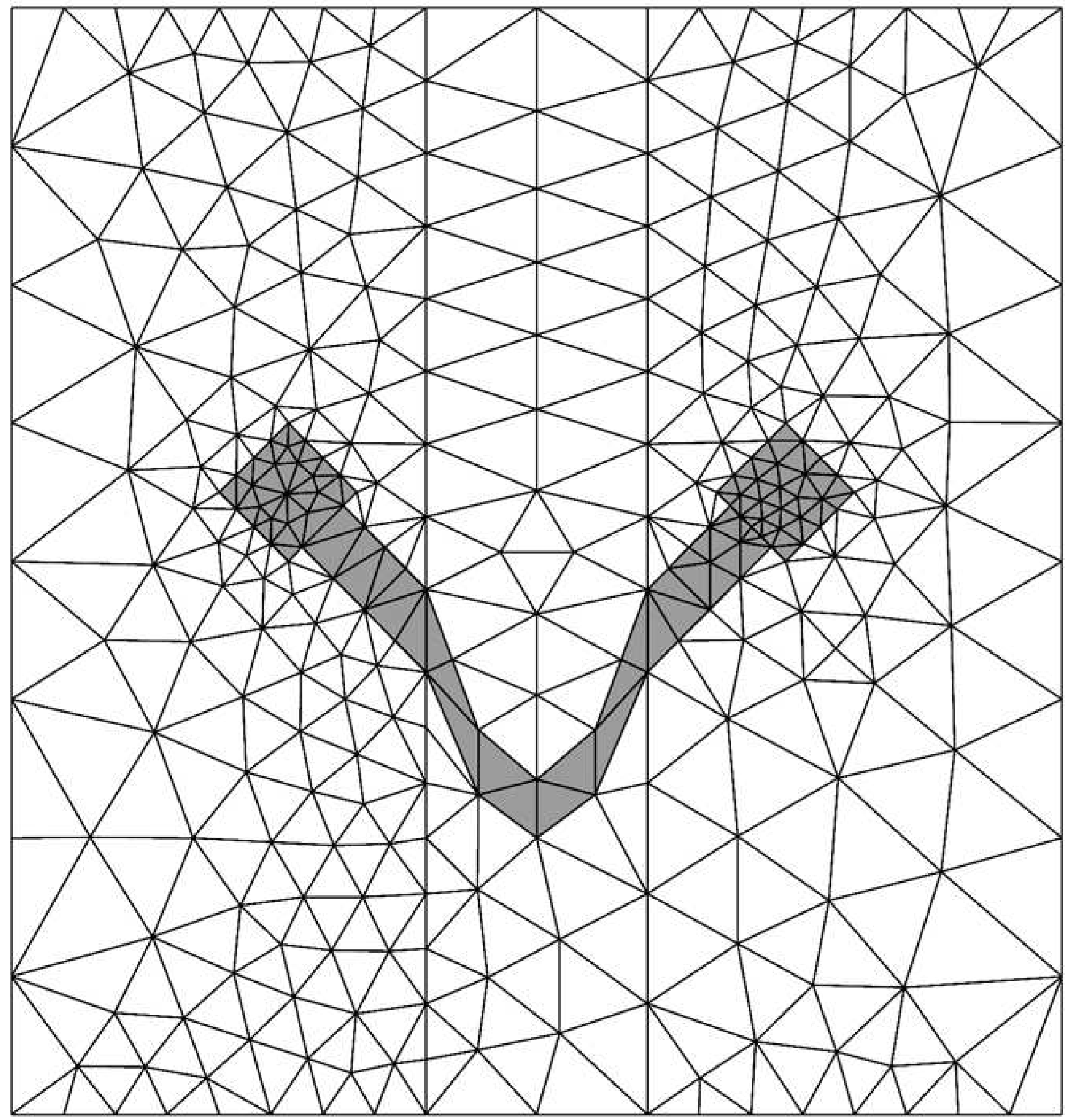
No reason to get stucken on the intuitive charts and metric.

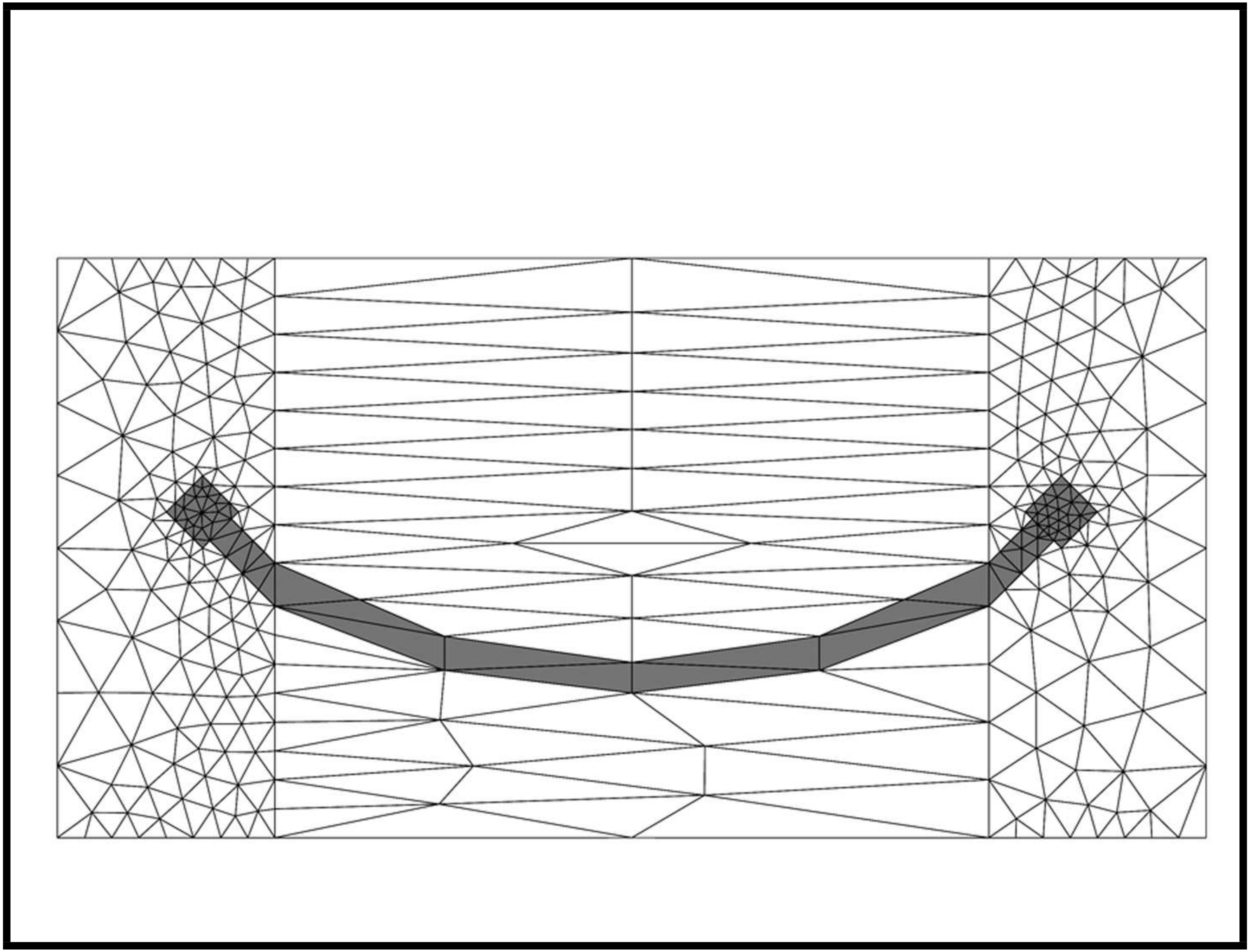
Example: Electric power line



Create the finite element kind of mesh using metric which fits the needs of the problem.

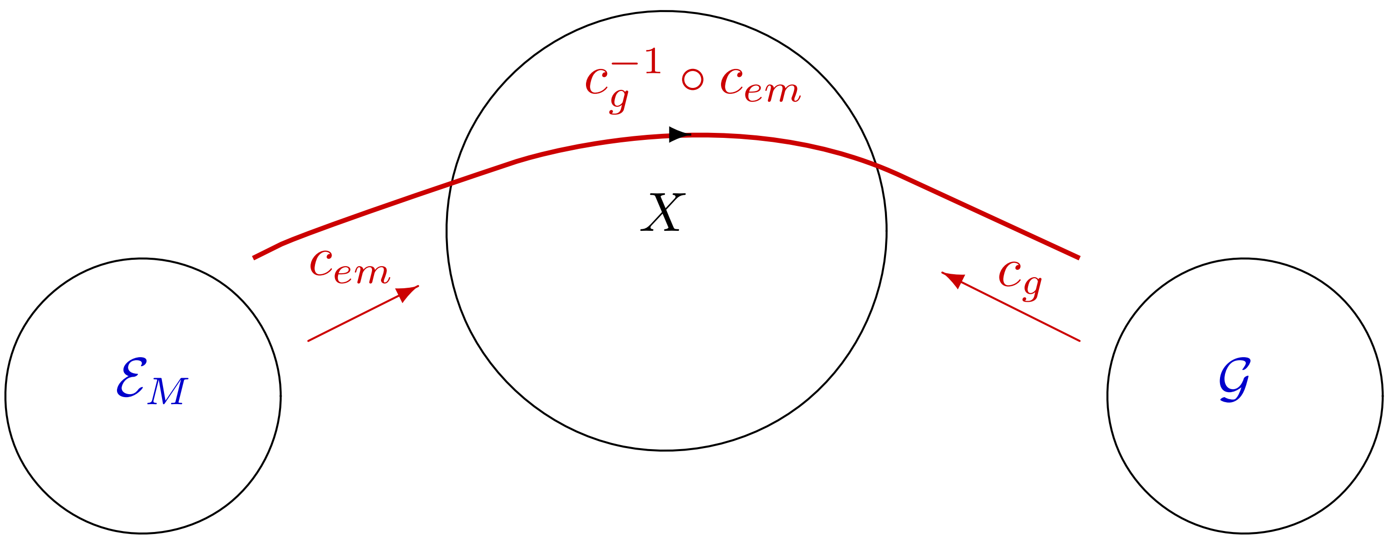
*Computers and software systems have no intuition*





Unexplored idea: Transition maps work both ways

*Parametrize by fields and solve for geometry*



Thank you for the invitation and attention.