EM Field Simulation Based on Volume Discretization:

Finite Integration and Related Approaches

Rolf Schuhmann

Fachgebiet Theoretische Elektrotechnik (TET), Universität Paderborn

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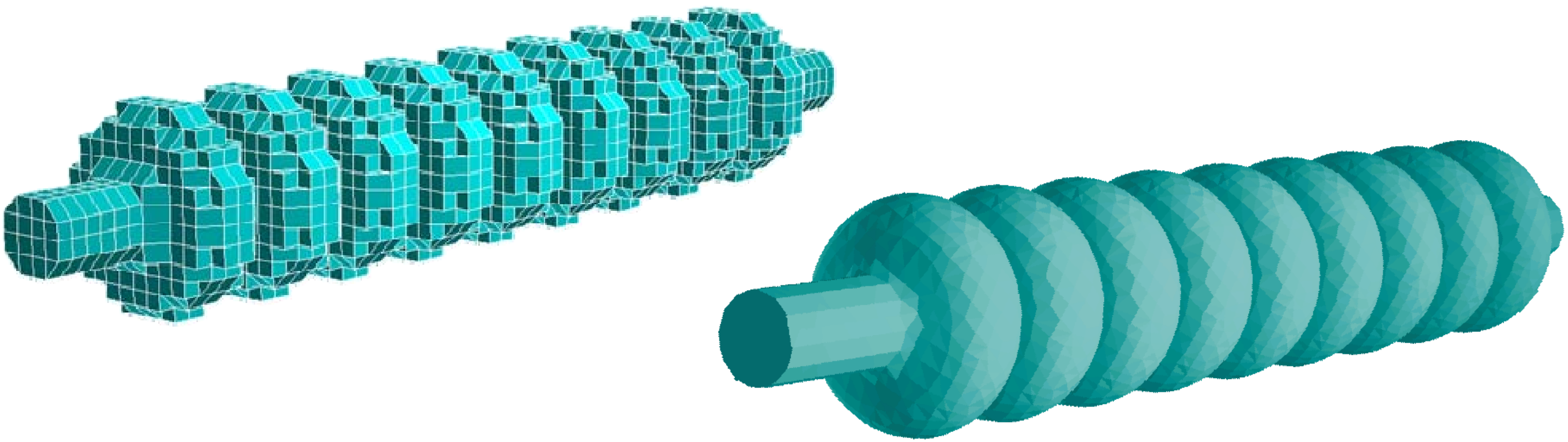
Outline

Basic Approaches of Volume-Grid Methods

– Finite Integration / Finite Differences / Finite Elements
– *Pros and Cons …*
– *Similarities and Differences …*

Advanced Topic

– Conformal Modeling *(with Igor Zagorodnov)*



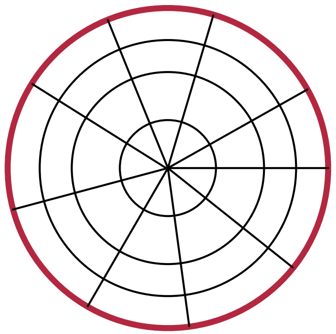
Tasks of computational grids

– Allocation of state variables (DoF) of the method: potentials, field components, basis functions, …

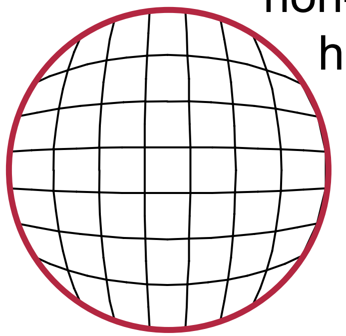
– Geometric Modeling (materials, boundaries)

not necessarily the same grid!

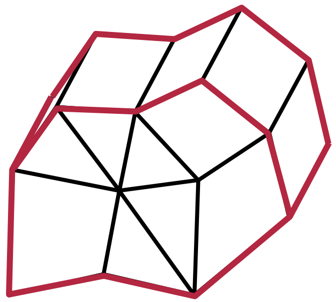
cylindrical / spherical

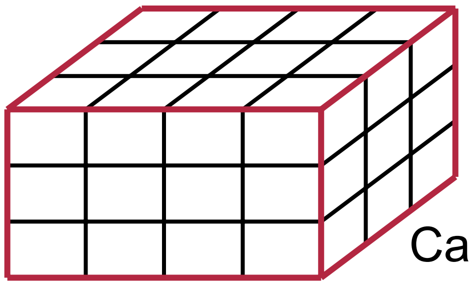


non-orthogonal hexahedra

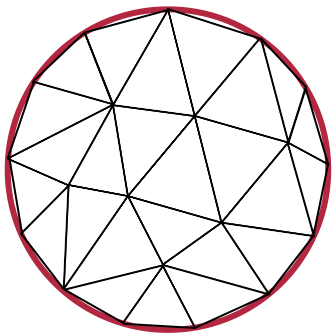


unstructured





Cartesian



triangles / tetrahedra

Idea of FIT: *Geometrical Discretization*

[Weiland 1977]

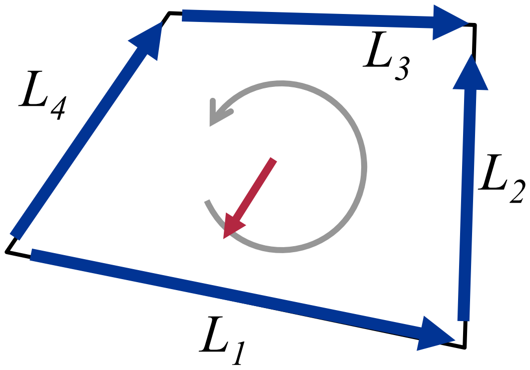
– *Faraday's Law*

$$\oint_{\partial A} \vec{E}(\vec{r},t)\cdot d\vec{s} = -\frac{d}{dt}\int_A B(\vec{r},t)\cdot d\vec{A} \qquad \forall A \subset \Omega$$

(arbitary faces)

$$\oint_{\partial A_n} \vec{E}(\vec{r},t)\cdot d\vec{s} = -\frac{d}{dt}\int_{A_n} B(\vec{r},t)\cdot d\vec{A} \qquad \forall A_n \in G$$

(facets of the grid)



"Discretization" = reduce to a finite number of unknowns

$$\widehat{\widehat{b}}_n := \int_{A_n} \vec{B}\cdot d\vec{A} \quad \text{= grid flux}$$

$$\widehat{e}_n := \int_{L_n} \vec{E}\cdot d\vec{s} \quad \text{= grid voltage}$$

$$\oint_{\partial A_n} \vec{E}\cdot d\vec{s} = \oint_{L_1} + \oint_{L_2} - \oint_{L_3} - \oint_{L_4} = -\int \dot{\vec{B}}\cdot d\vec{A}$$

$$+\widehat{e}_1 + \widehat{e}_2 - \widehat{e}_3 - \widehat{e}_4 = -\frac{d}{dt}\widehat{\widehat{b}}_n$$

(exact)

Finite Integration (FIT)

Idea of FIT: *Geometrical Discretization*

[Weiland 1977]

– *Faraday's Law*

$$+\widehat{e}_1 + \widehat{e}_2 - \widehat{e}_3 - \widehat{e}_4 = -\frac{d}{dt}\widehat{\widehat{b}}_n$$

all grid facets ⟹

$$\mathbf{C}\widehat{\mathbf{e}} = -\frac{d}{dt}\widehat{\widehat{\mathbf{b}}}$$

discrete Faraday's law

Building Blocks:

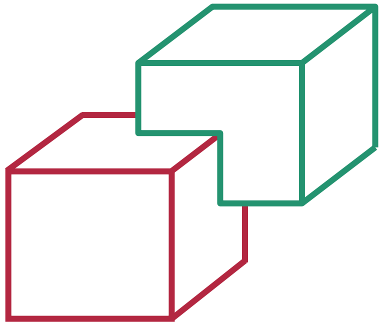
– *"Curl Matrix"* $\mathbf{C}$ ({0,±1}): edge orientations – role in local circulation
  - topological: describes structure of the grid (incidence matrix)
  - no metrics
– Integral state variables
  - grid voltage, grid flux
  - not further resolved at this point

→ generally available for arbitrary grid types

Finite Integration (FIT)

Further Steps:

– Primary and dual grid (*Cartesian or others*)



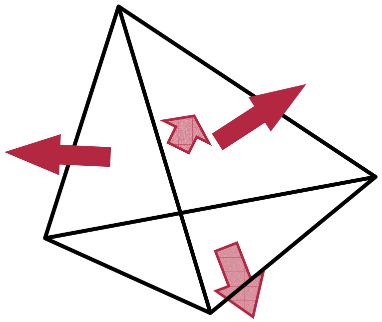
– extend set of state variables:

$$\widehat{\widehat{d}}_n := \int_{\tilde{A}_n} \vec{D}\cdot d\vec{A} \qquad \widehat{\widehat{j}}_n := \int_{\tilde{A}_n} \vec{J}\cdot d\vec{A} \qquad \widehat{h}_n := \int_{\tilde{L}_n} \vec{H}\cdot d\vec{s} \qquad q_n := \int_{\tilde{V}_n} \rho\cdot dV$$

– „dual curl matrix“ $\tilde{\mathbf{C}} = \mathbf{C}^T$

$$\tilde{\mathbf{C}}\widehat{\mathbf{h}} = \frac{d}{dt}\widehat{\widehat{\mathbf{d}}} + \widehat{\widehat{\mathbf{j}}}$$ discrete Ampere’s law

– “*div*”-operations: Surface integrals around cells



$$\tilde{\mathbf{S}}\widehat{\widehat{\mathbf{d}}} = \mathbf{q}$$

$$\mathbf{S}\widehat{\widehat{\mathbf{b}}} = \mathbf{0}$$

– ‘Topological’ consistency properties (for arbitrary grids):

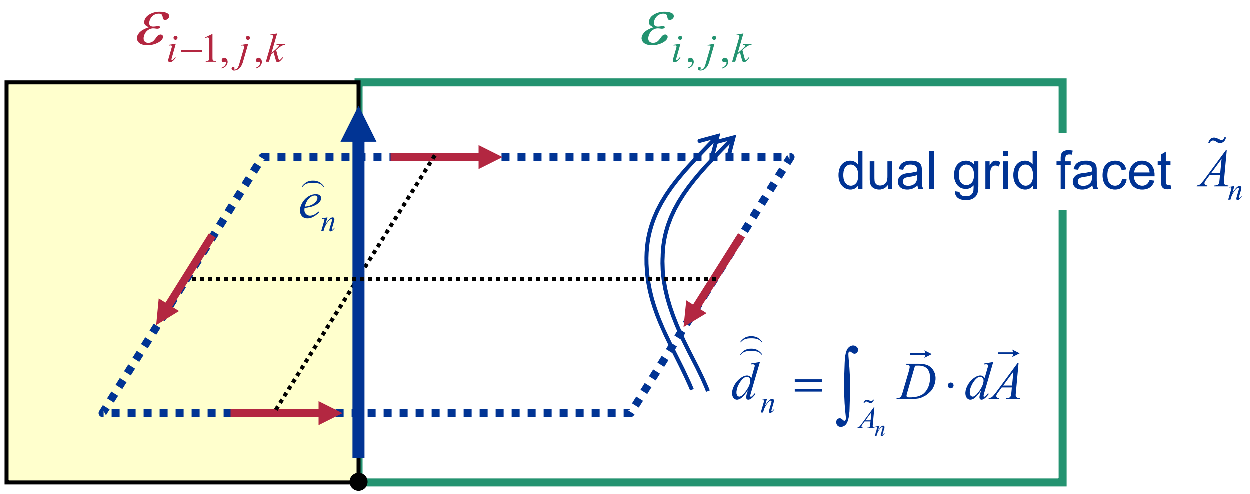
e.g. $\mathbf{S}\mathbf{C} = \mathbf{0}$

(corresponding to $\operatorname{div}\operatorname{curl} \equiv 0$)

Finite Integration Technique

(still missing:)

– Material relations, e.g. $\hat{e}_n \leftrightarrow \hat{\hat{d}}_n$



– Integral interpretation leads to canonical transformation formula (*incl.* area-averaging of permittivity over dual facet)

$$\hat{\hat{\mathbf{d}}} = \mathbf{M}_\varepsilon \hat{\mathbf{e}} \quad \text{with} \quad \mathbf{M}_\varepsilon = diag\left(\frac{\bar{\varepsilon}\tilde{A}_n}{L_n}\right)$$

(*here*: diagonal matrix)

Finite Integration: Matrix operators

- "Topological operators"
  - incidence matrices of the grids can be used as "discrete curl" and "discrete divergence"
  - *discretization, no approximation* → all continuous properties still valid

  Some catchwords:
  - a "natural discretization" of Maxwell's Equations
  - a *general framework* for "discrete electromagnetism on a grid"

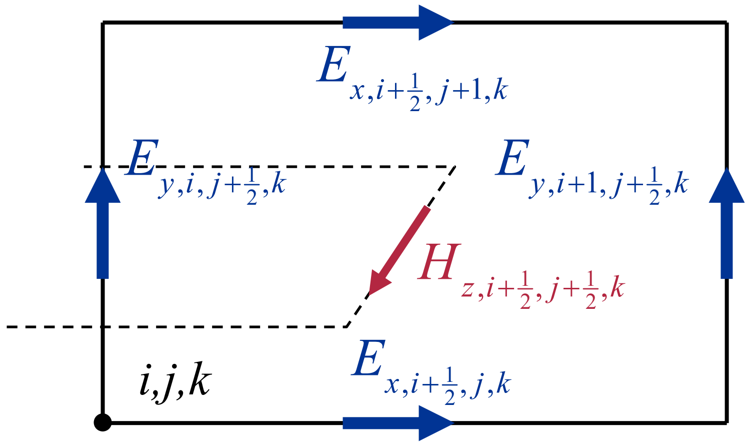
- "Material operators"
  - the approximations of the method
  - many ways to derive them
  - must fulfill key properties: *consistent*, *symmetric positive definite*

Finite Integration: Matrix operators

Primary Grid		Dual Grid
$\boldsymbol{\varphi}_E \in P$	$\longleftarrow - - - - - - \longrightarrow$	$\mathbf{q}_E \in \tilde{V}$
$\downarrow \mathbf{G}$		$\uparrow \tilde{\mathbf{S}}$
$\widehat{\mathbf{e}} \in L$	$\longleftarrow \mathbf{M}_\varepsilon \longrightarrow$	$\widehat{\widehat{\mathbf{d}}} \in \tilde{A}$
$\downarrow \mathbf{C}$		$\uparrow \tilde{\mathbf{C}}$
$\widehat{\widehat{\mathbf{b}}} \in A$	$\longleftarrow \mathbf{M}_\mu \longrightarrow$	$\widehat{\mathbf{h}} \in \tilde{L}$
$\downarrow \mathbf{S}$		$\uparrow \tilde{\mathbf{G}}$
$\mathbf{q}_m \in V$	$\longleftarrow - - - - - - \longrightarrow$	$\boldsymbol{\varphi}_m \in \tilde{P}$
exact	*Approximations*	*exact*

for RF-fields: **Finite Difference Time Domain (FDTD)**

- start with Maxwell's equations in their differential form: curl, $\partial/\partial t$ - operators
- 'Staggered grids' in time and space (Cartesian) [Yee 1966]



$$-\frac{\partial}{\partial t}(\mu H_z) = \left.\operatorname{curl}\vec{E}\right|_z = -\frac{\partial E_x}{\partial y} + \frac{\partial E_y}{\partial x}$$

staggered time grid (for $\frac{\partial}{\partial t}$) | staggered spatial grid ('dual grid') (for $\frac{\partial E_x}{\partial y}$, $\frac{\partial E_y}{\partial x}$)

$$\left.\frac{\partial}{\partial t} H_z\right|_{n+\frac12} \approx \frac{H_z^{n+1} - H_z^{n}}{\Delta t}$$

$$\left.\frac{\partial E_x}{\partial y}\right|_{j+\frac12} \approx \frac{E_{i+\frac12,j+1,k} - E_{x,i+\frac12,j,k}}{\Delta y}$$

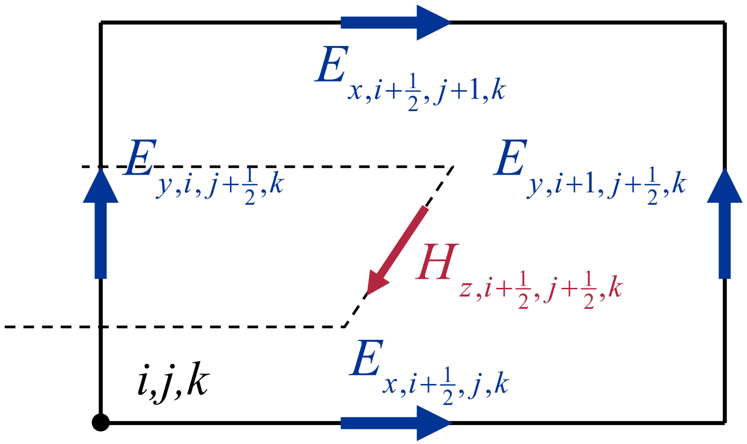
Discretization + Approximation in 1 step

Finite Differences

for RF-fields: **Finite Difference Time Domain (FDTD)**

– start with Maxwell's equations in their differential form: curl, $\partial/\partial t$ - operators

– 'Staggered grids' in time and space (Cartesian) [Yee 1966]



$$-\frac{\partial}{\partial t}(\mu H_z) = \operatorname{curl} \vec{E}\Big|_z = -\frac{\partial E_x}{\partial y} + \frac{\partial E_y}{\partial x}$$

↓ staggered time grid

↓ staggered spatial grid ('dual grid')

$$H^{n+1}_{z,i+\frac{1}{2},j+\frac{1}{2},k} = H^{n}_{z,i+\frac{1}{2},j+\frac{1}{2},k} - \frac{\Delta t}{\mu}\left(-\frac{E^{n+\frac{1}{2}}_{x,i+\frac{1}{2},j+1,k} - E^{n+\frac{1}{2}}_{x,i+\frac{1}{2},j,k}}{\Delta y} + \frac{E^{n+\frac{1}{2}}_{y,i+1,j+\frac{1}{2},k} - E^{n+\frac{1}{2}}_{y,i,j+\frac{1}{2},k}}{\Delta x} \right)$$

„Update Equations"

Finite Integration (FIT) and **Finite Differences (FDTD)**

– FIT on Cartesian grids + leapfrog time discretization:

$$\widehat{\mathbf{h}}^{n+1} = \widehat{\mathbf{h}}^{n} - \Delta t \, \mathbf{M}_{\mu}^{-1} \, \mathbf{C} \, \widehat{\mathbf{e}}^{n+\frac{1}{2}}$$

– update

conceptually different, but
„***computationally equivalent***“

Δy

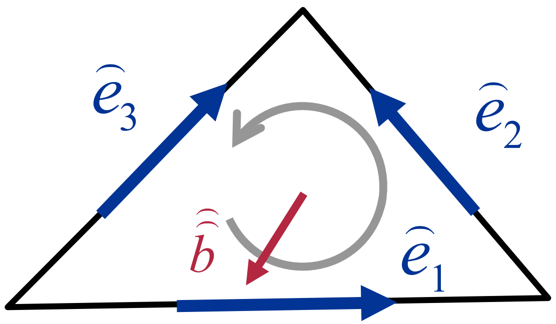
$\mu \Delta x \Delta y$

$\mathbf{M}_{\mu}^{-1}$ $\quad$ $\mathbf{C}\widehat{\mathbf{e}}^{n+\frac{1}{2}}$

$\widehat{h}_z \approx H_z \cdot \Delta \tilde{z}$

$$H^{n+1}_{z,i+\frac{1}{2},j+\frac{1}{2},k} = H^{n}_{z,i+\frac{1}{2},j+\frac{1}{2},k} - \frac{\Delta t}{\mu}\left(-\frac{E^{n+\frac{1}{2}}_{x,i+\frac{1}{2},j+1,k} - E^{n+\frac{1}{2}}_{x,i+\frac{1}{2},j,k}}{\Delta y} + \frac{E^{n+\frac{1}{2}}_{y,i+1,j+\frac{1}{2},k} - E^{n+\frac{1}{2}}_{y,i,j+\frac{1}{2},k}}{\Delta x} \right)$$

Triangular (tetrahedral) grids: (FDTD ??)



$$\oint_{\partial A} \vec{E} \cdot d\vec{s} = -j\omega \int_A \vec{B} \cdot d\vec{A}$$

$$\widehat{e}_1 + \widehat{e}_2 - \widehat{e}_3 = -j\omega \widehat{\widehat{b}}$$

- Operators **S**,**C** canonical ⇒ all topological properties fulfilled
- “Only” needed: new material matrices

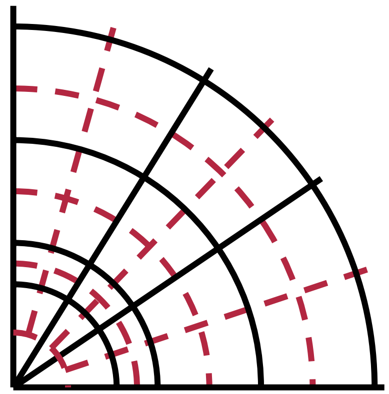
- Orthogonality of primary and dual grid: Standard-approach (→ diagonal matrix)
- Loss of this orthogonality: special solution needed
- Necessary for stability: symmetric positive definite

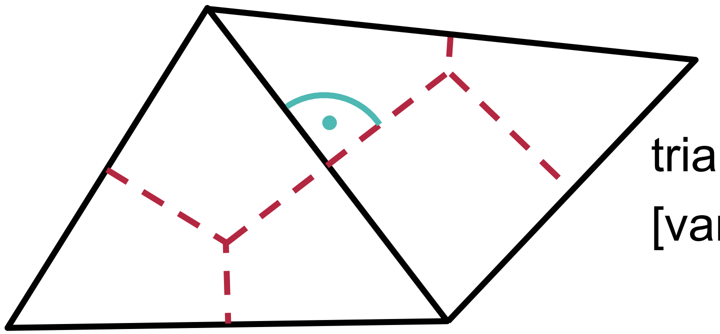
FIT: Other Grids

Extension to other grids

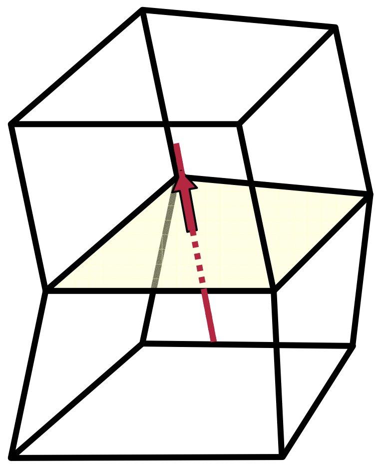
- ☞ So far implemented:

Cartesian, cylindrical (2D/3D)
[1983]





triangular (with orth. dual grid)
[van Rienen 1985]



non-orthogonal hexahedral
[1998]

FIT: History of Notations

1966 Yee's Method (FDTD)

1975 E. Tonti: "On the Formal Structure of Physical Theories"

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30 years of FIT
in 2007 !

$$j\,\omega\, D_{\tilde G}\, D_{\varepsilon}\, \underline{e} - R_{\tilde G}\, D_{\mu}^{-1}\, \underline{b} = \underline{c}\,,$$
$$j\,\omega\, D_{G}\, \underline{b} + R_{G}\, \underline{e} = 0\,.$$

1977 (1st paper)

AEÜ 31

$$D_h\,(-\mu_0\,\dot{h}) \quad = \quad R_e\; e$$
$$D_e\,(\varepsilon_0\;\dot{e} + j) \quad = \quad R_h\; h$$

1980, „Finite Integration"

Proc. XI Conf. High Energy Acc, Geneva.

FIT: History of Notations

1980 Taflove: Yee's Method named "FDTD" IEEE T-EMC 22

1980/1986 Nedelec: Edge elements in FE Num.Math. 35/50

$$\mathbf{C}\mathbf{D}_s\mathbf{e} = -\mathbf{D}_A\mathbf{b}$$

1986: Curl matrix notated "**C**"
URSI EMTS Budapest

$$e_i = \int_{L_i} \vec{E} \cdot d\vec{s}; \qquad b_i = \int_{A_i} \vec{B} \cdot d\vec{A}$$

$$-\dot{\mathbf{b}} = \mathbf{C}\mathbf{e}$$

$$\mathbf{S}\mathbf{b} = 0$$

$$\mathbf{i} + \dot{\mathbf{d}} = \tilde{\mathbf{C}}\mathbf{h}$$

$$\tilde{\mathbf{S}}\mathbf{d} = \mathbf{q}$$

1996: Grid voltages and fluxes
Int. J. Num Mod. 9

(in PhD-Theses since 1992)

but still the same basic formulas and discretization approach as in 1977 !

Back to time domain:

$$\widehat{\mathbf{h}}^{n+1} = \widehat{\mathbf{h}}^{n} - \Delta t \, \mathbf{M}_{\mu}^{-1} \, \mathbf{C} \, \widehat{\mathbf{e}}^{n+\frac{1}{2}}$$

$$\widehat{\mathbf{e}}^{n+\frac{3}{2}} = \widehat{\mathbf{e}}^{n+\frac{1}{2}} + \Delta t \, \mathbf{M}_{\varepsilon}^{-1} \, \tilde{\mathbf{C}} \, \widehat{\mathbf{h}}^{n+1}$$

Update-Equations of explicit scheme

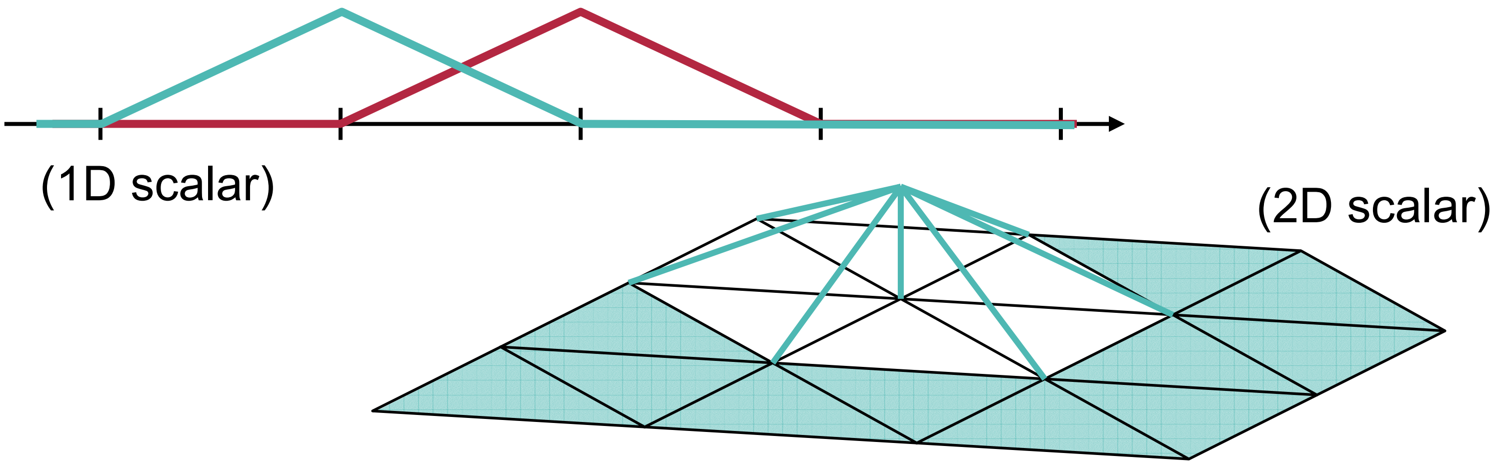
Inverse material operators needed !

– ok for diagonal matrices, but this needs dual-orthogonal meshes

→ A motivation for research in this field:
to find an explicit time-domain scheme for general tet-grids

Finite Elements:

– represent solution by basis functions with „compact support“
– low requirements on (type of) computational grid



– low polynomial order (linear, quadratic, ...) → rate of convergence

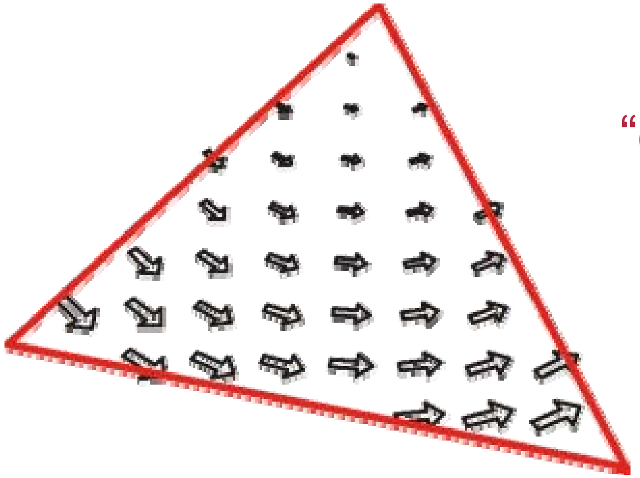
➡ *„discretization of (solution) space“*

(vs. discretization of operators in FD, FIT)

Finite Elements

Edge Elements (Whitney forms)

- A set of functions for vector-valued problems on simplicial meshes [Whitney 1957, Nedelec 1980]
- tangential continuity



"edge functions": $\int_{L_i} \vec{w}_j \cdot d\vec{s} = \delta_{ij}$

FE FIT

$$\Rightarrow \int_{L_i} \vec{E} \cdot d\vec{s} = \int_{L_i} \sum_j E_j \vec{w}_j \cdot d\vec{s} = E_i = \widehat{e}_i$$

state variables equivalent ☺

Finite Elements

„Whitney Complex“

– defines nodal (W^0), edge (W^1), face (W^2), and volume (W^3) functions

– complete sequence property:

$$W^0 \xrightarrow{grad} W^1 \xrightarrow{curl} W^2 \xrightarrow{div} W^3$$

functional spaces

$$\mathbf{W}^0 \xrightarrow[\mathbf{G}]{} \mathbf{W}^1 \xrightarrow[\mathbf{C}]{} \mathbf{W}^2 \xrightarrow[\mathbf{S}]{} \mathbf{W}^3$$

algebraic coefficients

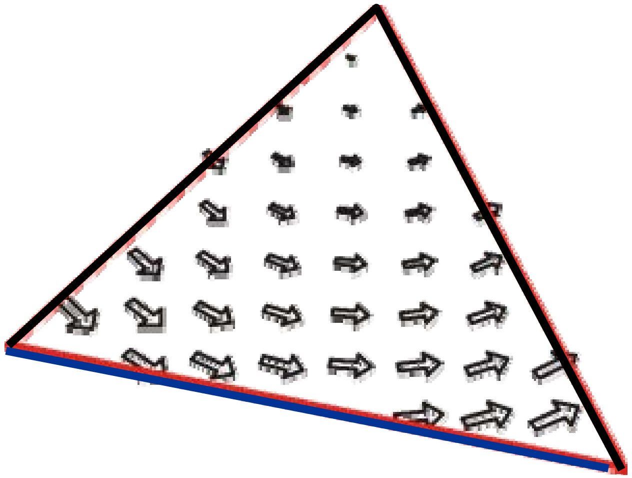
the same incidence matrices as in FIT !

– consistent discretization, e.g. no ghost modes !

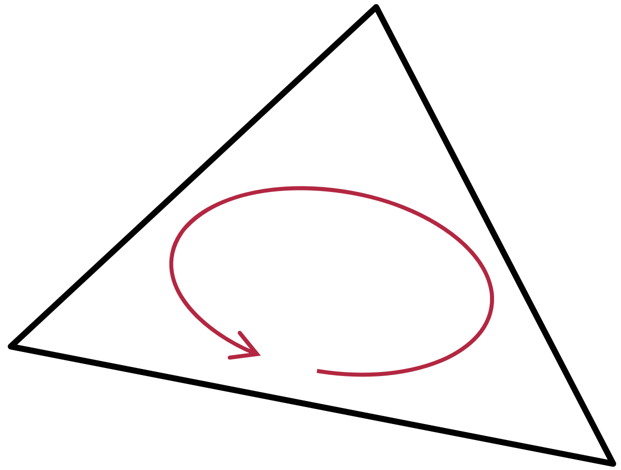
[Bossavit 98]

"Curl-Operation": $\nabla \times \vec{A} = \vec{B}$

FE



FIT



$$\nabla \times \left(\sum A_j \vec{w}_j^{(1)} \right) = B_n \vec{w}_n^{(2)}$$ (face function)

$$\sum \pm \widehat{a}_j = \widehat{\widehat{b}}_n$$

use Curl-matrix in FE !

Finite Elements: Application to Wave Equation

$$\text{curl}\,\mu^{-1}\,\text{curl}\,\vec{E} - \omega^2 \varepsilon \vec{E} = -i\omega \vec{J} \qquad \vec{E} = \sum_i E_i \vec{w}_i$$

– Weighted residuals: $\int \vec{w}_j \cdot \ldots \, dV \quad \forall j$

$$\Rightarrow \quad \mathbf{A}(E_i) - \omega^2\, \mathbf{B}(E_i) = \mathbf{r}$$

$$A_{ij} = \int \mu^{-1} (\text{curl}\,\vec{w}_i) \cdot (\text{curl}\,\vec{w}_j)\, dV$$

$$B_{ij} = \int \varepsilon \vec{w}_i \cdot \vec{w}_j \, dV$$

FIT: $\tilde{\mathbf{C}}\mathbf{M}_\mu^{-1}\mathbf{C}\,\widehat{\mathbf{e}} - \omega^2\, \mathbf{M}_\varepsilon \widehat{\mathbf{e}} = \mathbf{r}$ analogy (for identical grids) ?

$\widehat{\mathbf{e}}$	$\leftrightarrow$	(E_i)	degrees of freedom / state variables
$\tilde{\mathbf{C}}\mathbf{M}_\mu^{-1}\mathbf{C}$	$\leftrightarrow$	$\mathbf{A}$	stiffness matrix
$\mathbf{M}_\varepsilon$	$\leftrightarrow$	$\mathbf{B}$	mass matrix

Finite Elements: Matrices

Stiffness Matrix: $A_{ij} = \int \mu^{-1} (\operatorname{curl} \vec{w}_i) \cdot (\operatorname{curl} \vec{w}_j)\, dV$

– can be identified as

$$(A_{ij}) = \tilde{\mathbf{C}} \mathbf{M}^{FE}_{\mu^{-1}} \mathbf{C} \quad \text{with} \quad M^{FE}_{\mu^{-1},ij} = \int \mu^{-1} \vec{w}_i^{(2)} \cdot \vec{w}_j^{(2)}\, dV$$

(2nd FE mass matrix for face functions)

Mass Matrix:

– FIT: diagonal for dual-orthogonal grids

– FE: always non-diagonal

- Cartesian grids: *mass lumping leads to FIT-equivalent matrix!*
  [e.g. Rylander/Bondeson 2000]
- Tetrahedral grids: mass lumping fails
  [Bossavit / Kettunen 1999, 2001]

- (no explicit time domain scheme!)

Finite Elements vs. Finite Integration

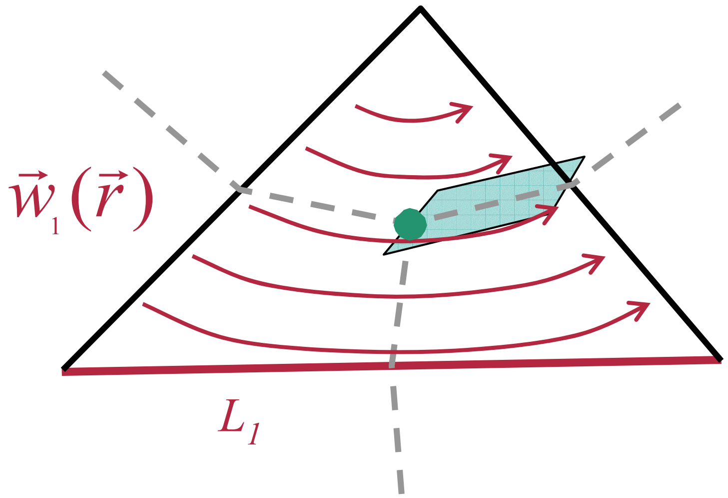
- (Lowest order) Whitney functions allow FIT framework in FE
  - topological matrices (incidence relations of grid)
  - material operators

- Hybridization:
  - both methods allow hybrid grids
  - analysis of properties by FE- and/or FIT-theory

  - Example: Hybrid method (Rylander/Bondeson [2000] and Edelvik/Ledfelt [2002])
    - unstructured grids for complex geometries
    - structured (FDTD-) grids elsewhere
    - common stability analysis by re-interpretation of FDTD part as „FE with mass lumping“

Others…

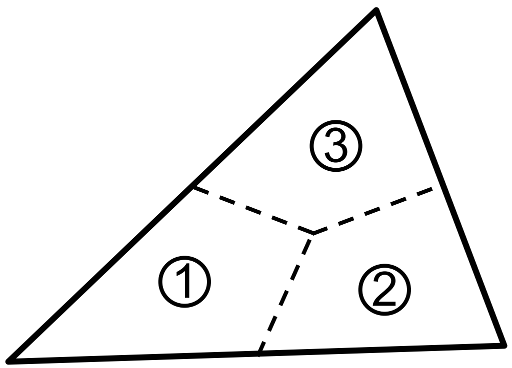
- Basis functions (Whitney or others), but matrices not from FE Galerkin approach
- Example: „FIT with Whitney“ (2D) [2001]



derive material operator from exact integration of basis function on dual face

needs modified dual grid for stability

- Example: „Microcell Method“ (2D) [Tonti 1975, Marrone 2001]



barycentric dual grid,
piecewise constant fields in „micro cells“

Others…

– Hybrid: FIT + Microcell Method [Cinalli et al., Darmstadt, 2004]

– **FE: higher order**
  - „higher order Whitney forms“: more DoF per edge / facet
  - hierarchical functional spaces

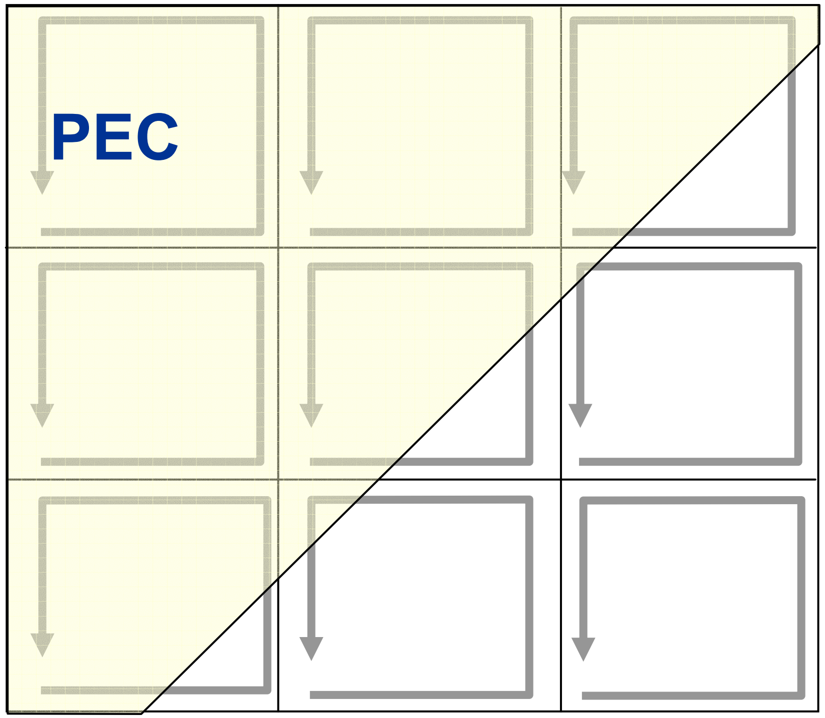
– **FIT: higher order**
  - no additional DoF
  - larger stencils in material operators

no FE-FIT correspondence; geometric interpretations?

– **Finite Volumes, Discontinuous Galerkin, Generalized FE, ...**

Conformal Modeling in a Cartesian Yee-Grid

– allow PEC-interfaces within cells



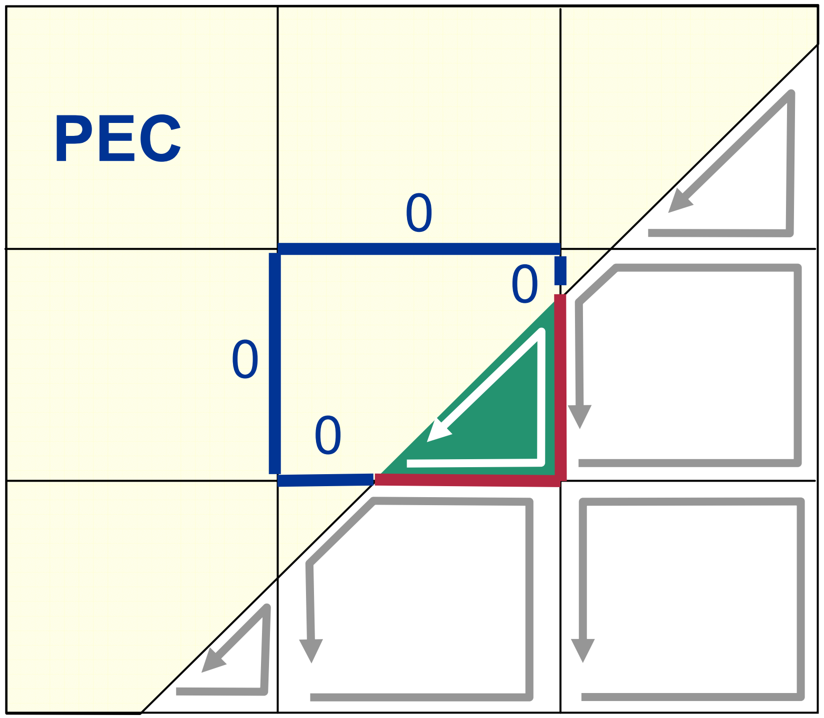
[Thoma 1997] („FIT + *Partially Filled Cells*"), [Dey, Mittra 1997] („*Conformal FDTD*")

Conformal Modeling in a Cartesian Yee-Grid

- allow PEC-interfaces within cells
- omit PEC parts $(\int \vec{E} \cdot d\vec{s} = 0)$

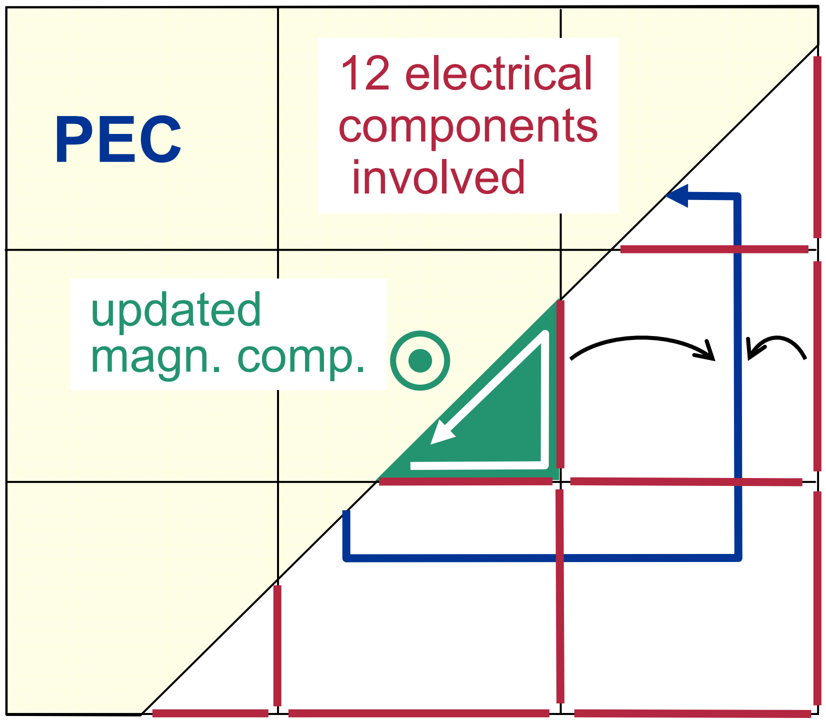
> → **reduced edge lengths**
>
> $L_{ijk} \rightarrow l_{ijk}$
>
> → **reduced facets**
>
> $S_{ijk} \rightarrow s_{ijk}$

- 2nd order accurate (PEC)
- still diagonal matrices!
- needs smaller time step



[Thoma 1997] („FIT + *Partially Filled Cells*“), [Dey, Mittra 1997] („*Conformal FDTD*“)

To keep full time step (numerical dispersion…)



Uniform Stable Conformal scheme (**USC**, 2003)

virtually enlarged cells (off-diagonal entries)

FE analogon ?? (would need new basis functions)

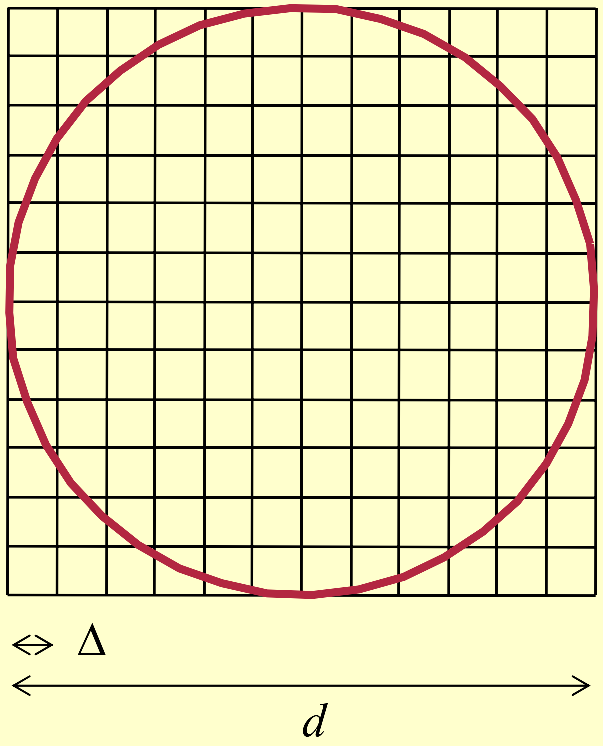
– simplified conformal scheme (SC 2006):
  - diagonal matrix with modified components
  - small loss of accuracy, still 2nd order

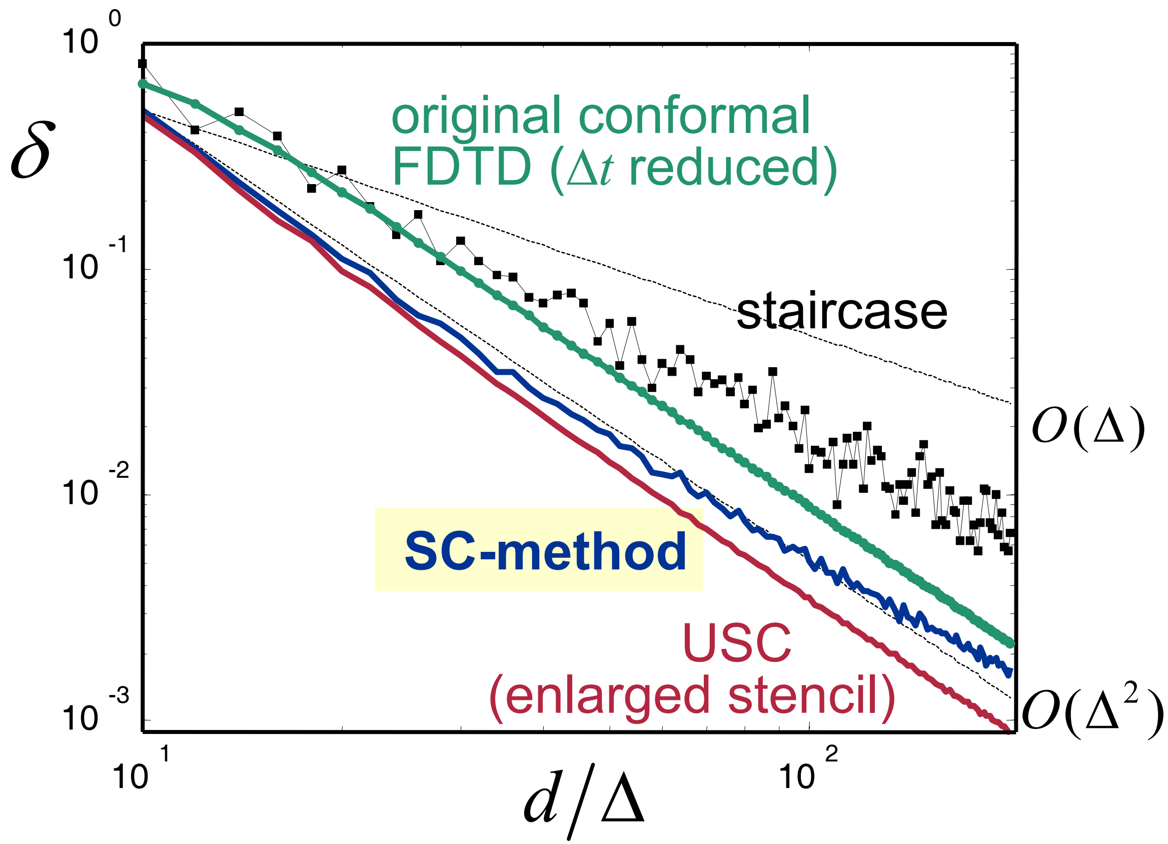
[I. Zagorodnov]

Convergence Analysis: Academic example

impress analytical field → iterate N time steps → δ = L_2-norm of field deviation

Cylinder (quasi-2D)





[I. Zagorodnov 2006]

Concepts of Finite Methods

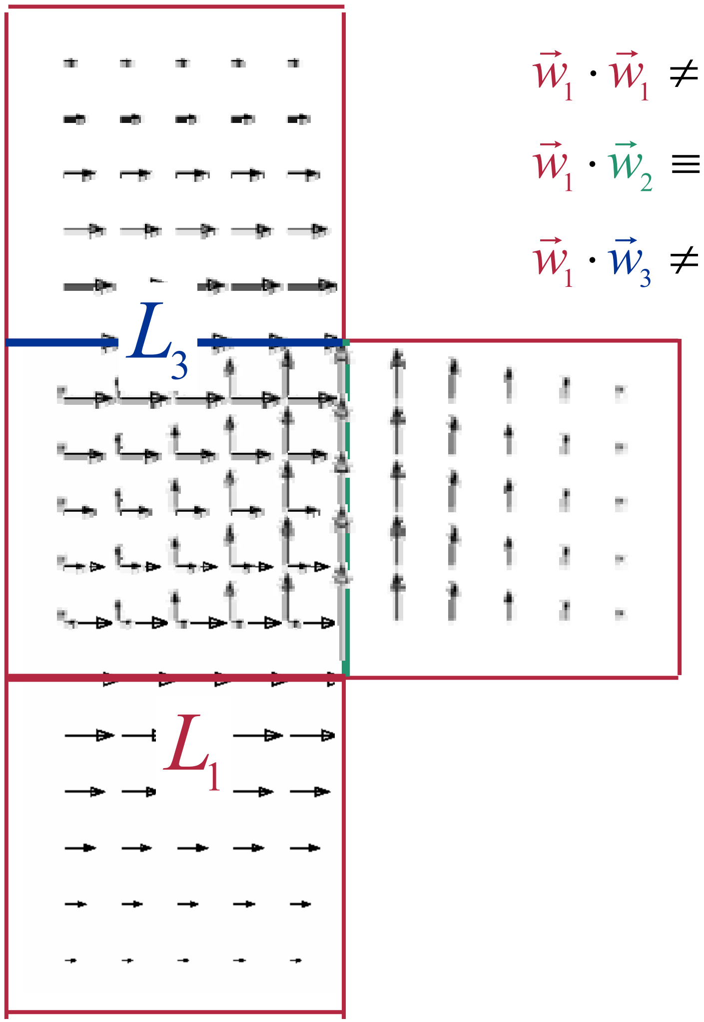
	continuous	discrete	approximation
FD	Derivatives $\lim_{h\to 0} \frac{f(x+\frac{h}{2})-f(x-\frac{h}{2})}{h}$	Differences $\frac{f(x+\frac{\Delta}{2})-f(x-\frac{\Delta}{2})}{\Delta}$	locally in difference formulas (Taylor series)
FIT	Circulations $\int_{\partial A} \vec{E}\cdot d\vec{s} \quad \forall A \subset \Omega$	Discrete Circulations $\sum_{i\in\partial_k} \widehat{e}_i \quad \forall A_k \subset G$	locally in material relations (Taylor series)
FE	Solutions space $H(\mathrm{curl}),\ \dim = \infty$	Functional Space $H_h^p(\mathrm{curl}),\ \dim = N$	globally (min. resid.): „optimal“ solution for chosen space and w.r.t. chosen inner product

Properties of Finite Methods

	FD / FIT	(Whitney-) FE
Basis functions	possible by re-interpretation	✓ main idea
Dual grid	✓ main idea	possible by re-interpretation
Topological Operators S,C	✓ main idea	possible by re-interpretation
Diagonal Material Matrix (explicit in Time Domain)	✓	
General Computational Grid	(✓)	✓
Higher Order Modeling	(✓)	✓
Subcell Methods	✓	

Mass Matrix: 1) Cartesian grids

$$B_{ij} = \int \varepsilon \vec{w}_i \cdot \vec{w}_j \, dV$$



$\vec{w}_1 \cdot \vec{w}_1 \neq 0$ (main diagonal)

$\vec{w}_1 \cdot \vec{w}_2 \equiv 0$

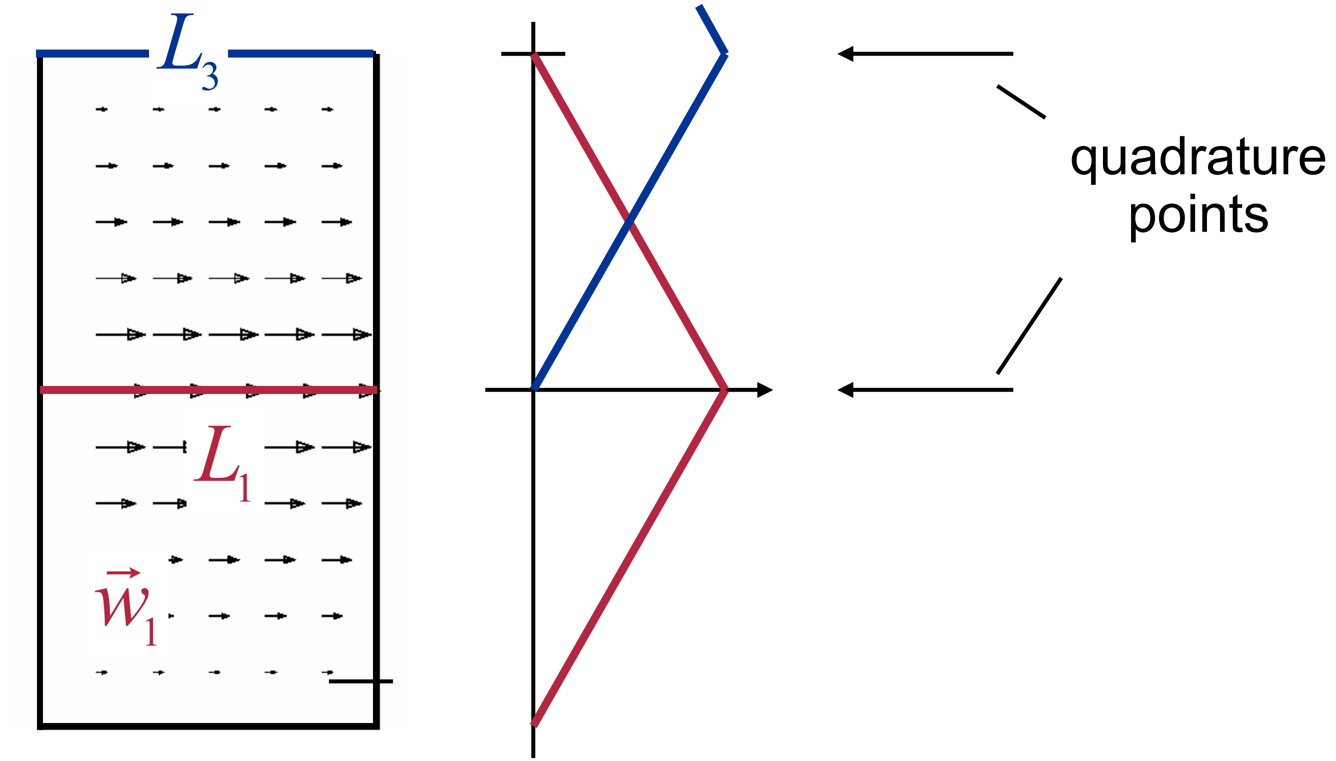
$\vec{w}_1 \cdot \vec{w}_3 \neq 0$ ⇒ no diagonal matrix !

$\mathbf{B} \neq \mathbf{M}_\varepsilon$ ☹

- eliminate off-diagonal entries by „mass lumping“
- idea: approximative evaluation of integrals (quadrature)

Mass Matrix: 1) Cartesian grids

$$B_{ij} = \int \varepsilon \vec{w}_i \cdot \vec{w}_j \, dV$$



(...) $\mathbf{B} = \mathbf{M}_{\varepsilon}$

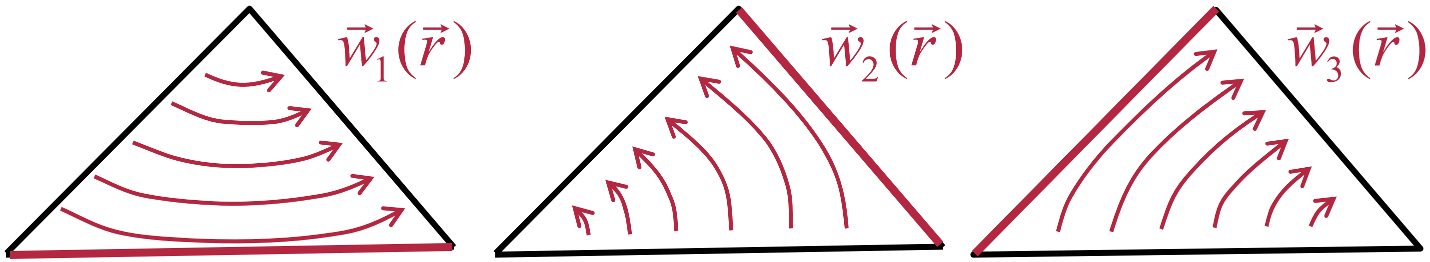
(...) $\mathbf{A} = \tilde{\mathbf{C}} \mathbf{M}_{\mu}^{-1} \mathbf{C}$

FE with mass lumping leads to FIT-equivalent equations! ☺

[e.g. Rylander/Bondeson 2000]

Mass Matrix: 2) Tetrahedral grids

$$B_{ij} = \int \varepsilon \vec{w}_i \cdot \vec{w}_j \, dV$$



- Dot products $\vec{w}_i \cdot \vec{w}_j$ always $\neq 0$, no diagonal matrix (neither in systems with orthogonal dual grid)

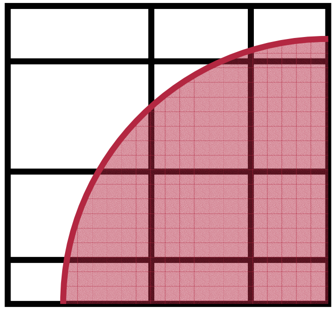
- Mass lumping? ⇒ can lead to indefinite matrices (in-stable !) [Bossavit / Kettunen 1999, 2001]

⇒ For tetrahedral / triangular grids no explicit time domain scheme with FE-matrices is available

FIT: Conformal Modeling

– Another way to gain flexibility in geometric modeling:

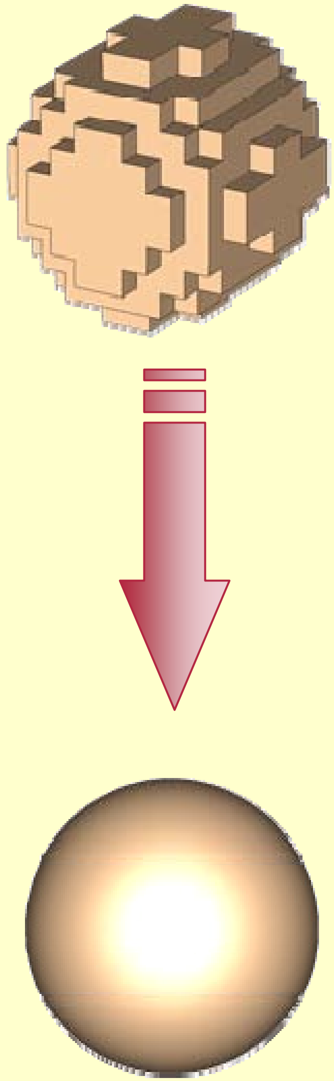
„Conformal modeling“ with Cartesian grids



allow **„partially filled cells“**

= interfaces to perfect conductors (PEC) within cells

task of grid: only allocation of components
material distribution practically independent of grid!



– main advantage: still Cartesian grids + diagonal material matrices

Analyze Stability Situation

- effective material parameters in reduced cells:

$$\varepsilon_{eff} = \varepsilon \frac{L_{ijk}}{l_{ijk}} \nearrow \qquad \mu_{eff} = \mu \frac{s_{ijk}}{S_{ijk}} \searrow \qquad \Delta t_{max} = \frac{\sqrt{\mu_{eff}\varepsilon_{eff}}}{\sqrt{\Delta x^{-2} + \Delta y^{-2} + \Delta z^{-2}}}$$

(edge reduction) (area reduction)

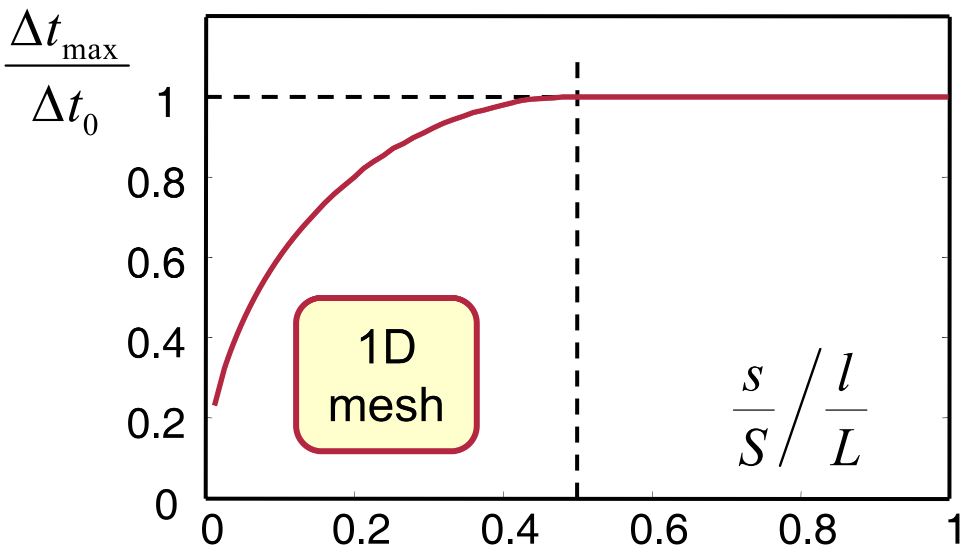
- formula for *critical cells*:

Δt has to be reduced, if

$$\frac{s_{ijk}}{S_{ijk}} < \frac{1}{2}\frac{l_{ijk}}{L_{ijk}}$$

relation between
facet- and edge reduction

(1D proof simple, 2D/3D empirical)



[I. Zagorodnov 2006]

Avoid Small "Effective" Cells

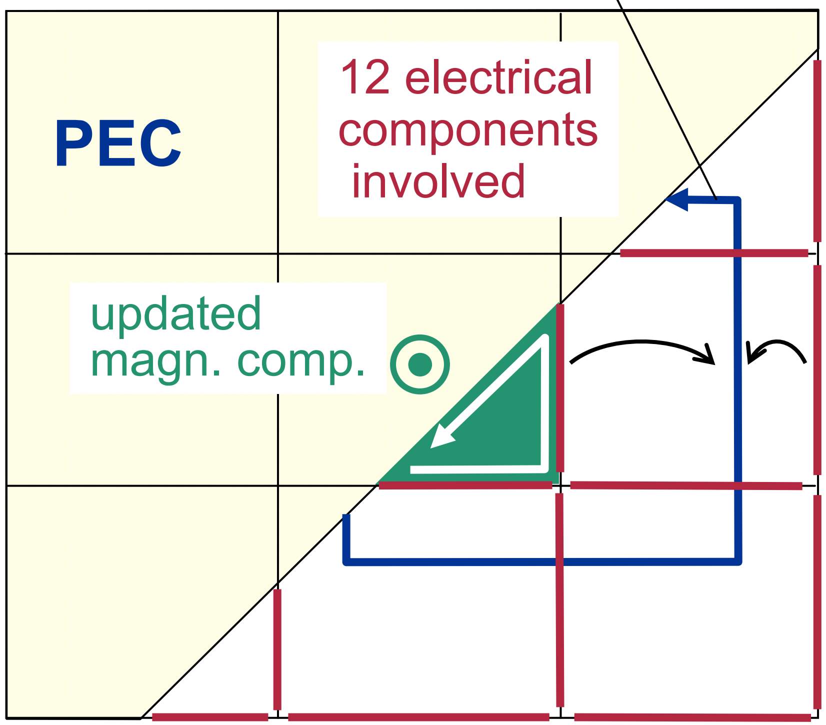
Uniform Stable Conformal scheme (USC, 2003)

- *virtually enlarge critical cells*: **enlarged local curl-operation** ("between" cells)
- unknown fields by interpolation (…)
- enlarged stencil, but only for a small number of critical cells

  (non-diagonal $\frac{1}{\mu}$ -matrix)
- still explicit (efficient), but complicated formulas

→ **no time step reduction !**

→ **full 2nd order accuracy of standard conformal FIT**



[with I. Zagorodnov]

SC-Scheme = Simpler Approach without Enlarged Stencil

– idea: directly enforce stability constraint
  (“stability correction” of conformal FIT-coefficients)

for critical cells only: $\frac{s_{ijk}}{S_{ijk}} < \frac{1}{2}\frac{l_{ijk}}{L_{ijk}} \xrightarrow{enforce} \frac{s_{ijk}}{S_{ijk}} = \frac{1}{2}\frac{l_{ijk}}{L_{ijk}}$

„s-method“: enlarge s_{ijk} (reduced area) in all coefficients concerned

„SC“-method“: reduce l_{ijk} (reduced edge) in all coefficients concerned

– some accuracy is lost, *but only as much as necessary*:
  → small number of cells: $N_{corr} \ll N_{PFC} \ll N_{tot}$
  → small amount of correction

[with I. Zagorodnov]