STRONG-STRONG BEAM-BEAM SIMULATIONS.

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Abstract

During the collision of two charged beams the strong non-linear electromagnetic fields of the two beams perturb each other. This effect is called beam-beam interaction. Of particular interest in present and future machines are studies of the behavior of equally strong and intense beams, the so-called strong-strong beam-beam interaction. After a careful definition of strong-strong beam-beam effects, I describe the applications where such studies are required. A major issue for strong-strong simulations are the computational challenges which are discussed. Finally I shall describe some of the modern techniques and procedures to solve them.

INTRODUCTION

The spectra of the barycentric motion and the mode frequencies of coherent beam-beam modes are well known and understood for the case of few bunches colliding head-on [1, 2]. In order to get higher luminosity, present and future colliders rely on a large number of bunches and multiple interaction points. The consequences are parasitic long-range interactions and a much richer spectrum of modes because many bunches couple through this non linear beam-beam force [3, 4]. This is in particular true when the collision points are not symmetrically distributed. Additional effects due to non-symmetric collision schemes [5, 6] or asymmetric machine optics [10] must be expected. The frame is further complicated due to PACMAN and super PACMAN effects, beam parameters variations (e.g. emittance and/or intensity fluctuations) as well as synchrotron motion [2]. It must be expected that these effects will lead to different coherent modes and in particular to different Landau damping behavior. In particular in the LHC 2808 bunches in each beam will be brought into collision in four experimental regions where they will experience four head-on collisions at the interaction points (IPs) as well as several long-range beam-beam interactions around the IPs. Moreover the accelerator layout presents several characteristics which break the symmetry between the collision points:

- Asymmetric configuration of the collision points
- Presence of a large number of parasitic long-range beam-beam interactions around the IP
- Unavoidable PACMAN and super PACMAN effects [7, 8]
- It is impossible to make the bunches collide exactly head-on [11, 12].

In the case of multiple head-on collisions these coherent modes can be analyzed with a linearized model for the beam-beam force searching for the eigenstates of the full single turn map. However, when the non linear long-range interactions are included, the linearized treatment is not adequate, for larger oscillating amplitudes one might expect a fairly large number of modes which may obscure tune measurements and/or feedback systems. The presence of a large number of modes due to the effect of local, parasitic interactions was already studied in [13, 14] with PACMAN effects and for a simplified LHC collision scheme. The COherent Multi Bunch Interactions program (COMBI) easily simulates a large number of bunches for any arbitrary collision or filling scheme as described in [15, 16]. The beam bunches can be simulated with different models depending on the effects of interest. A first modeling [15] was done using point like rigid bunches as in [13, 14] however, this simplified approximation cannot reproduce important effects such as Landau damping due to the natural tune spread of particles in a bunch. For this reason the COMBI program was extended to a multi particle version. A multi particle treatment reproduces damping mechanisms present in real machines and for this it is a good way to qualitatively understand the beam-beam interactions. One then can obtain quantitative results depending on the beam-beam field calculation used and on the particle distribution modeling. The particle distribution model and number and much more the field calculation used in the program are very important since they determine the speed of the simulations. Simulating several thousand bunches of 10^6 particles each interacting in several points along an accelerator for at least 2^{16} turns can take weeks and months of running time. It is foreseen to change the COMBI to allow parallel processing. In a first step I have used a Gaussian distribution for particles of a bunch and from this distribution the beam-beam kick is calculated at each interaction region by using the Gaussian particle weight and the particle barycenter. In particular I will focus here on the features of the three different methods. In addition I will present the main idea of the parallel version of COMBI with some preliminary results.

BEAM-BEAM MODELS

To understand multi bunches beam-beam coupling I use three different methods [17] depending on the different cases could help explaining the multi bunch interaction effects. In this report I will first describe the different methods and approximation used in COMBI and afterward I will compare simulation results obtained from the three models. Advantages and limitations of the different models will also be explained.

COMBI Analytical Linear Model (ALM)

Beam-beam coherent modes from localized interactions can be analyzed with a linearized model for the beam-beam force by searching the eigenmodes of the full one turn map [18]. For few equidistant bunches with multiple head-on collisions the effects are well understood [19, 10]. In this report I extend the method to more complicated beam filling schemes, such as bunch trains, and include long-range interactions. For the beam-beam force I use a linear approximation for the head-on as well as for the long-range interactions. This is justified when the separation is large enough and the amplitude of the oscillation small. The two counter rotating beams are described by a one column vector where I have the horizontal and vertical positions and angles of all bunches of beam 1 followed by all bunches of beam 2. In Eq. 1 I show only $x_{1^{b_1}}$ and $x'_{1^{b_1}}$ that represent the horizontal position and angle of bunch number one of beam one and below the same variables for beam two.

$$\left[\ x_{1^{b_1}}, x'_{1^{b_1}}, \ldots, x_{1^{b_2}}, x'_{1^{b_2}}, \ldots\ \right] \tag{1}$$

Through the arcs positions and angles are transformed by a simple rotation in phase space:

$$A_{x,y} = \begin{pmatrix} \cos(\Delta\mu_{x,y}) & \sin(\Delta\mu_{x,y}) \\ -\sin(\Delta\mu_{x,y}) & \cos(\Delta\mu_{x,y}) \end{pmatrix} \tag{2}$$

where $\Delta\mu_{x,y}$ represents the phase advance in the arc. In the case of n bunches I have a band diagonal matrix with the sub matrices A_x and A_y. Between the linear transfers I define a matrix for head-on and long-range BBIs. For both cases I use a linear approximation for the beam-beam kick and the sub matrix for two bunches colliding in the horizontal plane becomes:

$$\begin{pmatrix} x_1 \\ x'_1 \\ x_2 \\ x'_2 \end{pmatrix}_{s+1} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ -k_x & 1 & k_x & 0 \\ 0 & 0 & 1 & 0 \\ k_x & 0 & -k_x & 1 \end{pmatrix} \begin{pmatrix} x_1 \\ x'_1 \\ x_2 \\ x'_2 \end{pmatrix}_{s} \tag{3}$$

where k_x represents the horizontal linear beam-beam kick computed from the local derivative of the force. In case of head-on kick, k_x is given by:

$$k_x = \frac{2\pi\xi}{\beta_x} \tag{4}$$

where ξ is the beam-beam parameter and β_x the beta function at the collision point. The long-range kick depends on the beam separation one inputs [15]. The positions of the coupling terms in Eq. 3 depend on the bunch filling scheme and collision pattern. The one turn matrix M_{turn} is then obtained by multiplying the transfer and beam-beam matrices until one turn is completed.

The ALM enables to compute eigenvalues and eigenvectors of the system of coupled bunches and therefore to obtain the complete set of oscillation frequencies of the system dipolar modes. The calculation is fast and gives all oscillation modes. However the linear approximation of the beam-beam kick leads to an incorrect quantitative picture of the tune shifts in the tune spectra. The power of this method is that analyzing the eigenfrequencies and eigenvectors helps understanding bunch to bunch differences in the tune spectra produced with the other two models in COMBI [17]. ¿From the oscillation patterns provided by the ALM, it is possible to predict and understand the presence of different dominant modes for different bunches.

COMBI Rigid Bunch Model (RBM)

The rigid bunch approximation described in details in [15] is a first COMBI version that defines particle bunches as rigid objects where the particle distribution is assumed Gaussian with fixed RMS defined as a constant for all bunches and all time. From this values and from the bunch position the beam-beam kick is calculated by applying the formula for the coherent beam-beam kick in [5]. Between the beam-beam interactions usually a linear transfer (rotation in phase space) is applied however can be also a transverse kick from kickers, collimators or measurements devices. A Fourier analysis of the bunch barycenter turn by turn gives the tune spectra of the dipolar modes. The RBM is useful to study coupling of multi bunch beams and especially to study effects of collision scheme symmetries as well as of the beam filling structure. The processing time is short and gives the possibility to obtain sufficient amount of data on the mode frequencies for different and complicated interaction patterns. However, the model can only give qualitative results without taking into account damping effects. Many of the oscillating modes resulting from this model are expected to be damped when a multi particle simulation is used. Moreover the beam-beam effect is underestimated due to the approximation used for the particle distribution. A quantitative and correct approach requires multi particle bunch tracking even if the computing time will strongly increase. However this approximation will be used as comparison to better understand coupling and damping mechanisms in between bunches undergoing beam-beam interactions.

COMBI Multi Particle Simulations (MPS)

The COMBI-MPS [20] is a multi particle beam-beam code which allows to:

- Track individual macro particles of different bunches independently recalculating bunch parameters at each interaction point for a self consistent field evaluation
- Apply head-on and long-range beam-beam interactions at bunch encounters
- Give initial kicks to single bunches or a range of bunches to simulate excitation (e.g. for tune measurement)
- Analyze the motion and/or parameters (e.g. emittance, barycenter) of selected or a range of bunches of the beam structure,

- Show the complete set of possible coupled beam-beam modes

In the present version of COMBI-MPS bunches consist of Gaussian distribution of N_{tot} macro particles which lead to a 'soft' Gaussian approximation for the field calculation; with this approximation I can study just qualitatively the beam-beam interactions. A bunch is assumed to be populated by a Gaussian distribution of particles (10^4:10^6) with an initial given barycenter and squared transverse size. At each interaction each particle of the bunch involved receives a kick that depends on the particle distribution of the opposite bunch. Afterwords the merging particle distribution is still assumed Gaussian but with recalculated different barycenter and transverse size. However the program is not limited to this approximation and to get a fully self-consistent evaluation of beam-beam effects the field solver will be implemented to the HFMM (Hybrid Fast Multiple Method)[21]. A Fourier analysis of the bunch barycentre's turn by turn gives the tune spectra of the dipole modes while the bunch sizes give the emittance behavior of the bunches. As the RBM, the MPS is flexible and parameters such as tunes, number of bunches, filling scheme, collision scheme and crossing planes at the interaction points can be easily changed. The possibility to change the phase advance between collision points is very important and different phase advances are possible for the two counter rotating beams. Bunch intensities and emittance fluctuations can be simulated. The possibility to have different filling schemes for the two beams allows to study effects arising from beam asymmetries as well as demonstrating the so called PACMAN and super PACMAN effects which could play an important role in complex hadron colliders as for example the LHC [8]. In order to get all correct modes of the bunches coupled by head-on and long-range interactions, all individual interactions are simulated in full without lumping several interactions.

Advantages and disadvantages

All three models give useful and different informations about the beam-beam force coupling of multi bunch beams. Depending on the studies one needs to do it is possible to choose the model more appropriate to describe the phenomenon. These models have intrinsic advantages and disadvantages (Tab.1) and used together they can provide a deeper insight into the underlying physics.

In Tab.1 I schematically show for a defined property which model is more appropriate. For example if I need to simulate thousands of bunches colliding as for example in the LHC of course the multi particle simulation will take months while the rigid bunch and moreover the matrix will give really fast qualitative results. One the other hand the matrix model assumes a linear approximation of the beam-beam force and for this will not give information about the non linearities of the system while the multi particle will. Damping mechanism as well as higher order modes can be reproduced and studied only by using the multi particle treatment. A clear disadvantage of MPS is the large processing time required therefore to reduce it to a reasonable order MPS is now reconfigured for a multi processor system.

Parameters	ALM	RBM	MPS
calculation speed	++	0	–
non linear effects	-	+	++
Landau damping	-	-	++
higher order modes	-	-	++
correct field calculation	-	-	+(+)
flexibility	0	++	++

Table 1: Schematic summary on the different models performances and result goodness with respect to a given parameter one is interested in

VALIDATION AND RESULTS

To validate and demonstrate the programs features, I have tried to reproduce properties and well known effects of beam-beam interactions. As first step I analyzed single and multiple head-on interactions between bunches, the dependence on the collision scheme symmetries and the appearing of modes damping. Simulations are performed for equal charges colliding beams therefore frequencies are shifted downward from the unperturbed tune by the head-on collisions and upward for the long-range interactions. For the bunch population I used a sample of 10^4 macro particles per bunch for at least 2^{16} turns to produce tune spectra while for emittance studies samples of 10^6 macro particles per bunch tracked for at least 2^{16} turns are required. These choices are supported by a dedicated simulation campaign of the frequency resolution and stability of the results as a function of the number of macro particles processed and of the number of turns. In order to acquire a sufficient resolution and accuracy of the coherent modes with respect to the incoherent continuum background, the number of macro particles processed must be larger than 5000 and they must be tracked at least for 2^{14} turns. Both parameters can easily be changed using the input files.

Multiple Head-on collisions

For the head-on collisions I have simulated several cases from simple to more complex to see the rising of coherent modes and their shifts with respect to the unperturbed tune frequency as a function of the number of interactions.

In Fig.1 I show the horizontal tune spectra of two bunches undergoing variable number of head-on collisions. Red lines refer to results from the RBM while blue lines are results from the MPS. The abscissa shows the tune shift normalized to the linear beam-beam parameter ξ. For the rigid bunch model I expect a coherent maximum tune shift of $n \cdot \xi$, where n is the number of head-on collisions.

In the case of a simple linear transfer without beam-beam interactions I observe a coherent mode at the unper-

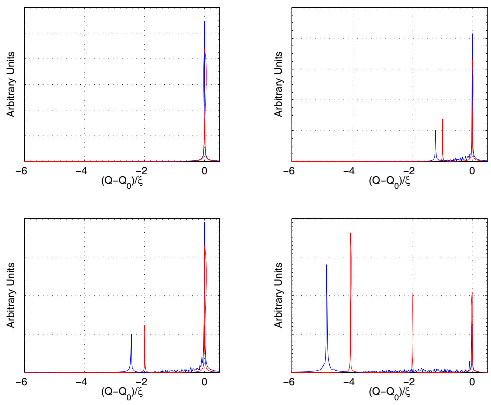

Figure 1: Tune spectra for different collision pattern using the two bunch approximations (rigid bunch model red lines and the multi particle model blue lines). A simple linear transfer along the machine shows the unperturbed σ mode (upper left). Linear transfer plus 1 head-on collision σ and π mode visible with in between the incoherent spectrum (upper right). Linear plus 2 head-on opposite in azimuth (lower left). Linear transfer plus 4 head-on collisions (lower right)

turbed tune Q_0 (see Fig.1 upper left). Turn after turn the distance between the bunches doesn't change and therefore there is no interaction in between. If I add a head-on collision, the bunches couple through the beam-beam force and start oscillating coherently at two different frequencies the σ and the π mode (see Fig.1 upper right). Differences in the absolute value of the tune shifts between rigid and soft spectra are well known and understood [1]. In rigid bunch results the π mode is shifted by one ξ from the unperturbed σ mode while with the multi particle Gaussian bunch approximation the π mode is shifted by Y times ξ, where Y is the so called Yokoya factor and is equal to approximately 1.21 as calculated in [1] for round Gaussian beams. For the MPS spectrum I find a continuum incoherent spectrum that goes from the σ mode down to one ξ. With two head-on collisions opposite in azimuth the beam-beam tune shift simply adds up and I find a maximum shift that is twice the one obtained with only one collision (Fig.1 lower left). The maximum tune shift obtained with the rigid bunch model is -2ξ while for the multi particle it is $-2Y\xi$. In the case of four head-on collisions (Fig.1 lower right) I obtain with the RBM a π mode at -4ξ while for the MPS it is at approximately at $-4Y\xi$. For the rigid model (Fig.1 lower right, red line) I have three modes: one at the unperturbed tune, one at -4ξ and an intermediate one at -2ξ. With the multi particle the picture is different (Fig.1 lower right, blue line): between the two coherent modes I only have the incoherent continuum spectrum that goes from the σ mode down to approximately -4ξ. This is a clear example of Landau damping effects due to the particle frequency spread caused by the non linear beam-beam interactions. This mechanism suppresses coherent modes in the neighborhood of the σ mode. Modes outside the incoherent spectrum [0,-4ξ] are still present while modes inside the incoherent continuum are completely damped, in agreement with the expectations. The same cases were solved with the ALM and the eigenfrequencies obtained with the matrix formalism lies at the same value obtained with the RBM. For head-on interactions only, the bunches oscillate in the small amplitude range of the beam-beam force which is linear in the approximation of small oscillation.

Eigenvectors to predict the oscillation pattern of bunch trains and bunch to bunch differences

As presented in [17] the analytical linear model based on a one turn map approach can be used to understand and predict bunch to bunch differences by looking at the eigenvectors of the system as done for the simplest case by Piwinski in [18]. In the case of one bunch beams colliding head-on in one IP, one obtains two possible oscillating states. As already explained, the tune spectrum will show two peaks corresponding to the unperturbed betatron frequency Q_σ and, shifted by the coherent beam-beam tune shift ξ, to the perturbed one Q_π. The related eigenvectors give for each eigenfrequency the relative phase and amplitude of the bunches. Two clear modes are known: the σ-mode at which bunches oscillate at same amplitude and in phase and the π-mode at which bunches oscillate at same amplitude but out of phase. In the case of four bunches

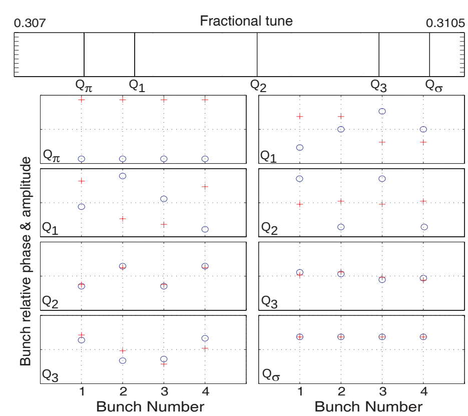

Figure 2: Eigenfrequencies and eigenvectors of a system of four bunches colliding head-on in two IPs.

per beam colliding head-on in two non-symmetric IPs the number of possible modes increases and one expects different oscillating patterns for each eigenfrequency. I find the σ and π modes and six intermediate modes for a total of five eigenfrequencies as in Fig. 2. The difference between the two extreme modes is 2ξ due to the two head-on collisions. These modes correspond to the mode in [18]. For a given eigenfrequency one can identify the contribution of individual bunches by inspecting the corresponding eigenvectors. The main goal of these studies is a better understanding of the bunch to bunch differences observed in the RBM and MPS when simulating bunch trains. Depending on the coupling, i.e. collisions, bunches of a train show different spectra. The Figs. 3 and 4 show the tune spectra obtained with the MPS "observing" only the first and third bunches of a train of five undergoing one head-on and one long-range interaction at one IP. Due to the long-range interaction all bunches are coupled and as a result sidebands around the σ and π modes appear. This coupling leads also in a breaking of symmetries therefore no mode degeneracies are visible (Fig. 5 for each eigenfrequency one eigenvector). The system has less degrees of freedom for the coherent motion. The spectrum of the first bunch in the train (Fig. 3) shows ten frequencies as obtained with the ALM (Fig. 5 top). Bunch number three shows only seven (Fig. 4). Bunches contribute differently to the different modes.

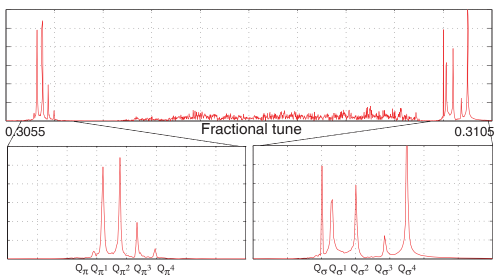

Figure 3: Top: MPS tune spectrum of the first bunch of a train of five undergoing head-on and long-range collision. Bottom: zoom of the sidebands around the π and σ-modes.

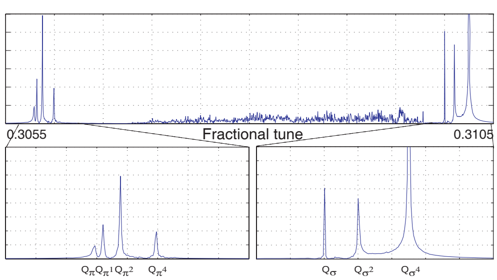

Figure 4: Top: MPS tune spectrum of the third bunch of a train of five undergoing head-on and long-range collision. Bottom: zoom of the sidebands around the π and σ-modes.

Looking at the eigenvectors in Fig. 5, the ones corresponding to the missing and/or smaller amplitude frequencies of Fig. 4, (Q_{π^1}, Q_{π^3},Q_{σ^1} and Q_{σ^3}), show that bunch three remains at zero level and is not contributing to the related coherent mode. The oscillating pattern of mode Q_{π^4} shows that bunch one of the train is contributing less with respect to bunch three which explains the differences in amplitude in Figs. 3 and 4. The analysis of the eigenfrequencies and especially of the eigenmodes obtained from the ALM model allows understanding the oscillation pattern of multi bunch modes for different eigenfrequencies. Moreover, using this model it is possible to predict the different responses of individual bunches to measurements. This enables us to correctly interpret the observations from single bunch measurements.

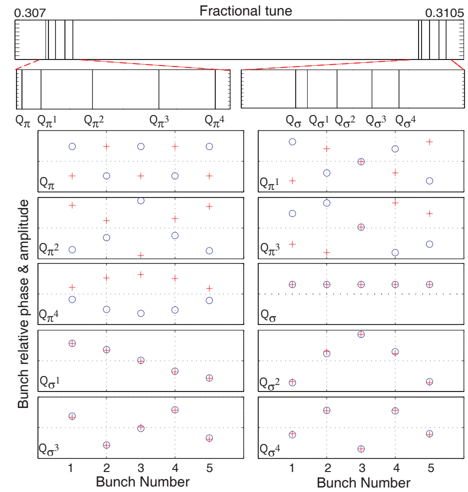

Figure 5: Eigenfrequencies and eigenvectors of bunch trains undergoing head-on and parasitic collision.

COMBI IN PARALLEL MODE

For a fully self-consistent beam-beam simulation one must use the MPS. The limiting factor of this model is that it is extremely time consuming. A simplified LHC simulation which includes 36 bunch beam described by a sample of 10^4 macroparticles colliding only head-on in the 4 IPs for about $5 - 6$ seconds physics run requires more than 1 week CPU time on a 3 GHz Pentium 4 PC. Therefore already at the design stage the COMBI-MPS was developed to allow an easy implementation of a multi processor architecture. A collaboration between CERN, TRIUMF and EPFL is established for the development of the parallel version of the COMBI code, a first version of the code is now under validation. The main idea of the parallel processing is sketched in Fig. 6. Bunches of the two beams are disheahributed amongst the available CPUs where each bunch is kept resident and where all calculations (i.e. beam-beam interactions, linear transfer, etc.) involving that bunch are performed. A MASTER CPU is used to take record of the beam stepping through the collider and collision pattern to address the bunches. When an action between two bunches is required the master comunicates it to the bunches involved which then exchange the relative information to proceed locally with the calculations. Simulations of multiple head-on beam-beam collisions were performed on the EPFL MIZAR cluster (448 CPUs) [22] and preliminary results are consistent with the expectations. The computing timing suggests a good scalability even if details of the code performance can be given only after dedicated studies which are on going at the moment.

CONCLUSIONS

The COMBI code has been developed and is used to study beam-beam coherent and incoherent effects in a

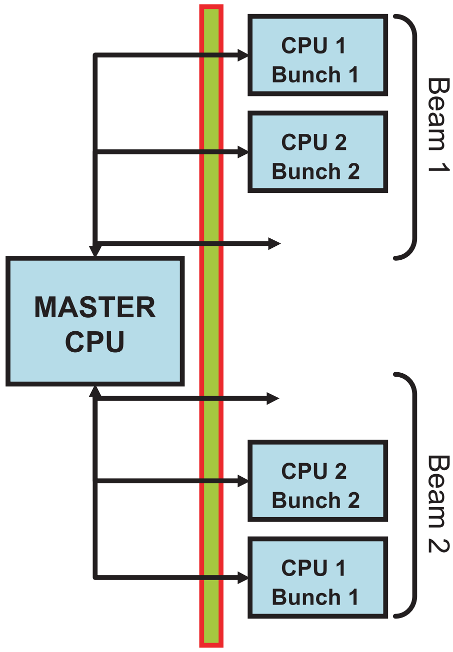

Figure 6: Sketch of the parallel implementation of COMBI-MPS code.

strong-strong regime for any beam filling scheme and collision pattern. The three complementary methods of COMBI are used to study coherent dipolar modes in complex multi bunches coupling cases and allow predicting bunch to bunch differences in tune spectra. Incoherent effects like emittance growth can be addressed as well as different damping properties of bunches can be studied. Instrumentation devices, kickers and collimators transverse kicks effects can be reproduced and higher order modes can be evaluated. The code therefore addresses the needs of predicting bunch to bunch differences (i.e. PACMAN and super PACMAN bunches) for diagnostic purposes and investigating beam-beam effects for different and complex beam filling schemes and collision pattern. A parallel version of the COMBI code is being developed and gives good preliminary results for the multiple head-on effects. Scalability studies are on going as well as a benchmark of the program results with experimental data from existing colliders. The parallel version of the COMBI code will allow a fully self consistent strong-strong beam-beam simulation of the complex LHC beam-beam interaction scenario.

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