A Space Charge Algorithm for the Bunches of Elliptical Crossection with Arbitrary Beam Size and Particle Distribution

A. Orzhekhovskaya, G. Franchetti, GSI, Darmstadt, Germany

Abstract

A general and precise method for calculating the electric field generated by an bunch with variable elliptical cross section is proposed. The particle distribution is fitted by a polynom of a proper degree. Field calculations by Kellog's formulae are made for this polynomial expansion. For an arbitrary 2D elliptic cross section of the bunch the analytic solution with hypergeometric function is derived. For a bunch with variable elliptical cross section we propose numerical methods. The high accuracy of these methods is benchmarked.

INTRODUCTION

The present work has been done in the frame of space charge modeling of high intensity beam for the new Facility for Antiproton and Ion Research (FAIR) in GSI [1]. Two new synchrotrons: SIS100 and SIS300 will be constructed and extend the chain of the existing accelerators, UNILAC and SIS18. The SIS100 scenario [1] foresees a beam loss control during a long term storage of 10^5 turns. However, space charge jointly with lattice nonlinearities has been proved dangerous for emittance growth and beam loss [2]. The effect of the space charge modeling on long term tracking is particularly challenging because of the artificial noise in the electric field deriving from standard Poisson solvers. For this reason an analytic model of the space charge for a bunched beam is developed. This model was also used in linac code benchmarking for the static comparison of different codes in frame of High Intencity Pulsed Proton Injector (HIPPI) project [3,4].

We restrict the modeling to the bunches with ellipsoidal symmetry, where the charge distribution is given by

$$\rho(x,y,z) = \frac{Q}{4\pi abc}\hat{n}(t), \qquad (1)$$

$$t = \frac{x^2}{a^2} + \frac{y^2}{b^2} + \frac{z^2}{c^2},$$

where Q is the total charge of the bunch, a, b, c are the bunch axis and t is the isodensity parameter. Function $\hat{n}(t)$ is normalized to

$$\int_0^\infty \hat{n}(t)dt = 1.$$

The following general formula for the electric field created by 3D ellipsoidal bunch is derived in [5]

$$E_x = \frac{Qx}{2}\int_0^\infty \frac{\hat{n}(\hat{T})ds}{(a^2+s)^{3/2}(b^2+s)^{1/2}(c^2+s)^{1/2}}, \qquad (2)$$

with

$$\hat{T} = \frac{x^2}{a^2+s} + \frac{y^2}{b^2+s} + \frac{z^2}{c^2+s}.$$

We consider the distribution $\hat{n}(t)$ on the interval $[0, T_{max}]$ so to describe all the physical space where a particle may be found during storage. The choice of T_{max} follows from the rms beam sizes and the beam pipe so $T_{max} \sim (x_{pipe}/a)^2$. For standard scenarios the pipe is approximately 10 times larger than the rms size of the beam, this yields $T_{max} \sim 100$. The quantities a, b, c in Eq. 1 and Eq. 2 are the rms sizes of the bunched beam. For distribution $\hat{n}(t)$ of polynomial form

$$\hat{n}(t) = \sum_{l=0}^{N} c_l t^l \qquad (3)$$

the electric field for 3D ellipsoidal bunch can be derived by substituting Eq. 3 into Eq. 2 [6]

$$E_x = \tfrac{Qx}{2}\sum_{l=0}^{N} c_l \sum_{i+j+k=l} \tfrac{l!}{i!j!k!} I_{i+1,j,k} x^{2i} y^{2j} z^{2k}, \qquad (4)$$

where

$$I_{i,j,k} = \int_0^\infty \frac{dt}{(a^2+t)^{0.5+i}(b^2+t)^{0.5+j}(c^2+t)^{0.5+k}}. \qquad (5)$$

The integrals $I_{i,j,k}$ are constants if the particle has isodensity parameter $t < T_{max}$. We can interpolate any arbitrary distribution $\hat{n}(t)$ on interval $[0, T_{max}]$ by a polynomial function $\hat{n}_p(t)$ of certain order N. The coefficients $c_0, ..., c_N$ of polynom Eq. 3 are defined by $N+1$ Chebyshev nodes

$$t_k = T_{max}\left\{\frac{1}{2} + \frac{1}{2}\cos\left[\frac{(2k+1)\pi}{2N+2}\right]\right\},$$

$$k = 0, .., N.$$

The obtained Lagrange polynom has the smallest maximal absolute error

$$M = max_{t\in[0,T_{max}]}|\hat{n}(t) - \hat{n}_p(t)|$$

in the class of polynoms of order N [7]. The optimal order of polynom N is chosen by calculating the dependence of M from N for every distribution $\hat{n}(t)$ and interval $[0, T_{max}]$. For example, for a Gaussian distribution on the interval [0,100] the optimal order of the interpolating polynom is $N = 22$ [6].

ACCURACY OF SOLUTION FOR 3D ELLIPSOIDAL BUNCH

In [6] it was proposed to calculate functions $I_{i,j,k}$ (Eq. 5) numerically by using Gauss Quadrature for 96 points. By comparison with Gauss Quadrature for 80 points it was found that the relative error in calculating $I_{i,j,k}$ is

$$\frac{|I^{(96)}_{i,j,k} - I^{(80)}_{i,j,k}|}{I^{(96)}_{i,j,k}} < 10^{-5}.$$

Now we apply the Lobatto method [8] for calculating Eq. 5 up to a given absolute accuracy. Let $I^{Lob}_{i,j,k}$ be the value of $I_{i,j,k}$ computed with Lobatto method with fixed accuracy 10^{-15}; $I^{Gauss}_{i,j,k}$ is value of $I_{i,j,k}$ computed by Gauss Quadrature for 96 points. The relative error

$$M = \frac{|I^{Lob}_{i,j,k} - I^{Gauss}_{i,j,k}|}{I^{Lob}_{i,j,k}} < 10^{-5}$$

for all i, j, k, where $i+j+k < N+1$ and $N = 22$, confirms the estimation of error found in [6]. The errors in interpolation $|\hat{n}(t) - \hat{n}_p(t)|$ and in the calculation of $I_{i,j,k}$ lead to a further error in Eq. 4. To estimate this influence we apply Lobatto method with accuracy 10^{-15} to compute the integral Eq. 2 for Gaussian distribution $\hat{n}(t) = e^{-t}$. Then we interpolate this distribution on the interval $[0, T_{max}]$ by the polynom $\hat{n}_p(t)$ and compare E^{Lob}_x with E_x computed by Eq. 4. This test has been done for 10^6 random test particles uniformly distributed on the interval $[0, T_{max}]$ with $|x| < 10a$, $|y| < 10b$, and $|z| < 3c$. Fig.1 shows the

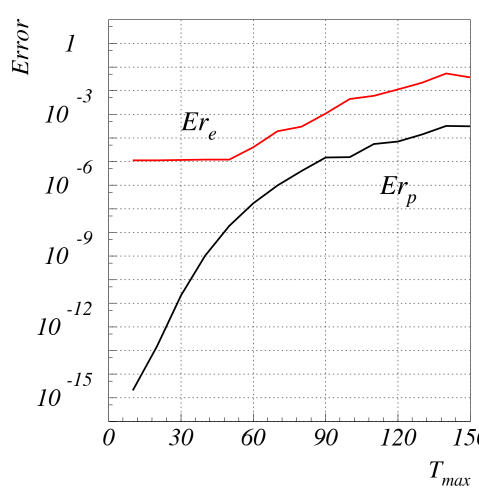

Figure 1: Dependence of maximum relative error of field calculation Er_e and maximum absulute error of distribution interpolation Er_p on the length of interval $[0, T_{max}]$ for $a = 0.001, b = 0.01, c = 10$.

dependence of maximum relative error

$$Er_e = max_{(x,y,z),t\in[0,T_{max}]} \frac{|E^{Lob}_x - E_x|}{E^{Lob}_x}$$

and maximum absolute error of interpolation

$$Er_p = max_{(x,y,z),t\in[0,T_{max}]} |\hat{n}(t) - \hat{n}_p(t)|$$

on $T_{max} \in [10, 150]$.

SOLUTION FOR 3D BUNCH OF VARIABLE ELLIPTICAL CROSSECTION

Until now we have considered charge distributions of type as in Eq. 1 with fixed sizes a, b, c. This type of distributions can be matched with a constant 3D focusing channel. In order to take into account the AG structure of a real ring we consider the long bunch ansatz: *for a sufficiently long bunch the transverse force is given by the local slice of bunch which resemble a piece of a coasting beam.* We change Eq. 1 with

$$\rho = \frac{\lambda(z)}{4\pi a(z)b(z)} \hat{n}\left(\frac{x^2}{a^2(z)} + \frac{y^2}{b^2(z)}\right), \qquad (6)$$

where $\lambda(z)$ is the local longitudinal charge density, $a(z)$ and $b(z)$ are the local transverse beam axis given by

$$a(z) = \sqrt{\epsilon_x \beta_x(z)}, \quad b(z) = \sqrt{\epsilon_y \beta_y(z)}. \qquad (7)$$

Note that Eqs. 6, 7 are applied for transverse matched beams. As in rings the functions $\beta_x(z)$ and $\beta_y(z)$ are periodic, we restrict the study to $z \in [0, P]$, where P is the length of one period. For each fixed z we have a 2D ellipse cross section with the particle distribution $\hat{n}(t)$, where

$$t = \frac{x^2}{a^2(z)} + \frac{y^2}{b^2(z)}.$$

The function $\hat{n}(t)$ is interpolated by the polynom $\hat{n}_p(t)$ of order N on the interval $[0, T_{max}]$. For 2D elliptic beam Kellog's formula Eq. 2 becomes

$$E_x = \frac{\lambda(z)}{2} x \sum_{l=0}^{N} c_l \sum_{i+j=l} \frac{l!}{i!j!} I_{i+1,j}(z) x^{2i} y^{2j}, \qquad (8)$$

where

$$I_{i,j}(z) = \int_0^\infty \frac{1}{[a^2(z)+t]^{1/2+i}[b^2(z)+t]^{1/2+j}} dt.$$

For each $z \in [0, P]$ and integer i, j we compute $a(z)$, $b(z)$ and then $I_{i,j}(z)$ by using hypergeometric function. However for large amount of particles this procedure is time consuming. In order to reduce the computational time for $I_{i,j}(z)$ we build a grid in z with ino nodes on the interval $[0, P]$. The functions $I_{i,j}(z)$ are previously computed for every i, j in grid nodes z_k, $k = 0, ..., ino$. Then they are interpolated at any arbitrary z. The choice of the interpolating method and number of nodes should be discussed additionally. The algorithm for computing the 3D electric field for an arbitrary particle (x', y', z') is following:

1. Initialization (is done once for all particles).

1.1 Choice of interpolation nodes z_k, $k = 0, ..., ino$.
1.2 Calculation of $\beta_x(z_k)$, $\beta_y(z_k)$ for any $k = 0, ..., ino$.
1.3 Calculation of correspondent $a(z_k)$, $b(z_k)$ with Eq. 7.
1.4 Calculation of $I_{i,j}(z_k)$ for any i, j, such that $i + j \le N + 1$, in $ino + 1$ nodes using the hypergeometric function:
if $a(z_k) > b(z_k)$, then

$$I_{i,j} = \frac{{}_2F_1(0.5 + i, i + j, 1 + i + j, 1 - b^2(z_k)/a^2(z_k))}{(i + j)a(z_k)^{2i+2j}},$$

if $a(z_k) \le b(z_k)$, then

$$I_{i,j} = \frac{{}_2F_1(0.5 + i, i + j, 1 + i + j, 1 - a^2(z_k)/b^2(z_k))}{(i + j)b(z_k)^{2i+2j}},$$

where

$${}_2F_1(l, m, n, k) = 1 + \frac{lm}{1!n}k + \frac{l(l + 1)m(m + 1)}{2!n(n + 1)}k^2 + ...$$

2. Calculation of E_x for an arbitrary particle (x, y, z).
2.1 Calculation of $I_{i,j}(z)$ for any i, j, such that $i + j \le N + 1$, using interpolating method.
2.2 Calculation of the electric field E_x for 2D case using Eq. 8.

APPLICATION TO SIS18

In the case of SIS18 the functions $\beta_x(z)$, $\beta_y(z)$ have a period $P = 18$ m. We use the linear interpolation for computing $I_{i,j}(z)$. The value of $I_{i,j}(z)$ with z between the nodes z_k and z_{k+1} is computed as

$$I_{i,j}^{interp} = I_{i,j}(z) = I_{i,j}(z_k)+$$
$$+\frac{z - z_k}{z_{k+1} - z_k}[I_{i,j}(z_{k+1}) - I_{i,j}(z_k)].$$

For 300 equidistant longitudinal nodes the linear interpolation gives a maximum relative error $|I_{i,j} - I_{i,j}^{interp}|/I_{i,j}$ of 10^{-8} (the number of nodes was chosen doing few tests).

APPLICATION TO SIS100

In the case of SIS100 the period of $\beta_x(z)$, $\beta_y(z)$ is of $P = 180.6$ m (status 2006). The presented calculations and tests have been done for $P = 198.6$ m (status 2005), but preliminary check for the new parameters shows that no remarkable difference in accuracy is expected. Functions $I_{i,j}(z)$ has big oscillations: the maximum variation of $I_{i,j}(z)$ for $i, j > 10$ is up to 10^{10}. In this condition the linear interpolation doesn't give good approximation. Therefore we use a quadratic interpolation. We interpolate $I_{i,j}(z)$ for certain z inside 1-st period of $\beta_x(z)$, $\beta_y(z)$: let z_k be the closest node to z, then $I_{i,j}(z)$ is obtained from

$$c_1 = z_{k-1}, \quad c_2 = z_k, \quad c_3 = z_{k+1},$$

$$d_1 = I_{i,j}(z_{k-1}), \quad d_2 = I_{i,j}(z_k), \quad d_3 = I_{i,j}(z_{k+1}),$$

$$I_{i,j}(z) = \frac{(z - c_2)(z - c_3)}{(c_1 - c_2)(c_1 - c_3)}d_1+$$

$$+\frac{(z - c_1)(z - c_3)}{(c_2 - c_1)(c_2 - c_3)}d_2 + \frac{(z - c_1)(z - c_2)}{(c_3 - c_1)(c_3 - c_2)}d_3.$$

The optimal number of equidistant nodes $ino = 5000$ was chosen with few tests in order to make smaller the relative error of $I_{i,j}$ in $z \in [0, P]$

$$M(z) = \max_{i+j<N+1} \frac{|I_{i,j}^{real}(z) - I_{i,j}^{interpol}(z)|}{I_{i,j}^{real}(z)}.$$

As shown on Fig.2, for 10^5 random values of $z \in [0, P]$ and for all i, j, such that $i + j \le N + 1 = 23$, the error $M(z)$ is less than 2.24×10^{-3} and for most of tested z-values $M(z) < 10^{-4}$.

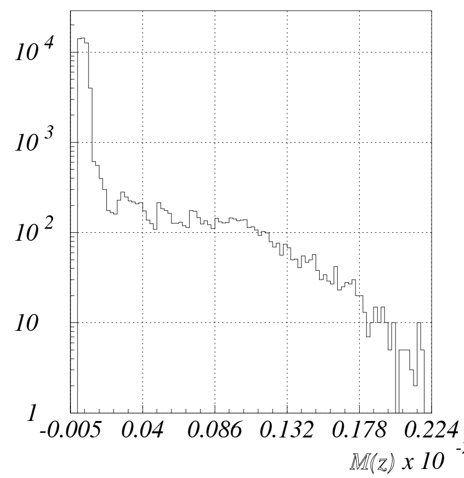

Figure 2: Maximum relative error $M(z)$ of $I_{i,j}(z)$ calculation on one SIS100 period

BENCHMARKING FOR SIS100

1. We repeat the test described in [6] with Gauss law

$$\vec{\nabla}\vec{E}(x, y, z) = 4\pi\rho(x, y, z) \quad (9)$$

to the method for calculating E_x, E_y to SIS18 and SIS100. We fix the logitudinal position $z \in [0, P]$. In this 2D ellipse cross section we interpolate Gaussian distribution $\hat{n}(t) = e^{-t}$ by the polynom $\hat{n}_p(t)$ on interval $[0, T_{max}]$ and take a set of 10^3 test particles (x, y) with

$$t = \frac{x^2}{a^2(z)} + \frac{y^2}{b^2(z)} < T_{max}.$$

For each particle we calculate E_x and E_y. Then from Eq. 6 and Eq. 9 we reconstruct distribution function $\hat{n}_{rec}(t)$ and compare it with the interpolating polynom $\hat{n}_p(t)$. For 10^3 test values of $z \in [0, P]$ the reconstruction error

$$M(z) = \max_{0<t<T_{max}} \frac{|\hat{n}_p(t) - \hat{n}_{rec}(t)|}{\hat{n}_p(t)}$$

is less than 10^{-3}.

2. In this test we fix the longitudinal position $z = 0$ and study the case of an axisymmetric beam ($a(z) = b(z)$). Locally we can consider this bunch as a piece of coasting beam. The analytical value of the electric field is [9]

$$E_{analyt} = \frac{\lambda(0)}{4\pi\epsilon_0 r}(1 - e^{-r^2/2a^2}).$$

We compute E_x, E_y and calculate $E = \sqrt{E_x^2 + E_y^2}$. For the set of 10^4 test particles uniformly chosen and satisfying $t < T_{max} = 150$ the relative error is

$$\frac{|E_{analyt} - E|}{E_{analyt}} < 10^{-5}.$$

3. We substitute E_x, E_y into the analytical formulae for E_x, E_y in the case of small x, y. As x, y are close to zero the following equation is found

$$E_x(x, y, z) = \frac{\lambda(z)b(z)x}{\epsilon_0[a(z) + b(z)]} + O(x). \qquad (10)$$

From Eq. 10 and the similar equation for $E_y(x, y, z)$ we find

$$\frac{yE_x}{xE_y} = \frac{b(z)}{a(z)}.$$

For 10^5 test particles $x \sim y \sim 0$ close to the longitudinal axis this equation holds with the relative error 10^{-5}.

4. In this test we compare 2 methods of calculation of the electric field: for 3D ellipsoidal bunch with rms sizes a, b, c and for a bunch with variable transverse elliptical crossection. When $c \ll a$ we can consider the ellipsoidal bunch locally as a piece of coasting beam. Then we can use the method for 3D ellipsoidal bunch and calculate $E_x^{3D}(x, y, z)$ for an arbitrary particle. Alternatively, we can use the method for the bunches of variable elliptical crossection with $\beta_x(z), \beta_y(z) = const$ and compute $E_x^{AG}(x, y, z)$. In the condition $c/a \to \infty$

$$E_x^{3D}(x, y, z) = E_x^{AG}(x, y, z). \qquad (11)$$

We now test if Eq. 11 holds for a finite aspect ratio c/a. Fig.3 shows the variation

$$D = \max_{(x,y,z)} \frac{E_x^{3D}(x, y, z)}{E_x^{AG}(x, y, z)} - \min_{(x,y,z)} \frac{E_x^{3D}(x, y, z)}{E_x^{AG}(x, y, z)}$$

for 10^4 random test particles (x, y, z) with $|z| < 3c$ and for different aspect ratio. For example for $c/a > 10^3$ (as it is in SIS100) we found $D < 10^{-5}$.

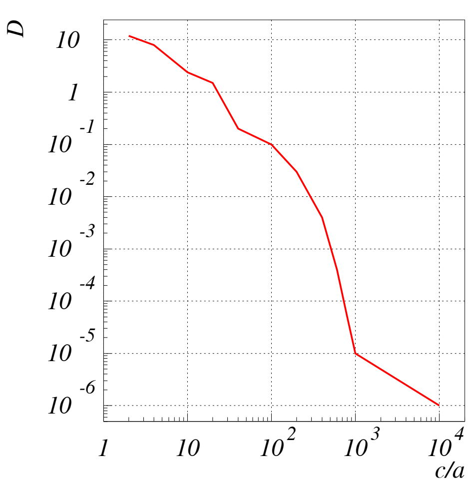

Figure 3: Variation D of ratio $E_x^{3D}(x, y, z)/E_x^{AG}(x, y, z)$ for $|z| < 3c$ and different values of c/a

CONCLUSION

A general and precise method for calculating the electric field in the case of an ellipsoidal bunch and a bunch of variable elliptical crossection is proposed. The arbitrary space charge distribution $\hat{n}(t)$ is fitted on the interval $[0, T_{max}]$, with $T_{max} = 100$ by the polynom of optimal order N using Chebyshev nodes. For Gaussian distribution $N = 22$. The electric field is calculatated for this polynomial expansion, that makes the general formulae for the electric field simple. Special functions $I_{i,j}$ have to be calculated only once that allows to increase the speed of calculations. For the bunch of variable elliptical cross section we propose a numerical method which uses longitudinal grid. The main part of the methods with the highest CPU consumption is done ones for all particle. The computer libraries with implemented these methods are included into MICROMAP library [10]. The tests of these methods demonstrate their high accuracy. Speed estimation shows benefits at least in 10 times for proposed method against Lobatto algorithm.

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