GEOMETRICAL METHODS IN COMPUTATIONAL ELECTROMAGNETISM

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Abstract

For almost one century (see [6]), it has been known that vector fields E, H, D, B, etc., in the Maxwell equations, are just "proxies" for more fundamental objects, the differential forms e, h, d, b, etc., that when integrated on lines or surfaces, as the case may be, yield physically meaningful quantities such as emf's, mmf's, fluxes, etc. This viewpoint helps separate the "non-metric" part of the equations (Faraday and Ampère), fully covariant, from the "metric" one (the constitutive laws), with more restricted (Lorentz) covariance. The usefulness of this viewpoint in computational issues has been realized more recently: it eases the way towards the "mimetic discretization" research program, which aims at providing the programmer with a "discretizing toolbox", parts of which can be assembled to create discrete models (consistent and convergent) of various electromagnetic situations. Some of those involving *forces*, where one must deal with energy and momentum transfer, are addressed.

DISCRETIZATION PRINCIPLES

Non-metric equations, discrete form

In 3D Euclidean space, with dot product " $\cdot$ ", norm $|\,|$, and standard orientation, let's consider the equations $\partial_t B + \mathrm{rot}\, E = 0$ and $-\partial_t D + \mathrm{rot}\, H = J$ (with current density J, not necessarily divergence-free, considered here as a data), completed by constitutive laws $H = \nu B$ and $D = \varepsilon E$, and initial conditions $B = 0$ and $E = 0$ at time $t = 0$. This makes a mathematically well-posed problem. Charge density Q derives from J by $Q(t) = -\int_0^t (\mathrm{div}\, J)(s)\, ds$, and Lorentz force, as needed to couple these equations with others, is $F = Q(E + v \times B)$. "Geometrical methods" discussed here put emphasis on integral quantities such as the emf along an oriented curve c (derived from E by $\int_c \tau \cdot E$, where τ is the unit tangent vector), the magnetic flux embraced by an oriented surface S (derived from B by $\int_S n \cdot B$, where n is the unit normal vector), etc., rather than on vector fields such as E and B. Accordingly, symbols e and b, associated with the physical entities "electric field" and "magnetic field" will denote the *maps* $c \to \langle \text{emf along } c \rangle$ and $b \to \langle \text{flux embraced by } S \rangle$, these bracketed real quantities being themselves denoted by $\int_c e$ and $\int_S b$. (Alternatively, for the sake of readability, I may use $\langle c\,;\, e \rangle$ and $\langle S\,;\, b \rangle$ instead.) In this mood, the first equation above, Faraday's law, rewrites as

$$\partial_t \int_S b + \int_{\partial S} e = 0 \quad \text{for all surfaces } S \tag{1}$$

(endowed with an orientation), where ∂S denotes the boundary of S, with orientation induced by that of S.

No shade of metric (length, area, dot or cross product, or metric-linked operators such as rot) remains in (1), so space is now construed as the naked 3D affine space A_3 (no orientation, no metric, but still, a notion of parallel transport by translations). Which is up to the point: It happens that (1) offers the most direct route to the discretization of Faraday's law, as follows.

Unable to pack infinite objects such as spatial domains and fields in a computer's innards, we build a discrete model of the computational domain D by setting up a cellular mesh m, and of the fields by assigning time-dependent degrees of freedom (DoF) to cells of this mesh. (A *simplicial* mesh is assumed in what follows, but this is not essential.) Cells of the mesh fall in sets $\mathcal{N}$, $\mathcal{E}$, $\mathcal{F}$, $\mathcal{V}$ of nodes, edges, faces, volumes. Each has its own (arbitrary) orientation, and the overall structure is described by incidence matrices. For instance, $\mathbf{R}_f^e = \pm 1$ if edge e is part of the boundary of face f, the sign depending on whether orientations match or not. Hence an $(F \times E)$-matrix $\mathbf{R}$, where E and F are the numbers of edges and faces in m. Same for the $(E \times N)$-matrix $\mathbf{G}$ between nodes and edges, and the $(V \times F)$-matrix $\mathbf{D}$ between faces and volumes. (One notes that $\mathbf{RG} = 0$ and $\mathbf{DR} = 0$ and that, for simple topologies of domain D, $\ker(\mathbf{R}) = \mathrm{ran}(\mathbf{G})$ and $\ker(\mathbf{D}) = \mathrm{ran}(\mathbf{R})$, which should be enough to motivate, by analogy with grad, rot, div, the notation.) A degree of freedom $\mathbf{e}_e$, interpreted as an emf [resp. $\mathbf{b}_f$, as a magnetic flux], is attributed to each edge e [resp., face f]. Boundary conditions are taken care of by assigning given values to some of these DoF's, and the other ones (the genuine unknowns, to be computed) reside at edges or faces we shall call *active* for this reason.

Now instead of enforcing (1) for all surfaces S, let's be content (this is the gist of the discretization strategy), with surfaces that are made of an assembly of faces of m. By linearity, this amounts to enforcing it for *each* f. Since $\partial f = \sum_{e \in \mathcal{E}} \mathbf{R}_f^e\, e$, by definition of the incidence numbers (explanations about this expression are forthcoming), one must have $\partial_t \mathbf{b_f} + \sum_{e \in \mathcal{E}} \mathbf{R}_f^e\, \mathbf{e_e} = 0$ for all f, i.e., in compact form with DoF-arrays,

$$\partial_t \mathbf{b} + \mathbf{R}\,\mathbf{e} = 0. \tag{1'}$$

(Some nonzero right-hand side can pop up there, as the result of non-homogeneous boundary conditions, and of

the restriction of DoF-arrays, and incidence matrices as well, to active cells, as will be done from now on.)

Ampère's relation, in a similar manner, could be discretized as $-\partial_t \mathbf{d} + \mathbf{R}\,\mathbf{h} = \mathbf{j}$, with h edge-based and currents d and j face-based. We won't do that (being content to point at the possibility, which may help with charge-conservation issues in charged-particle dynamics), because the fact that $\mathrm{F} \neq \mathrm{E}$, as a rule, would make the discretization of constitutive laws intractable. Instead, we assign DoF's $\mathbf{h}_f$ and $\mathbf{d}_e$ to *dual* cells $\tilde{f}$ and $\tilde{e}$ (Fig. 1), built from the "primal" mesh m by selecting a "center" (some point) inside each primal cell and by joining them as the figure suggests. (Two choices for such centers are popular: barycenters of primal cells, and centers of Euclidean circles, spheres, etc., circumscribed to them.) This way, dual cells are in 1–1 correspondence with primal ones (which will make easier, below, to discretize ν and ε), and each one inherits from its primal mate a so-called *outer* orientation (i.e., an orientation of a subspace that complements its support). This is auspicious as regards entities such as d and j, for these are to do with "substantial" fluxes (electric charge), whose very definition refers to a definite crossing direction (i.e., outer orientation) of surfaces they traverse.

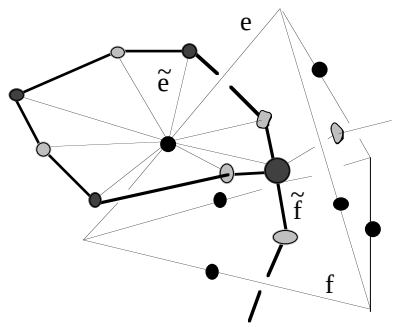

Figure 1. Centers, and how they generate a dual mesh.

Indeed, the expression of Ampère's law, in the style of (1), is

$$-\partial_t \int_\Sigma \mathrm{d} + \int_{\partial\Sigma} \mathrm{h} = \int_\Sigma \mathrm{j} \quad \text{for all surfaces } \Sigma \qquad (2)$$

endowed with *outer* orientation. (For constrast, the kind of orientation we had in (1) is dubbed *inner*.) Consistency in the discretization strategy suggests to enforce (2) for all Σ obtained by assembly of *dual* faces, and only those, and the same linearity argument as above leads to

$$-\partial_t \mathbf{d} + \mathbf{R}^{\mathrm{T}}\mathbf{h} = \mathbf{j}, \qquad (2')$$

with T for "transpose", since as one will readily see, incidence matrices for the dual mesh are $\mathbf{D}^{\mathrm{T}}$, $\mathbf{R}^{\mathrm{T}}$, $\mathbf{G}^{\mathrm{T}}$. In this equation, intensities $\mathbf{j}_e$ are computed, from the given J, by taking its flux across each $\tilde{e}$.

Remark. The duality between submanifolds (curves, surfaces, ...) and fields in (1) and (2) has physical bearing: One may consider the integration domains such as $\partial\mathrm{S}$ or Σ as models for *probes*—measuring devices—which measure something (emf, flow of electric charge) about the (physical) field, and integrands, e or j, as mathematical models for the field itself. This can be made even more striking by introducing *chains* (weighted linear combinations) of curves, surfaces, etc., hence vector spaces, and *cochains*, elements of their duals: chains model *probes*, cochains model *fields*, and duality products $\langle \text{chain} ; \text{cochain} \rangle$ model *observations* (real numbers, as a rule, read off from the probes' dials). Think for instance of a voltmeter, with its connecting wires, as a 1-chain, of a fluxmeter as a 2-chain, with reference to dimension. (The generic name is "p-chain".) ◊

Remark. A benefit of building such linear structures is that ∂, the boundary operator, now becomes a linear map, from p-chains to (p – 1)-chains. This justifies using the above expression $\partial \mathrm{f} = \sum_{\mathrm{e} \in \mathcal{E}} \mathbf{R}_{\mathrm{f}}^{\mathrm{e}}\,\mathrm{e}$. This also prompts one to consider the dual d of ∂, defined by $\langle \mathrm{c} ; \mathrm{d}\omega \rangle = \langle \partial \mathrm{c} ; \omega \rangle$ $\forall$ c. Hence a *local* version of (1) and (2), $\partial_t \mathrm{b} + \mathrm{de} = 0$ and $-\partial_t \mathrm{d} + \mathrm{dh} = \mathrm{j}$, which is closer to the received (Heaviside's) notation, especially if one notices that grad, rot, div, are avatars of d, in Euclidean garb. Note however that the integral form (1)(2) is the shortest way to discretization. ◊

Remark. Not only linearity, but *continuity*, of maps e, b, h, etc., is an issue, especially in convergence proofs. Hence a need, fortunately satisfied by recent mathematical developments [4], for *topologies* on chain spaces, a subject computer users can safely ignore. The vocabulary (should one speak of "cochains", or of "differential forms"?) is still unsettled. Here I'll use the shortest moniker, "p-forms", for cochains of degree p, i.e., dual objects with respect to p-chains. *Straight* and *twisted* p-forms, respectively, are dual to inner- and outer-oriented p-chains. Thus, b, e, and also a (the 1-form whose proxy vector A is the vector potential), are straight, whereas h, d, j, and q (the 3-form whose proxy is Q above) are twisted. (Qualifiers *even* and *odd* would do as well, and are in use.) ◊

Remark. In $\partial \mathrm{f} = \sum_{\mathrm{e} \in \mathcal{E}} \mathbf{R}_{\mathrm{f}}^{\mathrm{e}}\,\mathrm{e}$, the 1-chain on the right-hand side, being based on 1-cells of m, is called a *cellular* chain (*simplicial* chain when the primal cells are simplices). Cellular p-chains make a finite-dimensional vector space, whose dual consists of DoF arrays based on p-cells. For this reason, "DoF array" and "cellular cochain" are synonyms. We shall need to distinguish such cellular chains and cochains ("discrete" objects) from chains and cochains in general ("continuous" objects, in the Numerical Modelling jargon). So we shall refer to the latter as *singular* chains and cochains, borrowing the term from Homology [5]. ◊

At this stage, we know how to discretize the "metric-free" part of the Maxwell equations, and hence, we expect metric notions to be involved in the next task, building discrete analogues—square matrices $\boldsymbol{\nu}$ and $\boldsymbol{\varepsilon}$—for the operators ν and ε that appear in the constitutive laws,

$$\mathrm{h} = \nu \mathrm{b}, \qquad (3) \qquad\qquad \mathrm{d} = \varepsilon \mathrm{e}. \qquad (4)$$

Metric features: Hodge operators

For *operators* they are, and must be: h is a twisted 1-form, b a straight 2-form, objects of different types, which rules out a simple proportionality relationship. (Same with (4), which will be left aside here, the transposition by analogy being easy: e is 1-straight, d is 2-twisted.) Since the relation $H = \nu B$ links proxy vectors, which depend on the metric, we need metric structure to *define* the ν of (3)—the operator— in the first place. And just as "discretizing vector fields", and differential operators, was *not* the right way to obtain (1') and (2'), introducing a dot product does not make it easy to pass from ν to $\boldsymbol{\nu}$. Instead, we establish a correspondence between vectors and bivectors (the infinitesimal versions of curves and surfaces), first, then proceed by duality.

A *vector at* x, element of the tangent space of A_3 at point x, can as well be considered as (a) an oriented line through x (its support), plus (b) a segment on this line. By analogy, a bivector at x is made of (a) an oriented plane through x, (b) an area borne by this plane. Two independent vectors $\{v, w\}$, in this order (which specifies the plane's orientation), determine a 2-vector, denoted $v \vee w$. (It's safe to think of $v \vee w$ as a parallelogram, but be aware that $(v + \alpha w) \vee w$ is the same 2-vector, the latter actually being an equivalence class of figures of same oriented area in the plane.) A *twisted* vector is a line through x with outer orientation, i.e., a gyratory sense around it, plus a length. This will be enough for now—and how to define p-vectors and twisted p-vectors should be clear. (Notice how the distinction between straight and twisted multivectors is erased when ambient space is oriented.)

When a metric and orientation exist, there is a natural way to map a vector v to a 2-vector $w = \perp v$ (called its "perp" by some): In the plane through x orthogonal to v, take the class of figures whose common area equals the length of v. If an isotropic (relative to this metric) and hence scalar permeability μ_0 is given, define νv as $\perp v/\mu_0$. Observe now that if $B = \mu_0 H$, as vectors go, the flux of B across νv equals the mmf $H \cdot v$, a fact that can be written $\langle v ; h \rangle = \langle \nu v ; b \rangle$. So if we reuse the symbol ν for the operator *dual* to ν, defined by $\langle v ; \nu b \rangle = \langle \nu v ; b \rangle \ \forall\, v$, we have $h = \nu b$, where now ν does appear as an operator, called after Hodge in differential geometry.

To do away with metric and orientation now, let's assume a linear map ν is *given* (instead of being *derived*, as above), which sends twisted vectors to straight 2-vectors. Under the sole non-degeneracy condition $v \vee \nu v \neq 0$ (with due apologies for the visual pun), the extra structure over the tangent space provided by this "geometric Hodge" operator is enough to induce a metric, called the "ν-adapted" one. (The proof-idea for this "Hodge implies metric" result is simple: Noticing that all 3-vectors are scalar multiples of one of them in dimension 3, select a reference 3-vector Δ, define the norm $|v|$ by the equality $|v|^2\Delta = \lambda^2 (v \vee \nu v)$, and adjust the real parameter λ to make volumes and lengths compatible. Technical details on this are in [2].) Again, the Hodge operator ν we need, from 2-forms to twisted 1-forms, is the dual, defined by $\langle v ; \nu b \rangle = \langle \nu v ; b \rangle \ \forall\, v$. (Why a similar, but distinct operator ε will also be needed is easy to understand: the real action is in 3 + 1 dimensions.) We now know the status of ν and ε in (3)(4).

Discrete hodges

Why such a contrived way to confer metric on space? Because, again, this makes discretization almost automatic. The heuristic rule is that, everything to do with straight, resp. twisted, geometrical objects must be modelled with primal, resp. dual, simplices of the mesh. This worked well, above, with surfaces S or Σ. Now we need to handle twisted vectors and straight 2-vectors. Assume the mesh is made of a single tetrahedron T, cf. Fig. 2. (This is no serious restriction: once obtained the (4×4)-matrix $\boldsymbol{\nu}_T$ for that one, we'll have $\boldsymbol{\nu}$ for the whole m by assembly.) Now, the only allowed way to represent a twisted vector v is as a linear combination of the *dual* edges, $\tilde{f}$, of Fig. 2, i.e., as a twisted, cellular 1-chain, and the only way to express νv is as a weighted sum of primal faces f, i.e., as a straight cellular 2-chain. So let us set

$$\nu_T \tilde{f} = \sum_{f' \in \mathcal{F}} \boldsymbol{\nu}_T^{ff'} f', \qquad (5)$$

which defines, by linearity, a map ν_T from twisted cellular 1-chains to straight cellular 2-chains. How to fix the real weights $\boldsymbol{\nu}_T^{ff'}$ in (5) will soon be addressed, but for the time being, they are just coefficients that confer a metric-like structure on the "discrete space" m. They form a matrix $\boldsymbol{\nu}_T$ which is seen—reasoning again by duality—to represent also a map, from straight cellular 2-*co*chains such as $\mathbf{b}$ to twisted cellular *co*chains such as $\mathbf{h}$, of the kind that we need. At this stage, we have with (1'), (2'), and the discrete constitutive laws

$$\mathbf{h} = \boldsymbol{\nu}\mathbf{b}, \qquad (3') \qquad\qquad \mathbf{d} = \boldsymbol{\varepsilon}\mathbf{e}, \qquad (4')$$

a discrete model of the Maxwell system, living on m.

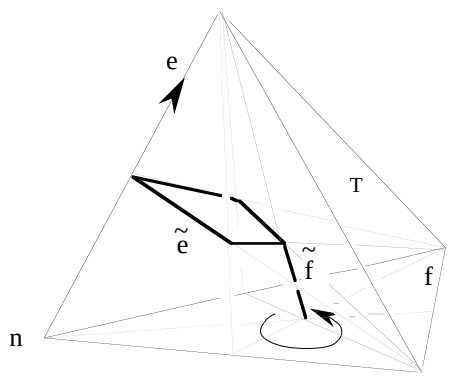

Figure 2. One-tetrahedron mesh (barycentric case).

DISCRETIZATION TECHNIQUES

Simulation runs of this discrete dynamical system are easy to do (by using leap-frog time-integration, for instance, which provides a generalization of Yee's scheme [8]), but the obvious question is, does (1'–4') properly *approximate* (1–4)? It can be proven that this is so provided the ν_T of (5) coincide with the restriction of the original Hodge operator ν to vectors such as $\tilde{f}$, within each tetrahedron T, which we write as

$$\nu\, \mathrm{vec}(\tilde{f}) = \sum_{f' \in \mathcal{F}} \boldsymbol{\nu}_T^{ff'}\, \mathrm{vec}(f'), \qquad (6)$$

where now vec(c) stands for the multivector naturally associated with the cell c in affine space. So this equality works as a *convergence criterion*, that $\boldsymbol{\nu}_T$ must satisfy. (Note that it entails its positive definiteness.)

In current practice, the mesh *m* is built inside a space which already possesses orientation and metric, albeit one which is neither ν- nor ε-adapted, and ν and ε are specified as *scalar* (and of course metric-dependent) coefficients. In that case, vectors and bivectors can be identified (via the perp operator), and (6) can be understood as a *vector* equality: $\mathrm{vec}(\tilde{f})$ is the vector traced out by $\tilde{f}$, and f is the vectorial area of face f, while ν is just the scalar reluctivity. One must then compute numbers $\boldsymbol{\nu}_T^{ff'}$ that satisfy (6), and this of course depends on how the above "centers", defining the dual mesh, are selected. Three cases:

(1) *Barycentric dual mesh.* It then happens that the mass matrix of Whitney face elements, $\boldsymbol{\nu}_T^{ff'} = \int_T \nu\ w^f \cdot w^{f'}$, satisfies (6). (The analogous criterion for $\boldsymbol{\varepsilon}$ is satisfied by the mass-matrix of edge elements, $\boldsymbol{\varepsilon}_T^{ff'} = \int_T \varepsilon\ w^e \cdot w^{e'}$. These $\boldsymbol{\nu}$ and $\boldsymbol{\varepsilon}$ are dubbed "Galerkin (discrete) hodges". Notice the use, optional, of proxy vector fields here.) See [1] for a proof. This sheds some light on the Galerkin approximation of Maxwell equations (which much predates, of course, the present theory), by explaining its kinship with the barycentric construction of the dual mesh.

(2) *"Circumcentric" dual mesh.* Now, since the line that joins the circumcenters of f and of T is orthogonal to f, vec(f) and $\mathrm{vec}(\tilde{f})$ are parallel, which makes it very easy to satisfy (6): Just set $\boldsymbol{\nu}_T^{ff} = \nu\ \mathrm{length}(\tilde{f})/\mathrm{area}(f)$, and all other entries (off diagonal) to 0. This is an essential feature of the MAFIA family of codes [3]. Thus having $\boldsymbol{\nu}_T$ *diagonal* is a great advantage (e.g., the Yee-like scheme becomes fully explicit), but not all primal meshes have a circumcentric dual, and mesh generators may have a hard time complying with this requirement.

(3) *Generic dual mesh.* With arbitrary placed centers, it is *still* possible to satisfy (6), in a non-unique way, but with weights $\boldsymbol{\nu}_T^{ff'}$ that do not necessarily make a symmetric matrix. (A further criterion that allows such symmetry is known, but still mysterious.) I mention this only for thoroughness, and to point out that whether a discrete Hodge *should* be symmetric is an open issue. (This is not required by the convergence proof.)

Last word on this, when ν is given as a tensor, there is always a metric (the ν-adapted one, up to a factor) that makes it diagonal, so the above considerations hold by shifting to this metric first. But note that it may change from an element to its neighbors.

Whitney forms

To carry on, we need the notion of wedge product of forms, which can be cheaply defined as follows. Assume a Euclidean structure, and let vector fields E and A (without any special physical significance) stand proxy for the 1-forms e and a. The 2-form $e \wedge a$ is then the one $E \times A$ stands proxy for. Let the vector field J stand proxy for the 2-form j. The 3-form $j \wedge e$ is then the one $E \cdot J$ stands proxy for. It's a simple exercise [1] to show that the *wedge product* $\wedge$ thus defined does not depend on the scaffolding Euclidean structure one has used.

Let k, l, m, n be the four vertices of a tetrahedron, and $\lambda^i(x)$ the barycentric weight of point x with respect to vertex i, where $i \in \{k, l, m, n\}$. Whitney forms, associated with the simplices $e = \{m, n\}$ and $f = \{l, m, n\}$, oriented by the ordering of their vertices, are

$$w^e = \lambda^m\, d\lambda^n - \lambda^n\, d\lambda^m,$$
$$w^f = 2(\lambda^l\, d\lambda^m \wedge d\lambda^n + \lambda^m\, d\lambda^n \wedge d\lambda^l + \lambda^n\, d\lambda^l \wedge d\lambda^m).$$

In the spirit of what precedes, where one tries to map features of the continuous model of electromagnetism to the discrete one living on *m*, Whitney forms can be understood as a device to map curves and surfaces to cellular chains: To curve c or surface S corresponds the 1-chain $\sum_{e \in \mathcal{E}} [\int_c w^e]\, e$ or the 2-chain $\sum_{f \in \mathcal{F}} [\int_S w^f]\, f$, which one can construe as simplicial approximations of c or S. Now, suppose a 2-form b is given. Then $\int_S b$ —or let's rather write that $\langle S\ ;\ b \rangle$ for better insight—is approximately $\langle \sum_{f \in \mathcal{F}} [\int_S w^f]\, f\ ;\ b \rangle \equiv \sum_{f \in \mathcal{F}} \langle S\ ;\ w^f \rangle \langle f\ ;\ b \rangle = \langle S\ ;\ \sum_{f \in \mathcal{F}} \mathbf{b}_f\, w^f \rangle$, where we have set $\mathbf{b}_f = \int_f b$. This shows that the 2-form $\sum_{f \in \mathcal{F}} \mathbf{b}_f\, w^f$, built from the DoF array $\mathbf{b}$ in the finite-element tradition, i.e., by using the w^f as interpolants to pass from the DoFs to a field, is a proper approximation for b. Whitney forms thus play a dual role: they map singular chains to cellular chains, and also, by duality, can be used to build singular cochains from cellular ones.

Which is what simulation requires: Solving (1'–4'), an ODE system, yields DoF's, but we want more: to be able to reconstruct *fields*, with controllable approximation. Whitney forms allow one to do just that.

A toolkit

Let's now stop to evaluate what we have. From an abstract viewpoint, a kind of reduction of standard exterior calculus (which dominates the theoretical background of classical electromagnetism) to some "discrete exterior calculus" (DEC), developed over a finite combinatorial structure (the mesh *m*). Founding DEC, in some form,

is an active research field these days [9]. (A nice feature of the approach sketched here is its reliance on a unique concept: the Whitney map from singular chains to cellular chains.) From a concrete viewpoint, we have a toolbox: Given a modelling problem governed by the Maxwell equations, one can erect a discrete, computer-manageable model, by first building a mesh, then picking and assembling the relevant parts. These are cellular chains (to model surfaces such as material interfaces, or curves like voltmeter connectors), matrices $\mathbf{G}$, $\mathbf{R}$, $\mathbf{D}$ (or $-\mathbf{D}^T$, $\mathbf{R}^T$, $-\mathbf{G}^T$) to replace grad, rot, div (depending on which kind of form, straight or twisted, the proxy vector field stands for), matrices $\boldsymbol{\varepsilon}$ and $\boldsymbol{\nu}$ (or its inverse $\boldsymbol{\mu}$) to replace ε and μ, matrix $\boldsymbol{\sigma}$ as a substitute for conductivity.

To give just one example (others can be found in [1]): The charge conservation relation, $\partial_t Q + \operatorname{div} J = 0$, becomes $\partial_t \mathbf{q} - \mathbf{G}^T \mathbf{j} = 0$, with $\mathbf{q}$ node-based and $\mathbf{j}$ edge-based. This is of course one of the Kirchhoff laws, which is not surprising: By its very principle, the discretization process constructs two interlocked, and interacting, *equivalent networks*, one on the primal mesh, with currents (the $\mathbf{j}_e$s) flowing between primal nodes along primal edges, according to the Kirchhoff-like equations

$$\mathbf{G}^T\mathbf{d} = \mathbf{q}, \quad \mathbf{d} = \boldsymbol{\varepsilon}\mathbf{e}, \quad \mathbf{R}\,\mathbf{e} = -\partial_t \mathbf{b},$$

the other one on the dual mesh, with magnetic fluxes (the $\mathbf{b}_f$s) flowing between dual nodes along dual edges, ruled by

$$\mathbf{R}^T\mathbf{h} = \mathbf{j} + \partial_t \mathbf{d}, \quad \mathbf{h} = \boldsymbol{\nu}\mathbf{b}, \quad \mathbf{D}\,\mathbf{b} = 0.$$

Moreover, if we adopt the notation $\mathbf{(b, h)}$ or $\mathbf{(j, e)}$, with **bold** parentheses in both cases, for such expressions as $\sum_{f \in \mathcal{F}} \mathbf{b}_f \mathbf{h}_f$ or $\sum_{e \in \mathcal{E}} \mathbf{j}_e \mathbf{e}_e$, there are ready-made interpretations for these quantities: $\mathbf{(b, h)}$, equal to $\mathbf{(\nu b, b)}$, is the discrete analogue of the integral $\int_D \nu b \wedge b$, or in more familiar terms, of $\int_D \nu B \cdot B$, so it represents (up to a factor 2, and with the linear constitutive laws we consider here) the *magnetic energy* inside the system. If one cares to localize this energy (though this, arguably, makes little sense), $\mathbf{b}_f \mathbf{h}_f$ is an approximation of (twice) the energy present in the "control domain" of face f, pictured in Fig. 3. Similarly, $\mathbf{(j, e)}$ corresponds to *Joule power*, i.e., $\int_D j \wedge e$, or $\int_D J \cdot E$, to which the control domain of edge e (Fig. 3) contributes $\mathbf{j}_e \mathbf{e}_e$. Electric energy is of course $\mathbf{(d, e)}/2$, etc.

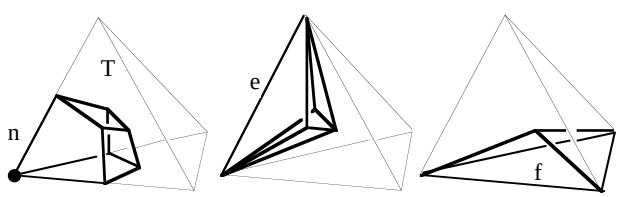

Figure 3. Parts of control domains of node n, edge e, face f, contained in T. (Other parts are contributed by adjacent volumes which share n, e, or f with T.)

Satisfying as this may be, it also raises further questions: What about the Poynting flux, $\int_{\partial D} e \wedge h$? Can it be recovered from DoF arrays by a simple formula? How should we understand classical expressions such as $v \times B$, or $J \times B$, or that for the Maxwell tensor, in terms of differential forms first (just to make a stepping stone towards the next point), then in terms of DoF arrays? All these are needed pieces in the toolkit.

The $v \times B$ *term in the Lorentz force*

Since all these missing pieces have to do with electromagnetic forces, in some way, it may help to understand what force is in the first place: Force, as a geometric object, is a *co*vector, not a vector, because it's a linear map from vectors (understood as virtual displacements) to reals (virtual power). That Coulomb force on a point charge Q be Qe, a 1-form, is therefore consistent. It means that $v \times B$ should stand proxy for a 1-form, too.

Let's assume we are dealing with a velocity *field* v, to which corresponds a flow $u_t : A_3 \to A_3$ determined by the differential equation $\partial_t y = v(y)$, $y(0) = x$, with $u_t(x) = y(t)$. As dragged by the flow, from time 0 to time t, a point x describes a piece of flow line ext(x, v, t), or *extrusion* of x by v from 0 to t, and a curve c sweeps a surface ext(c, v, t). Now $\langle \mathrm{ext}(c, v, t) ; b \rangle$ makes sense, for a 2-form b, and we can define a 1-form, denoted $i_v b$, by $\langle c ; i_v b \rangle = \lim_{t \to 0} t^{-1} \langle \mathrm{ext}(c, v, t) ; b \rangle$. It's not difficult to see that its proxy field is $-v \times B$. A convoluted definition if any, but as several times before, it happens to be the one that most easily can be mimicked at the discrete level, as follows.

The velocity field v is a map from points, x, to vectors, v(x), e.g., a vector-valued 0-form. Its natural discretization is thus by a vector $\mathbf{v}_n$ at each node of *m*, with linear interpolation within tetrahedra, i.e., $v(x) = \sum_{n \in \mathcal{N}} \mathbf{v}_n \lambda^n(x)$. Given $\mathbf{b}$, face-based DoF array, we want edge values for $i_v b$. By linearity, it's enough to be able to do that for the partial velocity field $\mathbf{v}_n \lambda^n(x)$. Let the flow of this field extrude edge e, and express this extrusion, in the $t = 0$ limit, as a linear combination $\sum_{f \in \mathcal{F}} \alpha^f_e f$ of faces. (At most two of them, hinged upon e, are concerned.) The edge value for $i_v b$ is then $\sum_{f \in \mathcal{F}} \alpha^f_e \mathbf{b}_f$. (See [2] for details.) Calling $i_v\mathbf{b}$ the edge-array thus obtained, we can now discretize a MHD problem, in Eulerian framework, with a given velocity field, where the conduction law is $J = \sigma(E + v \times B)$: Just transpose that as $\mathbf{j} = \boldsymbol{\sigma}(\mathbf{e} - i_v\mathbf{b})$.

The Laplace force

This may look more puzzling, because since both j and b are *two*-forms, one may wonder what $J \times B$ stands proxy for. But consider virtual power, $(J \times B) \cdot v$. For any given v, this is the proxy of a (twisted) 3-form, the *density* of virtual power, which one recognizes to be $i_v b \wedge j$. Force is therefore the *mapping* $v \to i_v b \wedge j$, that is to say, a *covector-valued* 3-form, which is at it

should be: Integrating this 3-form over a volume will give a covector, the resultant of body forces. Momentum, obtained by time integration, is of the same type, and stress is a (twisted) covector-valued 2-form, whose integral over a surface Σ (outer oriented) gives the flux of momentum through it. Now that we know how to deal with $i_\nu b$, discretization is easy: the virtual power of Laplace forces is $(i_\nu \mathbf{b}, \mathbf{j})$ and discrete force is the map $\mathbf{v} \to (i_\nu \mathbf{b}, \mathbf{j})$. Degrees of freedom for ν being vectors $\mathbf{v}_n$, one at each node, force appears as a node-base array of covectors $\mathbf{f}_n$, and virtual power is $\sum_{n \in \mathcal{N}} \langle \mathbf{v}_n ; \mathbf{f}_n \rangle$.

The Poynting flux

If two 1-forms ω and η are known by their edge DoFs $\boldsymbol{\omega}$ and $\boldsymbol{\eta}$, there is no difficulty with integrals $\int_f \omega \wedge \eta$, and hence with the "discrete wedge product" $\boldsymbol{\omega} \wedge \boldsymbol{\eta}$: Assuming for simplicity that edges of f are labelled 1, 2, 3, and oriented like ∂f (i.e., $\mathbf{R}_f^i = 1$, for i = 1, 2, 3), a simple manipulation of Whitney forms shows that $\int_f \omega \wedge \eta = [\boldsymbol{\omega}_1\boldsymbol{\eta}_2 + \boldsymbol{\omega}_2\boldsymbol{\eta}_3 + \boldsymbol{\omega}_3\boldsymbol{\eta}_1 - \boldsymbol{\eta}_1\boldsymbol{\omega}_2 - \boldsymbol{\eta}_2\boldsymbol{\omega}_3 - \boldsymbol{\eta}_3\boldsymbol{\omega}_1]/6$. (Similar formulas hold in all dimensions and degrees.) The wedge product defined by $(\boldsymbol{\omega} \wedge \boldsymbol{\eta})_f = \int_f \omega \wedge \eta$ is not associative, which is felt like a problem by some (see [7] for a discussion, and a proof that the non-associativity "vanishes in the continuous limit"), but is not our concern. What is more annoying is that we can't evaluate $\int_f e \wedge h$ by this method.

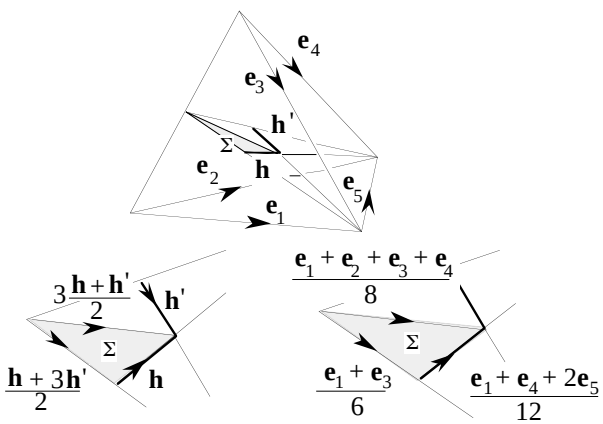

Figure 4. From DoF's $\mathbf{e}_i$, i = 1 to 4 and $\mathbf{h}$, $\mathbf{h}'$ (top), how to reconstruct edge DoF's for both e and h on the shaded triangle Σ (bottom), which is half of the dual face $\tilde{e}$. Both $\mathbf{h}$, $\mathbf{h}'$ are computed by integration of νb (well defined along these segments). All needed values are then obtained by affine interpolation.

Indeed, suppose $\mathbf{e}$ and $\mathbf{b}$ are known, hence their approximations $e = \sum_e \mathbf{e}_e w^e$ and $b = \sum_f \mathbf{b}_f w^f$. Then $h = \nu b$, though well-defined inside tetrahedra (if ν behaves) has no well-defined restriction to primal faces (because w^f has none). To evaluate $\int_S e \wedge h$ is thus an impossible challenge if surface S is modelled, as one is tempted to do in particular when $S = \partial D$, as a primal 2-chain. That was to be expected, for $e \wedge h$ is, like h, a *twisted* 2-form, and "twisted objects live on the dual mesh" in our discretization paradigm. This suggests to integrate $\int e \wedge h$ on *dual* faces, and Fig. 4 should, better than words, explain how. The idea is to first obtain the integrals of both e and h along the edges that bound the "small face" Σ, shaded, of Fig. 4, after which the above formula can be used. Since both e and $h = \nu b$ are affine functions of position (if one assumes ν constant inside each volume), their integrals along an oriented straight segment s are affine functions of s, which is the trick used in Fig. 4. We also took $dh = 0$ into account (note that $3(\mathbf{h} + \mathbf{h}')/2 = h + (\mathbf{h} + 3\mathbf{h}')/2$, in accordance with the Stokes theorem). If $dh = j$, nonzero, on the other hand, the flux of the twisted 2-form j through Σ, well defined, is thrown in.

By adding contributions of such small faces, one can compute $\int_\Sigma e \wedge h$ for any assembly Σ of dual faces.

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