Closed form formulas for the indirect space charge wake function of axisymmetric structures

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Analytical wake functions from wall impedance

- There are many ways to get analytical formulas for the (resistive-)wall impedance of simple geometries (cylindrical, flat, elliptic), but much less formulas are available for wake functions.

- In the case of indirect space-charge (ISC) – i.e. for a perfectly conducting cylindrical beam pipe (radius b & length L), the impedance is well-known:

ISC

$$Z_x^{dip}(\omega) = \frac{ik^2 Z_0 L}{4\pi\beta\gamma^4} \frac{K_1\left(\frac{kb}{\gamma}\right)}{I_1\left(\frac{kb}{\gamma}\right)}$$

$$Z_{\parallel}(\omega) = \frac{i\omega\mu_0 L}{2\pi\beta^2\gamma^2} \frac{K_0\left(\frac{kb}{\gamma}\right)}{I_0\left(\frac{kb}{\gamma}\right)}$$

$\omega > 0$: angular frequency

K_1, I_1: modified Bessel functions,

$k \equiv \frac{\omega}{v} = \frac{\omega}{\beta c}$: wave number,

$Z_0 = \mu_0 c$: free space impedance,

β & γ: relativistic velocity & mass factors.

… but not the corresponding wake function, typically computed numerically – except for ultrarelativistic beams (analytic formula in the form of a delta function – see e.g. A. W. Chao, John Wiley & Sons, 1993).

$\Longrightarrow$ An analytical expression for the ISC wake function is desirable for $\beta < 1$, for e.g. time domain, macroparticle simulations.

ISC wake function – fully analytical approach

- To get the wake functions one needs to compute the Fourier integrals of the previous expressions:

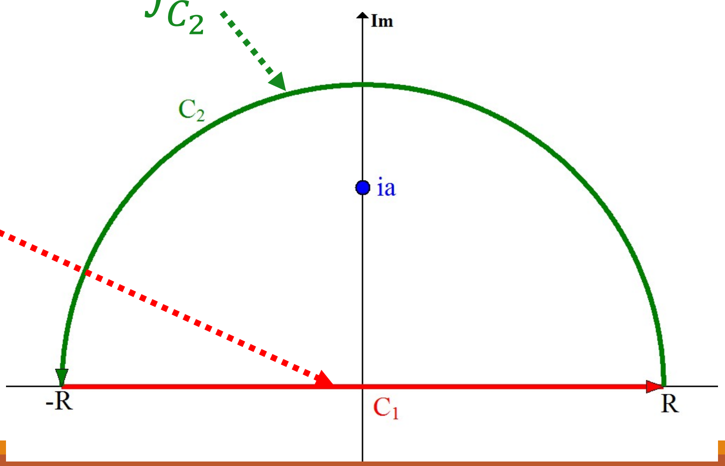
$$W_{\parallel}(t) = \frac{1}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t} Z_{\parallel}(\omega) d\omega$$

$$W_X^{dip}(t) = -\frac{i}{2\pi}\int_{-\infty}^{\infty} e^{i\omega t} Z_x^{dip}(\omega) d\omega$$

- This can be achieved using Cauchy's residue theorem with proper contours in the complex plane:

$$\int_{-R}^{R} e^{i\omega t} f(\omega) d\omega = \oint_{C_1+C_2} e^{i\omega t} f(\omega) d\omega - \int_{C_2} e^{i\omega t} f(\omega) d\omega$$

- Cauchy's theorem gives a residue sum for the first integral.
- The second one vanishes when $R \to \infty$ (Jordan's lemma).

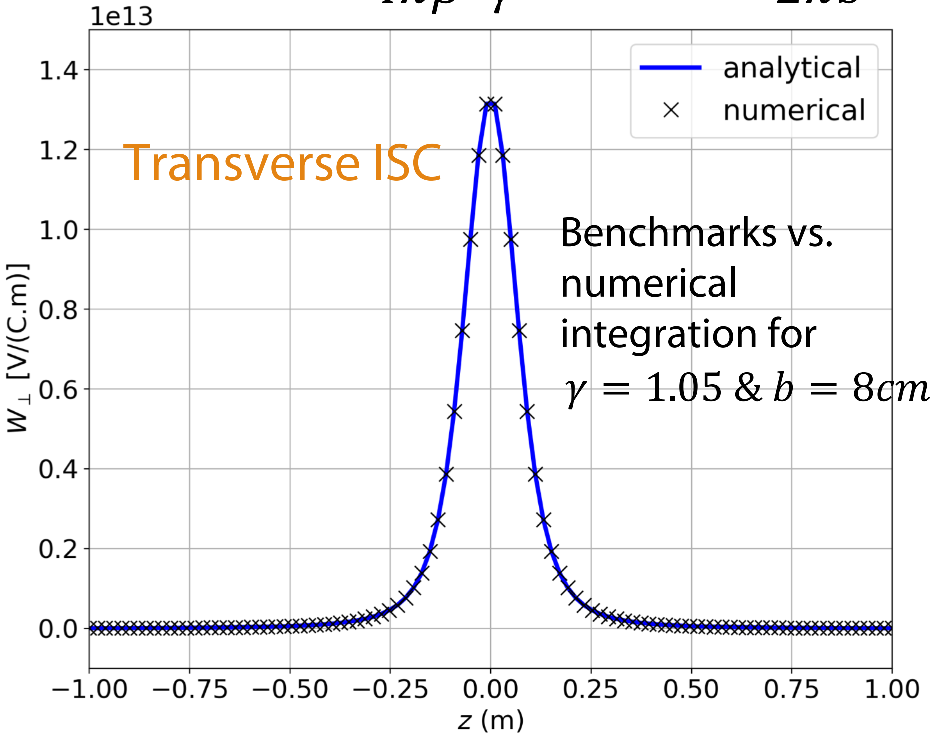


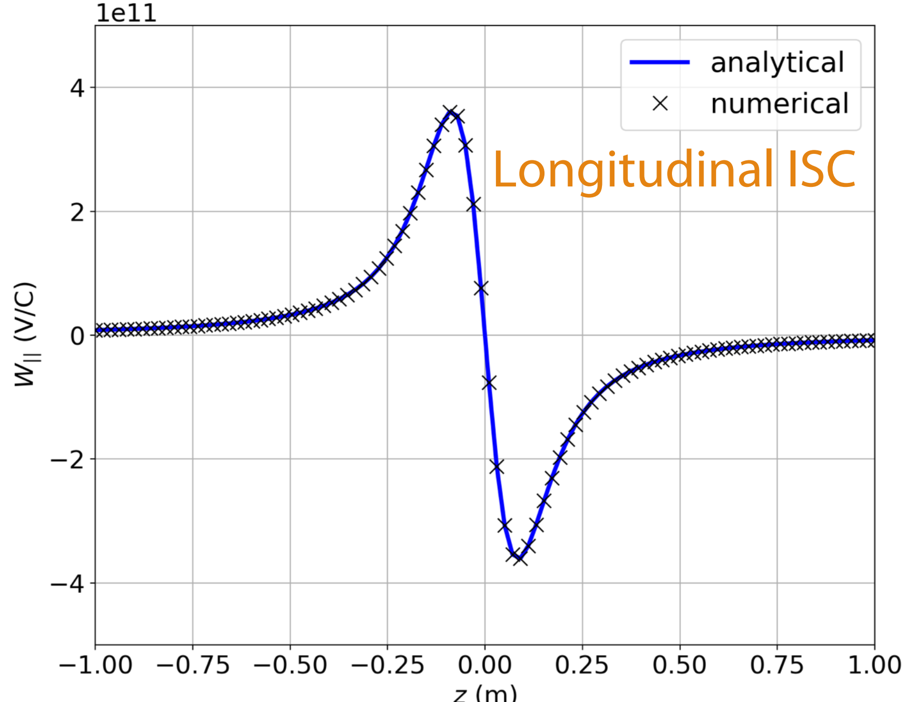
ISC wake function – fully analytical approach

- After some algebra one finally finds (with $j_{m,k}$ the k[th] zero of the J_m Bessel function):

$$W_x^{dip}(t) = \frac{\mathrm{sgn}(t) Z_0 L}{4\pi v^2 \beta\gamma^4}|t|^{-3} - \frac{i Z_0 c L}{2\pi\gamma b^3} sgn(t) \sum_{k=1}^{\infty} \frac{e^{\frac{-v\gamma j_{1,k}}{b}|t|}(j_{1,k})^2 K_1(sgn(t) i j_{1,k})}{J_0(j_{1,k}) - J_2(j_{1,k})}$$

$$W_\parallel(t) = \frac{-\mathrm{sgn}(t)\mu_0 L}{4\pi\beta^2\gamma^2}|t|^{-2} - \frac{Z_0 c L}{2\pi b^2}\mathrm{sgn}(t)\sum_{k=1}^{\infty}\frac{e^{\frac{-v\gamma j_{0,k}}{b}|t|} j_{0,k} K_0(sgn(t) i j_{0,k})}{J_1(j_{0,k})}$$





Transverse ISC wake function – simplified formula

$$Z_\perp(\omega) = \mathrm{j}\frac{Z_0 L \omega^2 K_1(\mathrm{a}\omega)}{4\pi\, c^2 \beta^3 \gamma^4 I_1(\mathrm{a}\omega)}$$

$$a = \frac{\mathrm{b}}{\mathrm{c}\beta\gamma}$$

$$\tau = \frac{\mathrm{z}}{\mathrm{v}}$$

a ω << 1

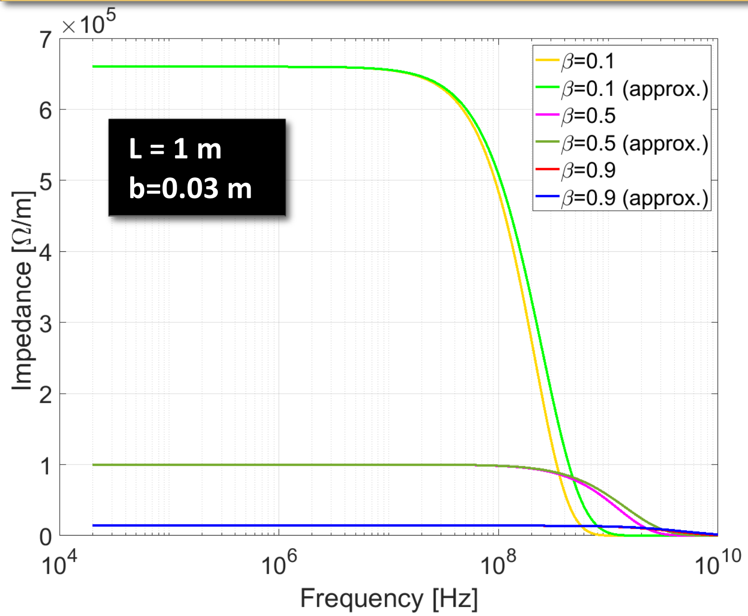
$$I_1(\mathrm{a}\omega) \approx \frac{\mathrm{a}\omega}{\mathbf{2}}$$

Approximation for small argument of I_1

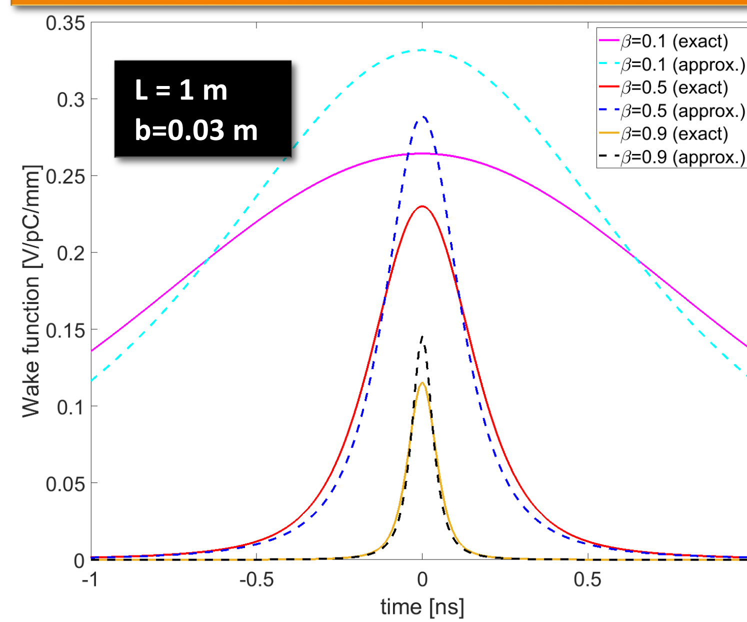
$$Z_\perp(\omega) \approx \mathrm{j}\frac{Z_0 L \omega K_1(\mathrm{a}\omega)}{2bc\pi\beta^2\gamma^3}$$

$$W_\perp(\tau) \approx \frac{L Z_0}{4\pi c^2 \beta^3 \gamma^4 (\tau^2 + a^2)^{3/2}}$$

Comparing the approximated and the exact impedance formula



Comparing the approximated wake function formula and the fully analytical approach



Transverse ISC wake function – corrected formula

$$W_{\perp}^{approx}(\tau) = \frac{LZ_0}{4\pi c^2\beta^3\gamma^4(\tau^2+a^2)^{3/2}}$$

$$a = \frac{b}{c\beta\gamma}$$

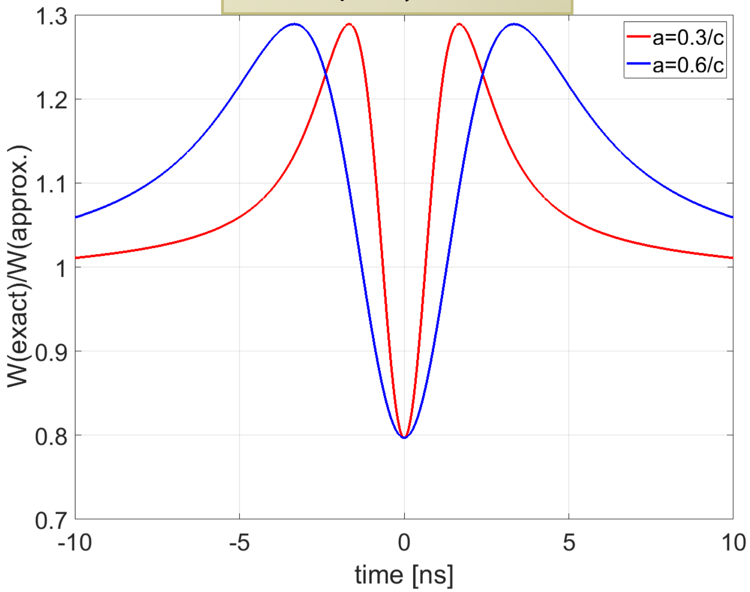
Using this peculiarity of the discrepancy function we can obtain a corrected formula for the ISC
⟹ we only need to compute once for all a reference discrepancy function.

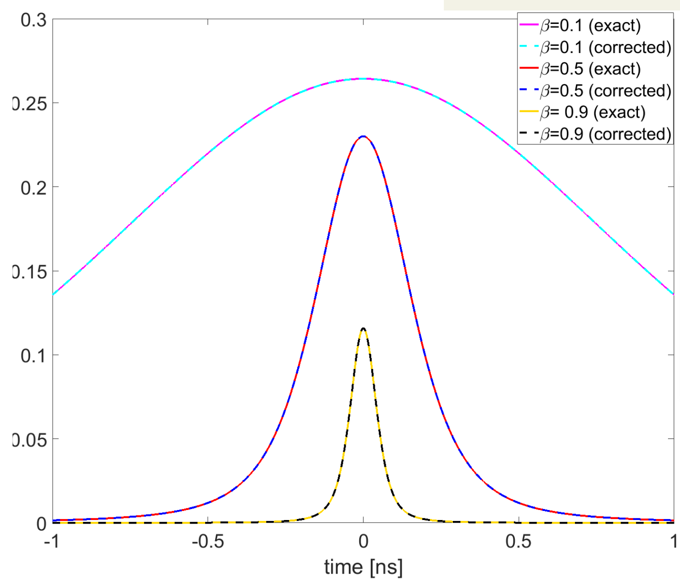
We can plot the ratio between the exact and the approximated formulas
⟹ discrepancy function *F*, a function of *(τ a)* only.

$$W_{\perp}^{corrected}(\tau) = F(\tau_n)\frac{LZ_0}{4\pi c^2\beta^3\gamma^4(\tau^2+a^2)^{3/2}}$$

$$\tau_n = \tau\frac{a_{ref}}{a} \qquad a_{ref} = \frac{b_{ref}}{c\beta_{ref}\gamma_{ref}}$$

Discrepancy function





Excellent agreement between corrected and exact formulas

Summary

- A fully analytical formula was found for the indirect space charge wake function, in both longitudinal and transverse.

- A simplified and efficient approximated formula could be derived in transverse. The exact result can be recovered thanks a multiplicative correction function that can be computed once for all.

Thank you for your attention!