Requirements and Results for Quadrupolar Mode Measurements

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Acknowledgements:

Simon Albright, Marcel Coly, Heiko Damerau, Marek Gasior, Tom Levens, Elias Métral, Guido Sterbini, Malte Titze, Panagiotis Zisopoulos

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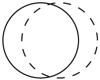
Motivation

1^{st} order



rigid dipolar centroid oscillation:

- Newton's third law, actio = reactio

⟶ no influence from direct space charge (SC)

Motivation

1st order

rigid dipolar centroid oscillation:

- Newton's third law, actio = reactio

⟶ no influence from direct space charge (SC)

2nd order

quadrupolar envelope oscillation:

- defocused by transverse space charge

⟶ frequency of envelope oscillation decreases with SC

⟹ measure direct space charge through frequency shift of beam size oscillations about matched $\sigma_{x,y}$

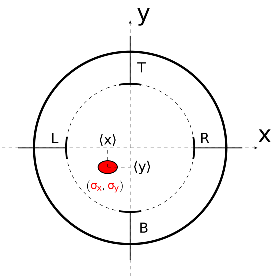
Outline

Content of this talk:

1. **Introduction**
   - spectrum of a quadrupolar pick-up
2. **Equipment** in LHC Injector Rings: Status and Plans
3. **Applications** and Ongoing Studies
   - quadrupolar beam transfer function to characterise high-brightness PS beams
     - ⟶ influence of chromaticity
     - ⟶ coherent dispersive mode
   - PS injection: transfer line mismatch

1. Introduction

Schematic Quadrupolar Pick-up



modified image taken from [1]

Evaluating the four pick-up signals as

$$(L + R) - (T + B)$$

results in the turn by turn signal

$$S_{\mathrm{QPU}}(i_{\mathrm{turn}}) \propto \langle x^2 \rangle - \langle y^2 \rangle = \sigma_x^2(i_{\mathrm{turn}}) - \sigma_y^2(i_{\mathrm{turn}}) + \langle x \rangle^2(i_{\mathrm{turn}}) - \langle y \rangle^2(i_{\mathrm{turn}}) \quad .$$

Some Historical Perspective

QPU in **time domain** for emittance measurements:

- 1983, R. H. Miller et al. at SLAC [2]
- 2002, A. Jansson at CERN in PS [3]

QPU in **frequency domain** for emittance measurements:

- 2007, C.Y. Tang at Fermilab [4]

QPU in **frequency domain** for space charge measurements:

- 1996, M. Chanel at CERN in LEAR [5]
- 1999, T. Uesugi et al. at NIRS in HIMAC [6]
- 2000, R. Bär at GSI in SIS-18 [7]
- 2014, R. Sing et al. at GSI in SIS-18 [1]

$\Longrightarrow$ all far away from coupling and coasting beams

CERN's proton synchrotrons peculiar:

1. close to coupling $\Longrightarrow$ quadrupolar mode frequencies change
2. bunched beam

QPU measurements at GSI by R. Singh, M. Gasior et al. [1]

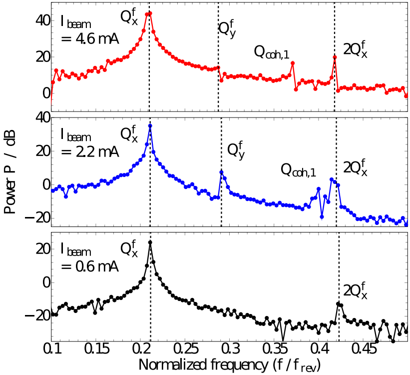


Figure 6: Shift of coherent quadrupole mode $Q_{coh,1}$ with beam current.

particle type	N^{7+}
E_{kin} (MeV/u)	11.45
I_{beam} (mA)	0.6 – 6
ϵ_x, ϵ_y (mm-mrad)	8, 12.75
Q_{x0}, Q_{y0}	4.21, 3.3

$$Q_x^f \mathrel{\hat{=}} Q_x$$

$$Q_y^f \mathrel{\hat{=}} Q_y$$

$$Q_{coh} \mathrel{\hat{=}} Q_\pm$$

⟶ far away from coupling resonance

⟶ coasting beam $\Longrightarrow$ sharp envelope peak

Incoherent KV Tune Shift

The Kapchinskij-Vladimirskij (KV) beam distribution has all particles at same incoherent space charge tune shift:

$$\Delta Q_{x,y}^{\mathrm{KV}} \doteq -\frac{K^{\mathrm{SC}} R^2}{4\sigma_{x,y}(\sigma_x + \sigma_y) Q_{x,y}} \tag{1a}$$

$$\doteq \frac{1 + \sigma_{x,y}/\sigma_{y,x}}{2Q_{x,y}} \Lambda \tag{1b}$$

space charge perveance $K^{\mathrm{SC}} \doteq \frac{q\lambda}{2\pi\epsilon_0 \beta\gamma^2 p_0 c}$

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$$\doteq \frac{1 + \sigma_{x,y}/\sigma_{y,x}}{2Q_{x,y}} \Lambda \tag{1b}$$

Connect Λ quantity to general 2D envelope mode expressions in terms of **observables**:

$$\Lambda = \frac{Q_+^2 + Q_-^2 - 4(Q_x^2 + Q_y^2)}{4 + 3(\sigma_x/\sigma_y + \sigma_y/\sigma_x)} \tag{2}$$

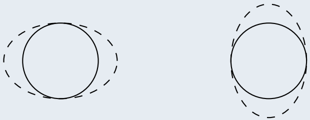
(Gaussian tune spread = 2x the RMS-equivalent KV tune shift!)

space charge perveance $K^{\mathrm{SC}} \doteq \dfrac{q\lambda}{2\pi\epsilon_0 \beta\gamma^2 p_0 c}$

Far Away vs. On the Coupling Resonance

2 eigenmodes for coherent quadrupolar betatron oscillation:

far away from coupling



(a) horizontal mode (b) vertical mode

Quadrupolar mode tunes:

$$Q_{\pm} = 2Q_{x,y} - \left|\Delta Q_{x,y}^{\mathrm{KV}}\right| \left(3 - \frac{\sigma_{x,y}}{\sigma_x + \sigma_y}\right)/2 \quad (3)$$

Far Away vs. On the Coupling Resonance

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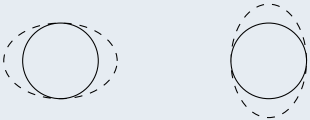

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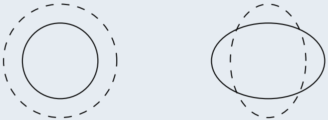
$$Q_\pm = 2Q_{x,y} - \left|\Delta Q_{x,y}^{\mathrm{KV}}\right| \left(3 - \frac{\sigma_{x,y}}{\sigma_x + \sigma_y}\right) / 2 \qquad (3)$$

full coupling



(a) breathing mode (b) antisym. mode

Quadrupolar mode tunes:

$$Q_+ = 2Q_0 - \left|\Delta Q_{x,y}^{\mathrm{KV}}\right| \qquad (4a)$$

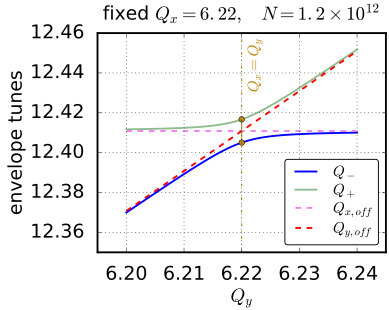
$$Q_- = 2Q_0 - \frac{3}{2}\left|\Delta Q_{x,y}^{\mathrm{KV}}\right| \qquad (4b)$$

(assuming round beams, $Q_{x,y} \equiv Q_0$)

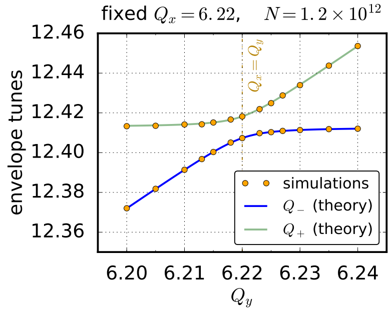
Peculiarity 1: Near Coupling Resonance

At vanishing lattice coupling, keep constant incoherent SC tune shift and fixed Q_x. Vary Q_y for a coasting round beam:

analytic expression



simulation results



Peculiarity 2: Bunched Beam Envelope Signal

Assumption (justification e.g. [6]):

- synchrotron motion much slower than betatron motion, $Q_s \ll Q_{x0,y0}$
  - $\longrightarrow$ 3D RMS envelope equation (Sacherer) decouples to 2D + 1D
  - $\Longrightarrow$ for a given longitudinal bunch slice, the coherent transverse quadrupolar oscillation depends on local line charge density $\lambda(z)$, longitudinal motion is quasi-stationary and independent

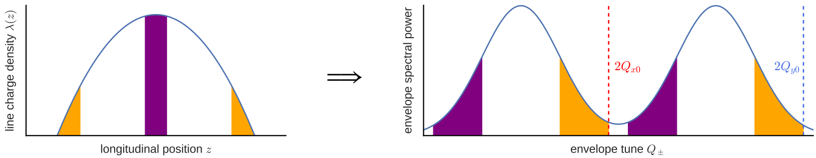
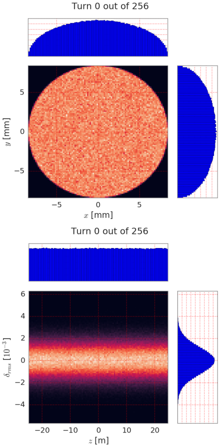


Figure: sketch of envelope detuning scaling with local line charge density

1. Introduction:
QPU Spectrum Simulations

Coasting: KV Beam

bunched	transv. distr.	synchrotron motion	dispersion	chromaticity
no	KV (uniform)	no	no	no



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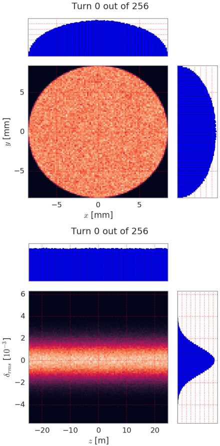
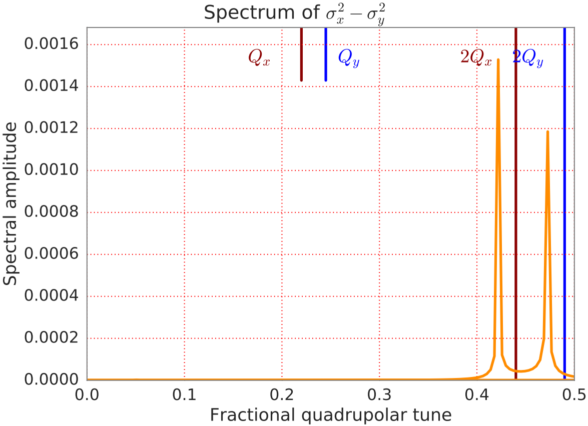
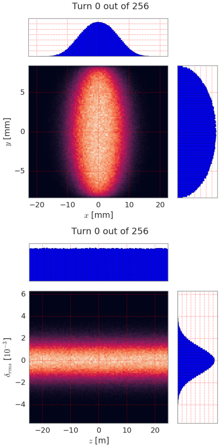
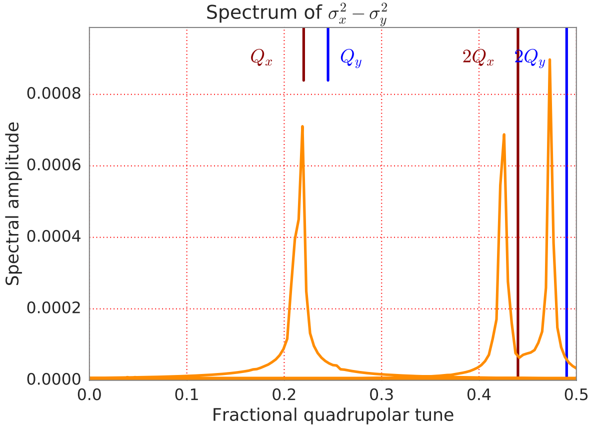




Coasting: KV Beam with Dispersion

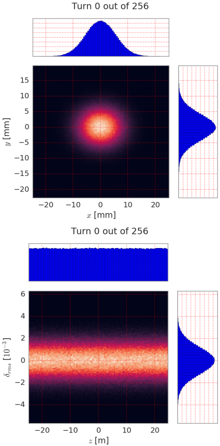
bunched	transv. distr.	synchrotron motion	dispersion	chromaticity
no	KV (uniform)	no	yes	no

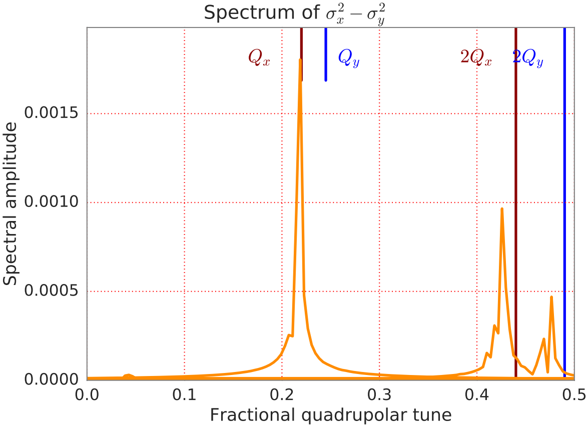




Coasting: RMS-equiv. Gaussian Beam with Dispersion

bunched	transv. distr.	synchrotron motion	dispersion	chromaticity
no	Gaussian	no	yes	no





Bunched KV Beam

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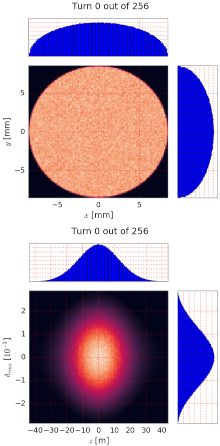
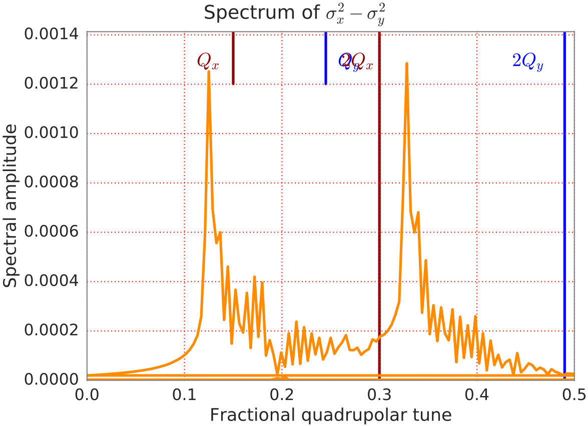
yes	KV (uniform)	no	no	no

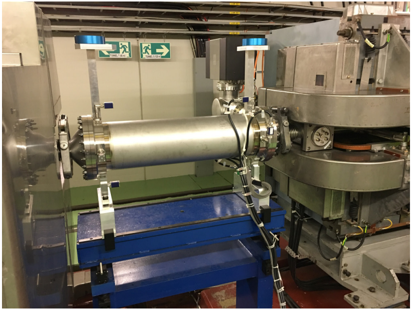




2. Equipment in LHC Injectors:
Status and Plans

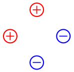
PS: Quadrupolar Pick-up

Stripline pick-up `PR.BQL72` is part of the BBQ system:



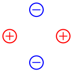
courtesy Tom Levens

Currently recabled to quadrupolar mode (started in 2016):



(a) dipolar

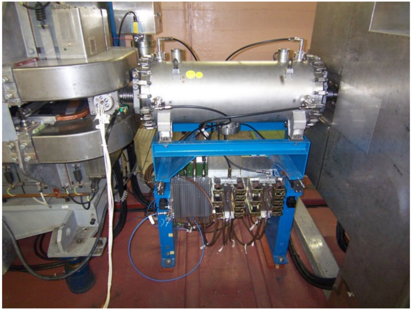
$\Longrightarrow$



(b) quadrupolar

PS: Transverse Feedback (TFB) as Quadrupolar Kicker

Kicker in section 97 is part of the new PS transverse feedback system:



courtesy Guido Sterbini

Since May 2018:

- source signal for quadrupolar excitation comes from BBQ system
  - ⟶ new dedicated card `BQL72_Q` to control excitation parameters separately from dipolar tune measurements

Plans

Pick-up side:

- in PS, upgrade `BQL72` with *3 channel frontend*, simultaneously extract
  - dipolar signals $\langle x \rangle$ and $\langle y \rangle$,
  - quadrupolar signal Q

  $\longrightarrow$ technical stop in June 2018
- in PSB, make use of brand-new (2018) stripline pick-ups

  $\longrightarrow$ install 3 channel frontend to include Q channel

  $\longrightarrow$ during 2018
- in SPS, upgrade existing BBQ system with Q channel

  $\longrightarrow$ install new 3 channel frontend during LS2

Kicker side:

- separate quadrupolar excitation signal path from rest of system

  $\longrightarrow$ possibility to operate dipolar feedback system in closed loop + simultaneous quadrupolar excitation (these coming weeks)

3. Application:
(a) quadrupolar beam transfer function (2017)

Goal of Study

Motivation

In the context of **strong space charge regime** with **LHC Injectors Upgrade** beam parameters: determine beam brightness (or incoherent KV tune shift) **directly** via coherent quadrupolar modes

Starting from nominal LHC beam-type set-up:

- (large) natural chromaticity: $Q'_x = -0.83Q_x$ and $Q'_y = -1.12Q_y$
- lattice is usually strongly coupled via skew quadrupoles to stabilise slow horizontal head-tail instabilities
  - $\longrightarrow$ decouple lattice during envelope measurements
  - $\Longrightarrow$ only space charge coupling in envelope tunes

$\longrightarrow$ measure *quadrupolar* beam transfer function to learn about space charge

Experimental Set-up

Ingredients:

- small time window of 15 ms with decoupled optics
- chirped **quadrupolar excitation** of beam via transverse feedback: external waveform generator connected to kicker plates
  - 12 ms long frequency sweep with 1 ms return
  - harmonic $h = 5$ with frequency range 2.19 MHz to 2.4 MHz

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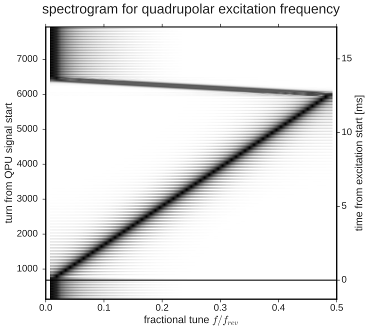
- chirped **quadrupolar excitation** of beam via transverse feedback: external waveform generator connected to kicker plates
  - 12 ms long frequency sweep with 1 ms return
  - harmonic $h = 5$ with frequency range 2.19 MHz to 2.4 MHz
- single bunch in PS with a factor 5 smaller incoherent KV tune shift compared to currently operational LHC beams, off coupling

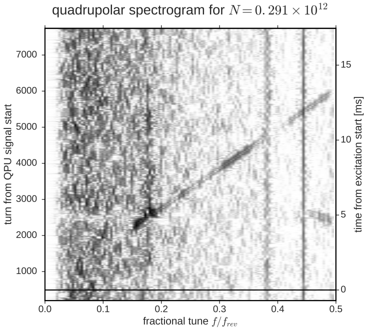
intensity	$N \approx 0.3 - 0.4 \times 10^{12}$ ppb
transverse emittance	$\epsilon_{x,y} \approx 2.3$ mm mrad
average betatron function	$\beta_x \approx \beta_y \approx 16$ m
average dispersion	$D_x \approx 3$ m
momentum deviation spread	$\sigma_\delta \approx 1 \times 10^{-3}$
bunch length	$B_L \approx 180$ ns
synchrotron tune	$Q_s = 1/600 = 1.67 \times 10^{-3}$
KV space charge tune shift	$\Delta Q_{x,y}^{\text{KV}} \approx 0.02$

Quadrupolar Excitation: Chirp

excitation signal



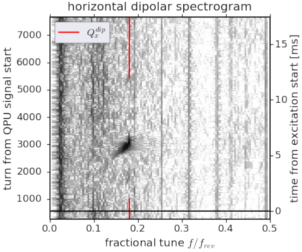
beam response (via QPU)

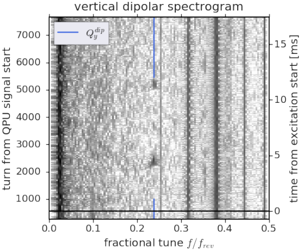


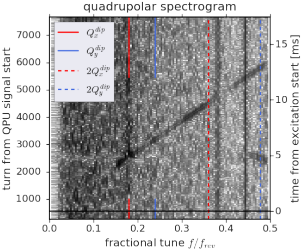
- distinct peaks around machine tunes $f < 0.25 f_{\text{rev}}$
- frequency bands around twice the machine tunes
- (disregard the constant frequencies, due to instrumentation)

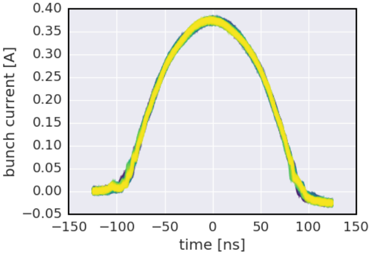
Measured Quadrupolar Beam Transfer Function

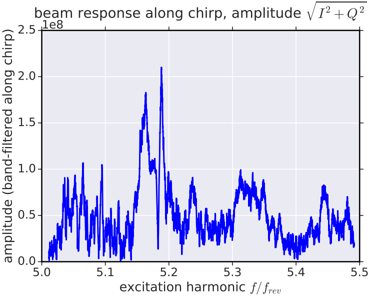
$N = 0.291 \times 10^{12}$



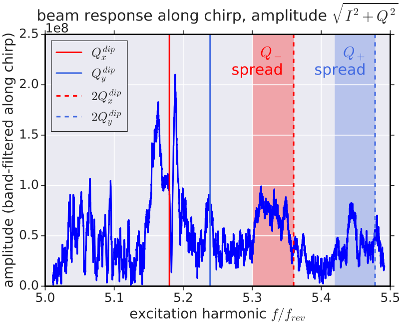








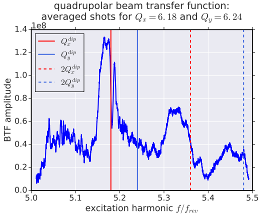
Observations in Spectrum



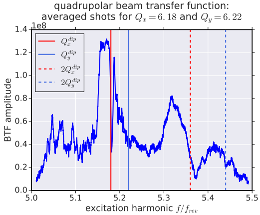
Observations:

- significant peaks around Q_x
  - $\longrightarrow$ dispersive coherent mode?
  - $\longrightarrow$ influence of chromaticity?
- envelope band below $2Q_x$ clearly visible
  - ⚠ would infer $\Delta Q_{x,y}^{\mathrm{KV}} \approx 0.04 - 0.05$ (factor 2 too large!)
  - $\longrightarrow$ difficult to extract maximum shift, always many peaks (chromaticity?)

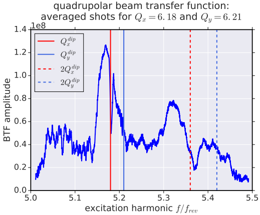
Tune Scan: BTFs Averaged over Shots



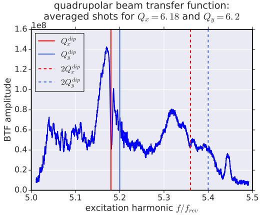
(a) $Q_x = 6.18,\ Q_y = 6.24$



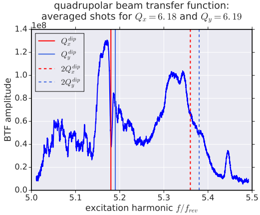
(b) $Q_x = 6.18,\ Q_y = 6.22$



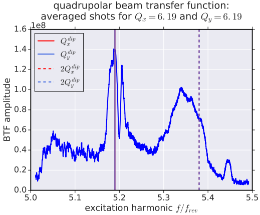
(c) $Q_x = 6.18,\ Q_y = 6.21$



(d) $Q_x = 6.18,\ Q_y = 6.20$



(e) $Q_x = 6.18,\ Q_y = 6.19$

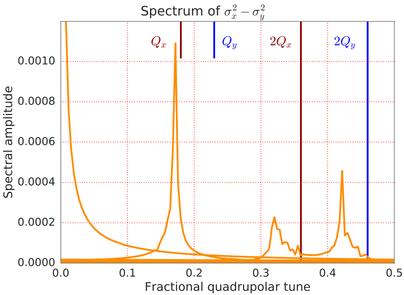


(f) $Q_x = 6.19,\ Q_y = 6.19$

… simulations?

Dispersive Coherent Mode

bunched	transv. distr.	synchrotron motion	dispersion	chromaticity
yes	KV (uniform)	yes	yes	no



- experimental parameters (here $N = 4 \times 10^{11}$ ppb)
- evident quadrupolar betatron bands below $2Q_{x,y}$
- coherent dispersive mode slightly below Q_x (shifted by space charge!)

$\longrightarrow$ however, only one peak is seen as opposed to experiment...

Including Chromaticity

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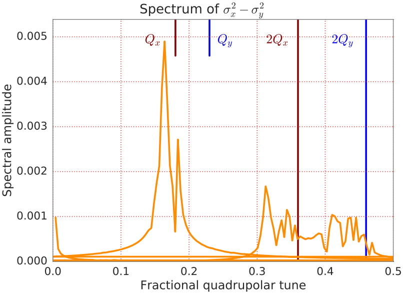

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⟶ including natural chromaticity ($Q'_x = -0.83Q_x$ and $Q'_y = -1.12Q_y$):

- broadens dispersive peak (here FFT undersamples sidebands)
- produces additional peaks, shifted dominant peak

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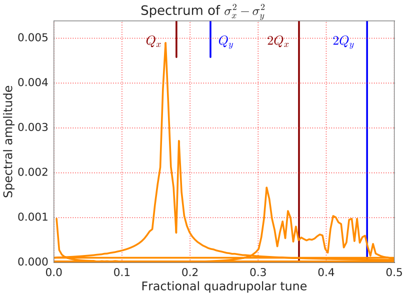

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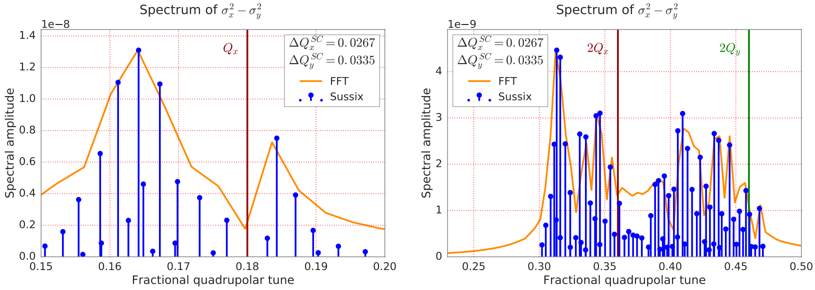
NB: simulations ran with 10×10^6 macro-particles on 150 longitudinal slices across the RF bucket (≈ 80 m) where space charge is solved on 128×128 grids (no significant transverse difference between 2.5D / 3D PIC)

Detailed Dispersive and Betatron QPU Spectrum

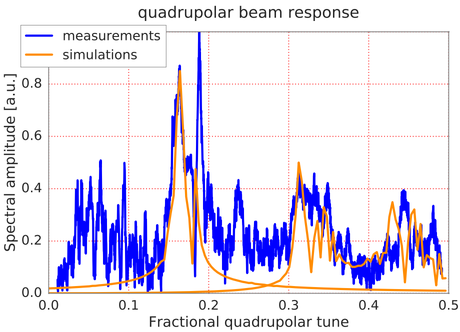


(a) dispersive part　　(b) betatron part

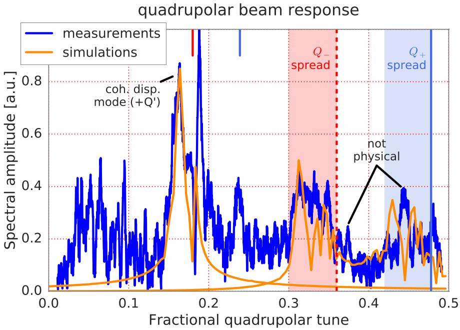
- Sussix tune analysis reveals regular sideband structure around dispersive mode (blue peaks)
- chromaticity also affects betatron spectrum, additional peaks distort betatron band (e.g. vertical extending beyond $2Q_y$)

  ⟶ with finite chromaticity, measuring betatron band width seems intricate

just giving it a try... Measurement vs. Simulation



just giving it a try... Measurement vs. Simulation



- horizontal quadrupolar betatron band below $2Q_x \approx 0.36$: similar width
- vertical quadrupolar betatron band (different Q_y in simulation and experiment): similar width

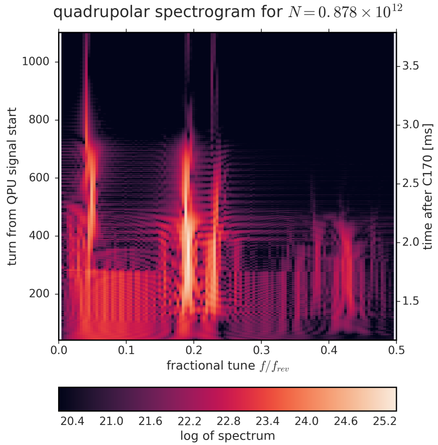
⟶ chromaticity seems to explain larger width of betatron bands (~factor 2) w.r.t. computation from 2D envelope equations (without dispersion)

- coherent dispersive mode peak at same frequency

  ⟶ width and sidebands closer to measurements (than without chromaticity)

3. Applications:
(b) PS injection oscillations (2018)

Injection Oscillations in QPU



Observations:

1. **coherent dispersive modes** around dipolar tunes $Q_x = 6.19$ and $Q_y = 6.23$
   - here transfer line into PS corrected for dispersion mismatch (cf. Vincenzo Forte's poster)

   ⟶ oscillation about both D_x and D_y, nominal settings only D_x
2. injection into strongly coupled optics

   ⟶ **Chernin's odd envelope modes** $Q_y - Q_x$, $Q_y + Q_x$ visible
3. **even envelope modes** gone $\lesssim 50$ turns, difficult!

Summary and Outlook

In conclusion:

- development of quadrupolar pick-up as powerful diagnostic tool
  - ⟶ for space charge also in bunched beams
  - ⟶ injection mismatch (betatron, dispersion, coupling)
- coherent **dispersive mode** identified as strong quadrupolar spectral component (especially for injection oscillations)
- **chromaticity** significantly impacts quadrupolar spectrum:
  - broadens betatron bands $\Longrightarrow$ complicates estimation of $\Delta Q_{x,y}^{\mathrm{KV}}$
  - shifts coherent dispersive mode and creates sidebands

Next steps for ongoing studies:

- further investigate injection oscillations and spectrum
- infer $\Delta Q_{x,y}^{\mathrm{KV}}$ with vanishing chromaticity in PS
- dedicated space charge experiments (e.g. resonance studies)

$\Longrightarrow$ theory for chromaticity impact on quadrupolar eigenmodes?

Thank you for your attention!

Appendix

References I

[1] R Singh et al. "Observations of the quadrupolar oscillations at GSI SIS-18". In: (2014).

[2] R H Miller et al. *Nonintercepting emittance monitor*. Tech. rep. Stanford Linear Accelerator Center, 1983.

[3] Andreas Jansson. "Noninvasive single-bunch matching and emittance monitor". In: *Physical Review Special Topics-Accelerators and Beams* 5.7 (2002), p. 072803.

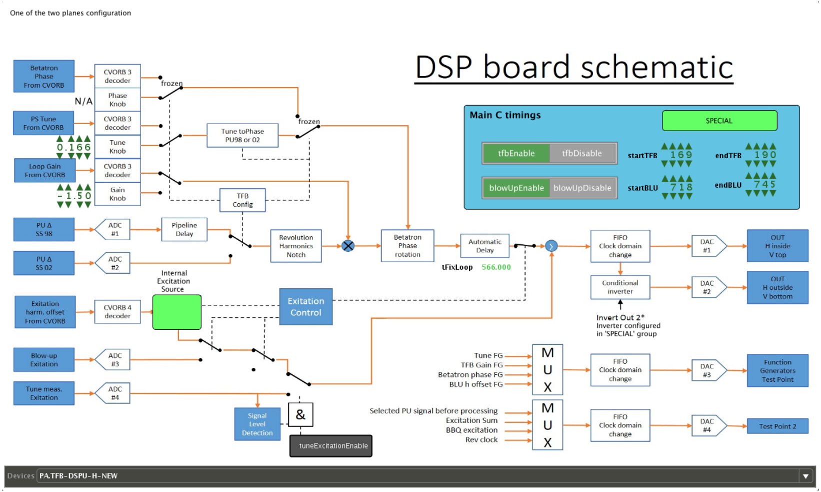
[4] Cheng-Yang Tan. *Using the quadrupole moment frequency response of bunched beam to measure its transverse emittance*. Tech. rep. Fermi National Accelerator Laboratory (FNAL), Batavia, IL, 2007.

[5] Michel Chanel. *Study of beam envelope oscillations by measuring the beam transfer function with quadrupolar pick-up and kicker*. Tech. rep. 1996.

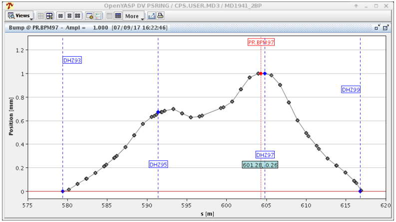
[6] T Uesugi et al. "Observation Of Quadrupole Mode Frequency And Its Connection With Beam Loss". In: KEK-99-98 (1999). URL: http://cds.cern.ch/record/472700.

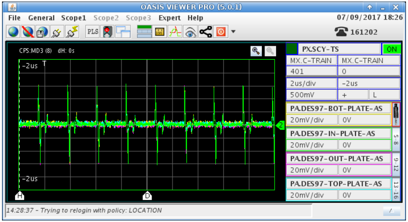
[7] R C Baer. "Untersuchung der quadrupolaren BTF-Methode zur Diagnose intensiver Ionenstrahlen". Universitaet Frankfurt, Germany, 2000.

TFB: Schematic Plan

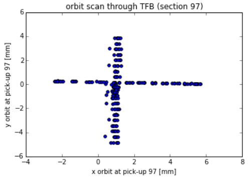


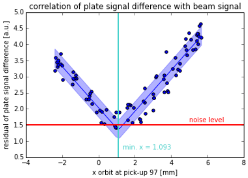
Set up a local bump through the TFB and measure the induced beam signal on the plates (effectively a BPM):

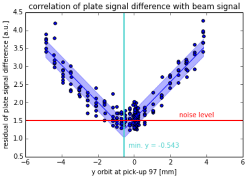




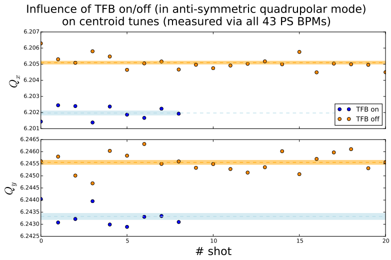
By scanning the orbit location one can minimise the difference signal:

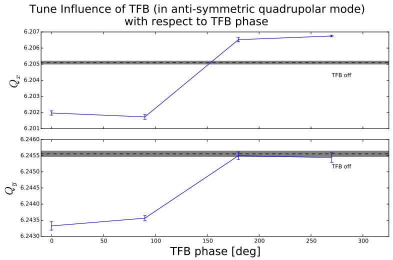




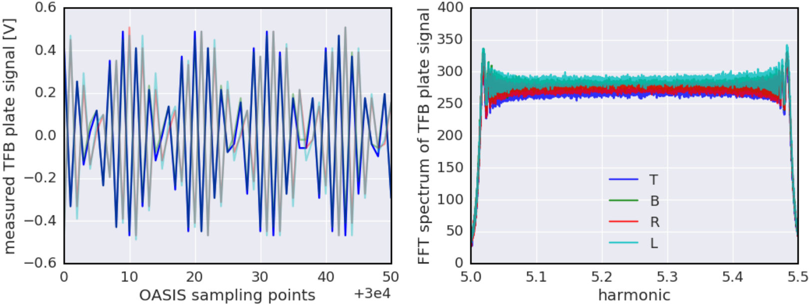


- TFB pulsing at f_{rev} becomes a static quadrupole to the beam
- varying the phase of the pulsing RF quadrupole changes the tune impact





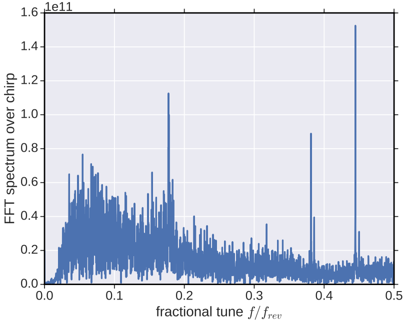
TFB: Quadrupolar Chirp

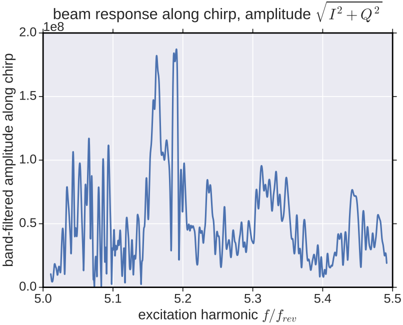


- top and bottom plates oscillate together in anti-phase to right and left plates of transverse feedback
  - ⟶ quadrupolar RF excitation (anti-symmetric mode)
- frequency swept during BTF measurement: 2.19 MHz to 2.4 MHz
  - ⟶ harmonic 5 to 5.5 (PS revolution frequency $f_{\text{rev}} = 437\,\text{kHz}$)

Extracting the Beam Response...

(a) FFT across up-chirp time is not such a useful idea...





(b) ... instead project and band filter along local excitation frequency

Approach: In-phase and Quadrature Components

Take

a) QPU time signal $S_{\mathrm{QPU}}(t)$

b) excitation signal $S_{\mathrm{exc}}(t)$ (sine wave with increasing frequency)

c) 90 deg shifted excitation signal $C_{\mathrm{exc}}(t) = S_{\mathrm{exc}}(t)|_{\phi\to\phi+90\,\mathrm{deg}}$

Assume immediate beam response to chirp:

1. **correlation**: find excitation start in $S_{\mathrm{QPU}}(t)$ by correlation with $S_{\mathrm{exc}}(t)$
2. **demodulation** of measured QPU time signal into

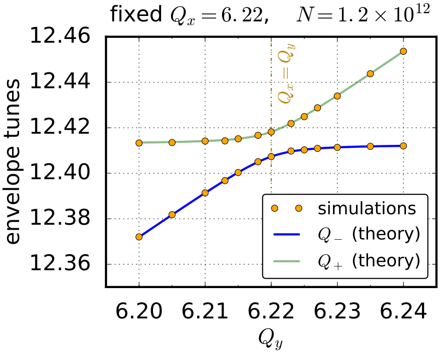
$$I(t) = S_{\mathrm{QPU}}(t) \cdot S_{\mathrm{exc}}(t) \qquad \text{(in-phase component)}$$

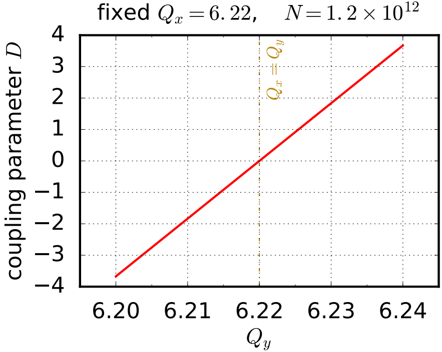
$$\text{and} \quad Q(t) = S_{\mathrm{QPU}}(t) \cdot C_{\mathrm{exc}}(t) \qquad \text{(quadrature component)}$$

3. **band filter** original $S_{\mathrm{QPU}}(t)$ around time-varying excitation frequency by low pass filtering $I(t)$ and $Q(t)$
4. **amplitude** of beam response along chirp amounts to $\sqrt{I^2(t) + Q^2(t)}$

Simulations for Tune Scan

Simulations with KV beams for $N = 1.2 \times 10^{12}$ confirm theory:

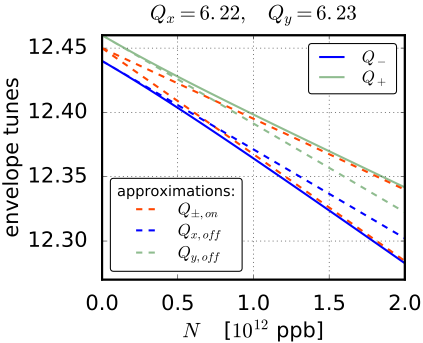


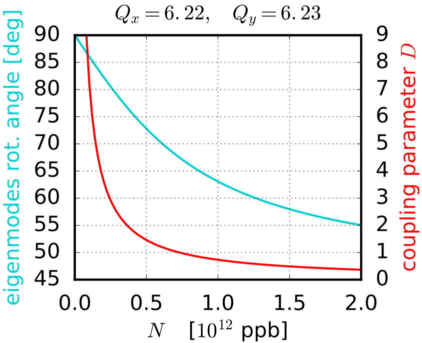


⟶ r.m.s. equivalent Gaussian beams (with same $\sigma_{x,y}$ like KV beams) exhibit same quadrupolar tunes as KV

⚠ Gaussian spectra broaden quickly

With slightly split tunes, approach full coupling by increasing bunch intensity:





$\Longrightarrow$ scan space charge tune shift $\Delta Q^{\mathrm{KV}}_{x,y}$ and verify theory

Envelope Equations

Envelope equations of motion (e.o.m.)

$$r_x'' + K_x(s)\,r_x - \frac{\epsilon_{x,\mathrm{geo}}^2}{r_x^3} - \frac{K^{\mathrm{SC}}}{2(r_x + r_y)} = 0 \quad , \tag{5a}$$

$$r_y'' + K_y(s)\,r_y - \frac{\epsilon_{y,\mathrm{geo}}^2}{r_y^3} - \frac{K^{\mathrm{SC}}}{2(r_x + r_y)} = 0 \tag{5b}$$

for transverse r.m.s. beam widths $r_{x,y} = \sigma_{x,y}$ have equilibrium

$$\frac{Q_x^2}{R^2}\,r_{x,\mathrm{m}} - \frac{\epsilon_{x,\mathrm{geo}}^2}{r_{x,\mathrm{m}}^3} - \frac{K^{\mathrm{SC}}}{2(r_{x,\mathrm{m}} + r_{y,\mathrm{m}})} = 0 \quad , \tag{6a}$$

$$\frac{Q_y^2}{R^2}\,r_{y,\mathrm{m}} - \frac{\epsilon_{y,\mathrm{geo}}^2}{r_{y,\mathrm{m}}^3} - \frac{K^{\mathrm{SC}}}{2(r_{x,\mathrm{m}} + r_{y,\mathrm{m}})} = 0 \tag{6b}$$

Linear Perturbation in Smooth Approximation

Constant focusing channel

$$K_{x,y} = \frac{1}{\beta_{x,y}^2} = \frac{Q_{x,y}^2}{R^2} = \text{const} \tag{7}$$

gives linearised e.o.m. for perturbation around equilibrium $r = r_\mathrm{m} + \delta r$

$$\frac{d^2}{ds^2}\begin{pmatrix}\delta r_x \\ \delta r_y\end{pmatrix} = -\underbrace{\begin{pmatrix}\kappa_x & \kappa_\mathrm{SC} \\ \kappa_\mathrm{SC} & \kappa_y\end{pmatrix}}_{\doteq(\kappa)} \cdot \begin{pmatrix}\delta r_x \\ \delta r_y\end{pmatrix} \tag{8}$$

$$\text{with} \quad \begin{cases} \kappa_{x,y} = 4\frac{Q_{x,y}^2}{R^2} - \frac{2\sigma_{x,y}+3\sigma_{y,x}}{\sigma_{x,y}}\kappa_\mathrm{SC} \\ \kappa_\mathrm{SC} \doteq \frac{K^\mathrm{SC}}{2(\sigma_x+\sigma_y)^2} \end{cases} \tag{9}$$

Definitions

Coupling Parameter

$$D \doteq \frac{\kappa_y - \kappa_x}{2\kappa_{\mathrm{SC}}} = 4\frac{Q_y^2 - Q_x^2}{K^{\mathrm{SC}} R^2}(\sigma_x + \sigma_y)^2 + \frac{3}{2}\left(\frac{\sigma_y}{\sigma_x} - \frac{\sigma_x}{\sigma_y}\right) \tag{10}$$

Rotation Into Decoupled Eigensystem

$$\begin{aligned} \tan(\alpha) &= \frac{1}{2\kappa_{\mathrm{SC}}}\left[\kappa_y - \kappa_x + \sqrt{4\kappa_{\mathrm{SC}}^2 + (\kappa_y - \kappa_x)^2}\right] \\ &= D + \sqrt{1 + D^2} \end{aligned} \tag{11}$$

Incoherent Tune Shifts

KV Space Charge Tune Shift

$$\Delta Q_{x,y}^{\mathsf{KV}} = -\frac{K^{\mathsf{SC}} R^2}{4\sigma_{x,y}(\sigma_x + \sigma_y) Q_{x,y}} \tag{12}$$

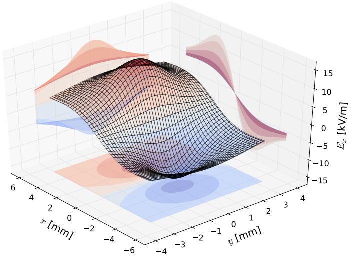
$$\text{with} \quad K^{\mathsf{SC}} \doteq \frac{q\lambda}{2\pi\epsilon_0 \beta\gamma^2 p_0 c} \tag{13}$$

R.m.s. Equivalent Gaussian Space Charge Tune Spread

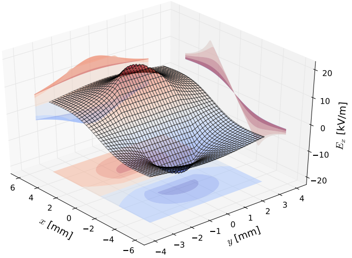
linearised Gaussian e-field = twice r.m.s. equivalent KV e-field

$$\Longrightarrow \qquad \max\left\{\Delta Q_{x,y}^{\mathsf{spread}}\right\} = 2\,\Delta Q_{x,y}^{\mathsf{KV}} \tag{14}$$

Gaussian vs. R.m.s. Equivalent KV



(a) Gaussian beam

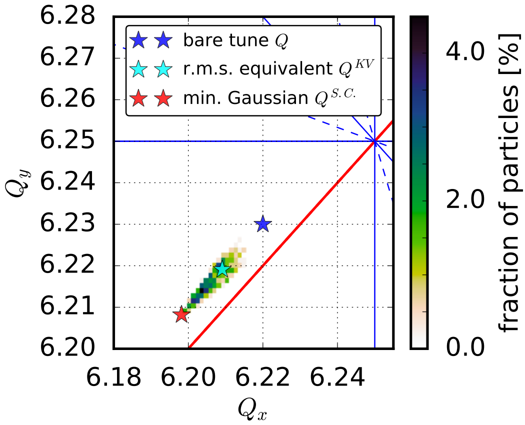


(b) r.m.s. equivalent KV beam

Figure: Electric fields in r.m.s. equivalent distributions with same $\sigma_{x,y}$

Incoherent Tunes and R.m.s. Equivalence

Incoherent tune spread of a coasting, transversely Gaussian distribution:



Quadrupolar Mode Formulae

Quadrupolar Mode Tunes (General Formula)

$$Q_{\pm}^2 = \frac{R^2}{2}\left[\kappa_x + \kappa_y \pm \sqrt{4\kappa_{\mathsf{SC}}^2 + (\kappa_y - \kappa_x)^2}\right] \tag{15}$$

$$= 2(Q_x^2 + Q_y^2) - \frac{K^{\mathsf{SC}} R^2}{(\sigma_x + \sigma_y)^2}\left[1 + \frac{3}{4}\left(\frac{\sigma_y}{\sigma_x} + \frac{\sigma_x}{\sigma_y}\right) \mp \frac{\sqrt{1+D^2}}{2}\right]$$

Quadrupolar Mode Formulae

Off-resonance $D \gg 1$ With Round Beam

$$Q_+ = 2Q_y - \frac{5}{4}|\Delta Q_y^{\mathsf{KV}}| \quad , \tag{16a}$$

$$Q_- = 2Q_x - \frac{5}{4}|\Delta Q_x^{\mathsf{KV}}| \quad . \tag{16b}$$

for $Q_y > Q_x$ otherwise exchange $x \leftrightarrow y$

Quadrupolar Mode Formulae

On-resonance $D \approx 0$ With Round Beam

$$Q_+ = 2Q_0 - |\Delta Q^{\mathrm{KV}}| \quad , \tag{17a}$$

$$Q_- = 2Q_0 - \frac{3}{2}|\Delta Q^{\mathrm{KV}}| \quad . \tag{17b}$$

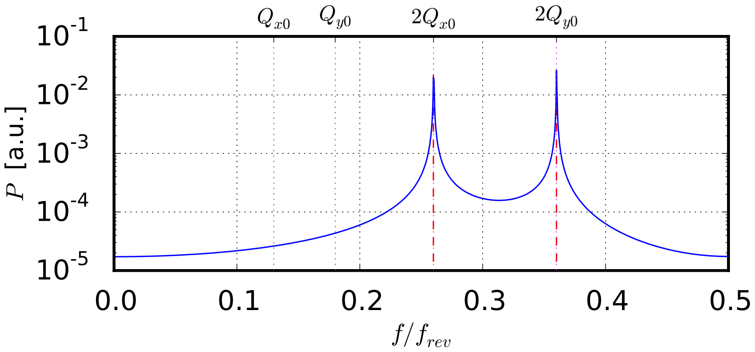
for $Q_0 \doteq Q_x = Q_y$ and $\Delta Q^{\mathrm{KV}} \doteq \Delta Q_x^{\mathrm{KV}} = \Delta Q_y^{\mathrm{KV}}$

QPU Simulations in SPS

simulation parameters:

- machine: SPS at injection
- $\gamma = 27.7$
- $\epsilon_x = \epsilon_y = 2.5\,\text{mm} - \text{mrad}$
- $N_b = 1.25 \times 10^{11}$
- $512 - 2048$ turns
- 2.6×10^5 macro-particles
- longitudinally matched Gaussian-type distribution
- betatron mismatch by 10% in both x, y

$\Longrightarrow$ injection oscillations need beam mismatched to both β_x, β_y to see clear peaks

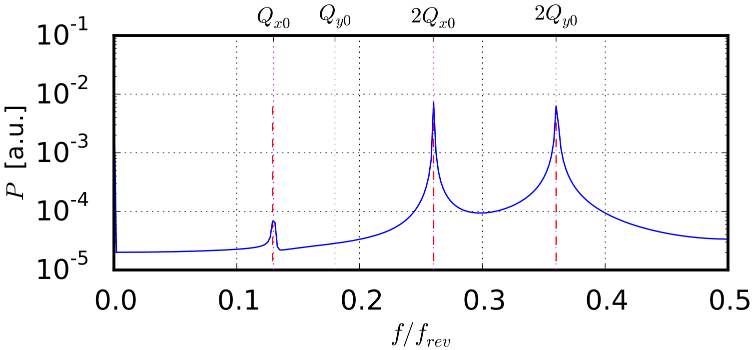


$\Longrightarrow$ $2Q_{x0}$, $2Q_{y0}$ from undepressed envelope oscillations

$\Longrightarrow$ including synchrotron motion: same spectrum (no coupling!)

QPU Spectrum: Include Dispersion

smooth approximation: constant $D_x = 2.96$ around the ring



$\Longrightarrow$ peak at Q_{x0} comes from dispersion

Reason for Dispersion Peak

$$\sigma_x(i_{\text{turn}}) = \sqrt{\langle x_i^2 \rangle_{\text{beam}} - \langle x_i \rangle_{\text{beam}}^2}$$

with $\quad x_i(i_{\text{turn}}) = \sqrt{\beta_x \epsilon_{x,i}^{\text{s.p.}}} \cos(2\pi Q_{x0} i_{\text{turn}} + \Psi_0) + D_x \delta_i$

Reason for Dispersion Peak

$$\sigma_x(i_{\text{turn}}) = \sqrt{\langle x_i^2 \rangle_{\text{beam}} - \langle x_i \rangle_{\text{beam}}^2}$$

with $$x_i(i_{\text{turn}}) = \sqrt{\beta_x \epsilon_{x,i}^{\text{s.p.}}} \cos(2\pi Q_{x0} i_{\text{turn}} + \Psi_0) + D_x \delta_i$$

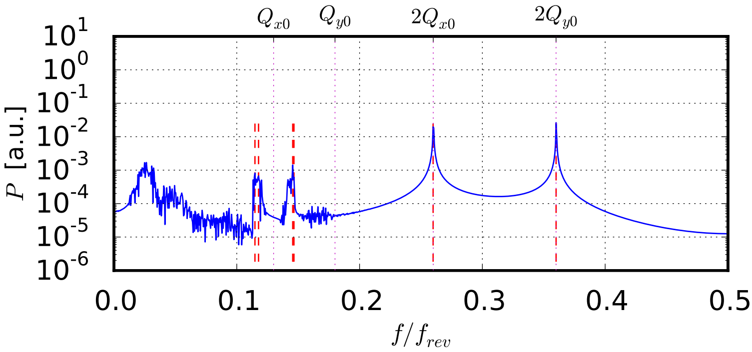
$$\stackrel{\sim}{\Longrightarrow} x_i^2 = \ldots \underbrace{\cos^{2}(2\pi Q_{x0} i_{\text{turn}} + \ldots)}_{\ldots \cos(2\pi 2 Q_{x0} i_{\text{turn}} + \ldots)} + \ldots D_x \delta_i \cdot \cos(2\pi Q_{x0} i_{\text{turn}} + \ldots) + \ldots$$

due to: $2\cos^2(\alpha) = \cos(2\alpha) + 1$

i.e. only for $D_x \neq 0 \implies$ peak at Q_{x0}

QPU Spectrum: Include Synchrotron Motion

synchrotron motion couples to betatron motion through non-zero $D_x = 29.6\,\mathrm{m}$ (smooth approximation!)

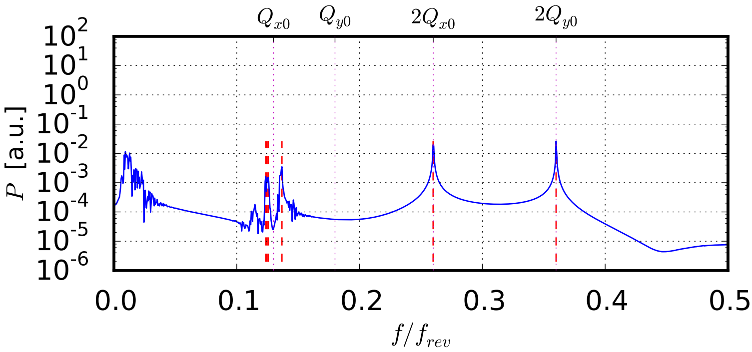


$\Longrightarrow$ peak separation at Q_{x0} from synchrobetatron coupling

- $Q_s = 0.017$ at injection for $V = 5.75\,\mathrm{MV}$

QPU Spectrum: Slower Synchrotron Motion

synchrotron motion couples to betatron motion through non-zero $D_x = 29.6\,\text{m}$ (smooth approximation!)



- $Q_s = 0.007$ changing $\gamma_{tr} = 17.95 \longrightarrow 25$ (while $\gamma = 27.7$)

$\Longrightarrow$ peak separation shrinks

Reason for Peak Separation with Q_s

$$x_i^2 = \ldots + \ldots D_x \delta_i \cdot \cos(2\pi Q_{x0} i_{\mathrm{turn}} + \ldots) + \ldots$$

with $\delta_i(i_{\mathrm{turn}}) = \hat{\delta}_i \cos(2\pi Q_s i_{\mathrm{turn}} + \ldots)$

Reason for Peak Separation with Q_s

$$x_i^2 = \ldots + \ldots D_x \delta_i \cdot \cos(2\pi Q_{x0} i_{\text{turn}} + \ldots) + \ldots$$

with $\quad \delta_i(i_{\text{turn}}) = \hat{\delta}_i \cos(2\pi Q_s i_{\text{turn}} + \ldots)$

$$\overset{\sim}{\Longrightarrow} x_i^2 = \ldots + \ldots \underbrace{\cos(2\pi Q_{x0} i_{\text{turn}} + \ldots)\cos(2\pi Q_s i_{\text{turn}} + \ldots)}_{\cos(2\pi(Q_{x0}-Q_s) i_{\text{turn}}+\ldots)+\cos(2\pi(Q_{x0}+Q_s) i_{\text{turn}}+\ldots)} + \ldots$$

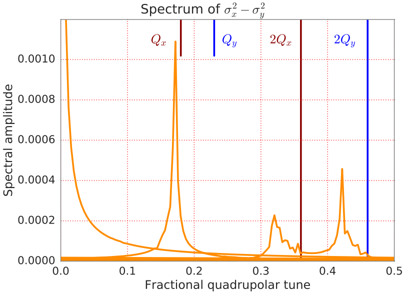
due to: $\quad 2\cos(\alpha)\cos(\beta) = \cos(\alpha-\beta) + \cos(\alpha+\beta)$

i.e. for $D_x \neq 0$ and $Q_s \neq 0$:

one peak at $Q_{x0} \Longrightarrow$ two peaks located at $Q_{x0} \pm Q_s$

PS: Bunched Beam with SC (Normal $V_{\mathrm{RF}} = 24\,\mathrm{kV}$)

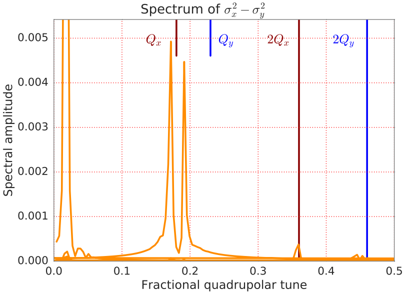
bunched	transv. distr.	synchrotron motion	dispersion	chromaticity
yes	KV (uniform)	yes	yes	no



- coherent dispersive mode with synchrotron motion splits into two peaks
- at usual $V_{\mathrm{RF}} = 24\,\mathrm{kV}$ we have $Q_S \approx 1/600$

PS: Bunched Beam with SC (Large $V_{RF} = 1.5\,\text{MV}$)

bunched	transv. distr.	synchrotron motion	dispersion	chromaticity
yes	KV (uniform)	yes	yes	no



- coherent dispersive mode with synchrotron motion splits into two peaks
- at usual $V_{RF} = 24\,\text{kV}$ we have $Q_S \approx 1/600$
- at (unrealistic) $V_{RF} = 1.5\,\text{MV}$ we have $Q_S \approx 0.0107$

$\Longrightarrow$ two peaks are clearly separated in quadrupolar spectrum