Analysis of spontaneous emission and its self-amplification in free-electron laser

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FEL equations

in the time domain, gain describing, the scaling and FEL simulations
invoking a average of the e- beam over a longitudinal distance
not applicable to the problems related spontaneous emission

Spontaneous emission studied with the L.-W potential equation
dose not describe the interaction of e-_ radiation

theory study of SASE is carried out in the frequency domain

Including the spontaneous emission in the same framework
is meaningful and convenient to FEL analysis

with the time domain approach:
we investigate spontaneous emission (incoherent and coherent)
and its self-amplification in FEL

Spontaneous Emission Equation

$$\left(\frac{\partial}{\partial z}+\frac{1}{c}\frac{\partial}{\partial t}\right)\tilde{a}_s=\frac{\lambda_s r_e a_u \delta_p}{\gamma \Sigma_e}e^{-i(k_s z-\omega_s t)}\sum_j \delta(z-z_j)e^{-ik_u z_j}$$

$$\delta_p=\begin{cases}1, & \text{circular polarization}\\ [J,J]_n, & n=1,3,5,\ldots\ \text{line polarization}\end{cases}$$

change the variables (z, t) ⇨ $Z=z-ct,\quad \tau=ct$

*j*th e- $z_j=\beta ct+z_0+\zeta_j=\beta\tau+z_0+\zeta_j$,

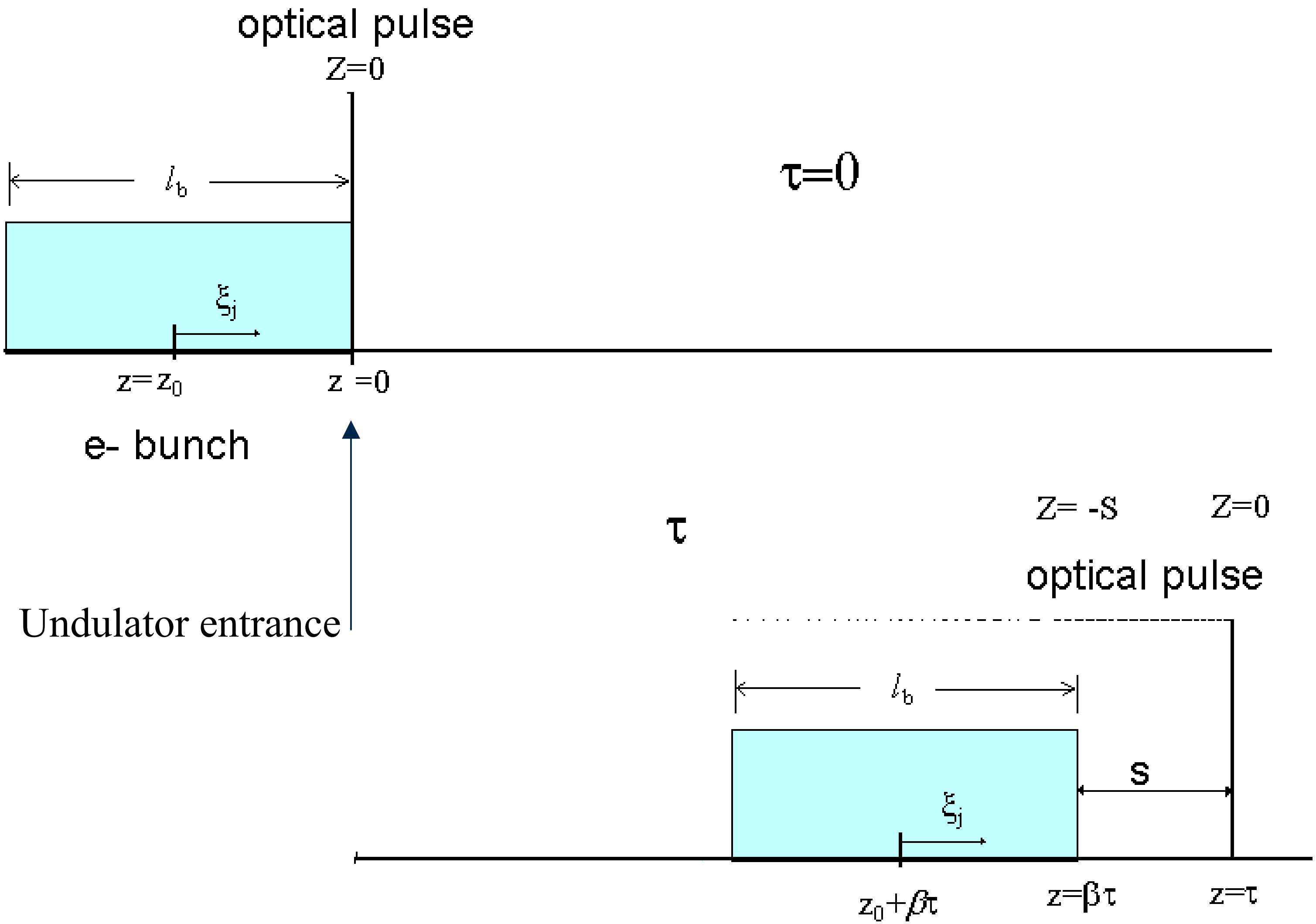
z_0 initial position of the reference point

ζ_j relative position respect to z_0 within the bunch for *j*th e-.

$$\frac{\partial}{\partial \tau}\tilde{a}_s=\frac{\lambda_s r_e a_u \delta_p}{\gamma \Sigma_e}e^{-ik_s Z}\sum_j \delta(S-(\zeta_j+z_0-Z))e^{-ik_u(z_0+\beta\tau+\xi_j)}$$

$S=(1-\beta)\tau$: slippage distance.

the coordinate variables relation



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ζ_j: Generally is dependent on the time

but for the spontaneous emission

e- have no interaction with optical field,

is independent on the time

Spontaneous emission

$$\tilde{a}_{sp}(Z,\tau) = \frac{r_e \lambda_u a_u \delta_p}{\gamma \Sigma_e} e^{i\frac{\dot{\phi}}{1-\beta}Z} \sum_j e^{-i\frac{k_u}{1-\beta}(\zeta_j + z_0)} H(S - z_0 - \zeta_j + Z) H(z_0 + \zeta_j - Z)$$

$$\tilde{a}_{sp}^2(Z,\tau) = \left(\frac{r_e \lambda_u a_u \delta_p}{\gamma \Sigma_e}\right)^2 \left\langle \left(\sum_j e^{-ik_s \zeta_j} H(S - \zeta_j - z_0 + Z) H(\zeta_j + z_0 - Z) \right)^2 \right\rangle$$

$<\quad>$: the ensemble average over bunches

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- incoherent spontaneous emission: (term *i*=*j*)

$$a_{SE}^2 = \left(\frac{r_e \lambda_u a_u \delta_p}{\gamma \Sigma_e}\right)^2 N_{e,l}$$

$$-l_b/2 < \zeta_j < l_b/2$$

$\boldsymbol{N_{e,l}}$ the number of e- in the distance $l=\zeta_2-\zeta_1$

$$\zeta_1 = \max[-l_b/2, Z - z_0]$$

$$\zeta_2 = \min[l_b/2, S + Z - z_0]$$

- coherent spontaneous emission: (cross term $i \neq j$)

$$\tilde{a}_{CSE}^2(Z,\tau) = \left(\frac{r_e \lambda_u a_u \delta_p}{\gamma \Sigma_e}\right)^2 \left\langle \sum_{j,l=1(j\neq l)}^{N_e} e^{-ik_s(\zeta_j-\zeta_l)} H(S-\zeta_j - z_0 + Z) H(\zeta_j + z_0 - Z) \right.$$

$$\left. * H(S-\zeta_l - z_0 + Z) H(\zeta_l + z_0 - Z) \right\rangle$$

$$\cong \left(\frac{r_e \lambda_u a_u \delta_p}{\gamma \Sigma_e}\right)^2 N_e^2 \left| \int f(\zeta) e^{-ik_s\zeta} H(S-\zeta - z_0 + Z) H(\zeta + z_0 - Z) d\zeta \right|^2$$

$f(\zeta)$: normalized e- density distribution function

Written together

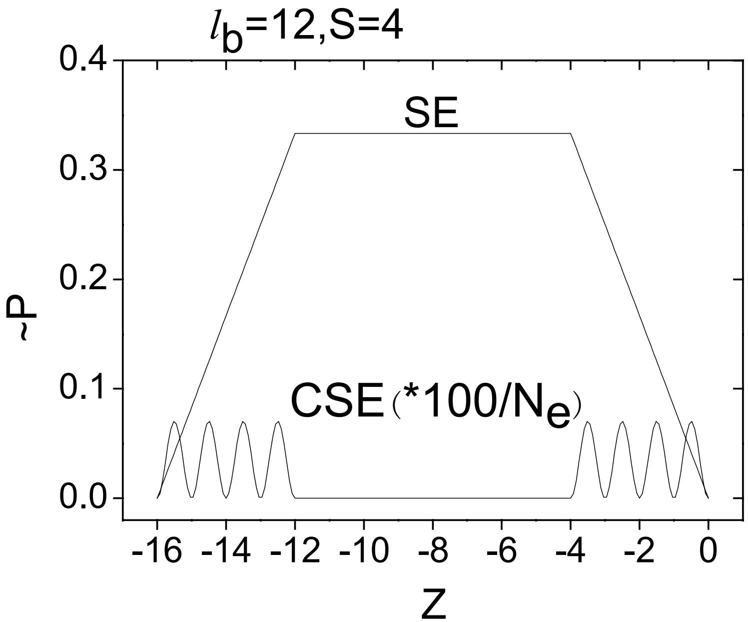
$$\tilde{a}_{sp}^2(Z,\tau) = \left(\frac{r_e \lambda_u a_u \delta_p}{\gamma \Sigma_e}\right)^2 \left(N_e \int_{\zeta_1}^{\zeta_2} f(\zeta)d\zeta + N_e^2 \left| \int_{\zeta_1}^{\zeta_2} f(\zeta) e^{-ik_s\zeta} d\zeta \right|^2 \right)$$

Both incoherent spontaneous emission and coherent spontaneous emission related with slippage distance

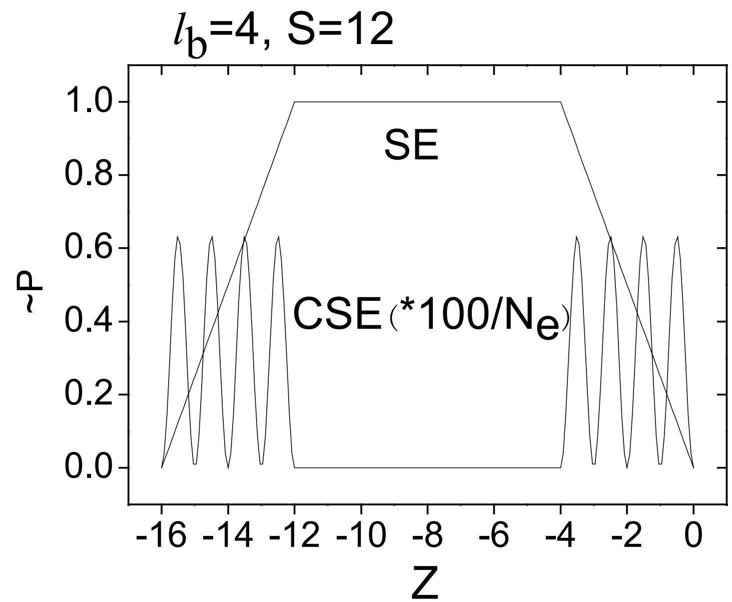
in the body of the radiation pulse

$$l = \zeta_2 - \zeta_1 = \begin{cases} l_b & \text{for short e- bunch } (l_b << S) \\ S & \text{for long e- bunch } (l_b >> S) \end{cases}$$

long e- bunch



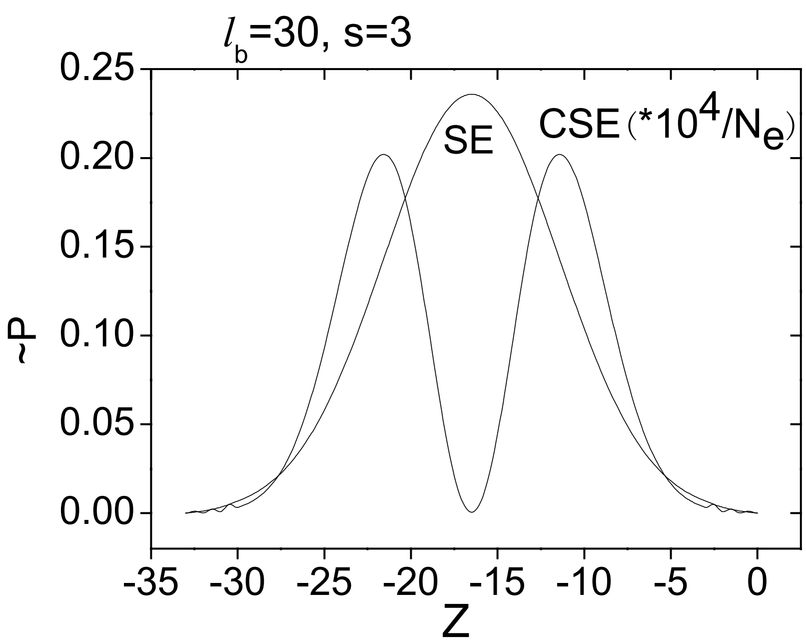
short e- bunch

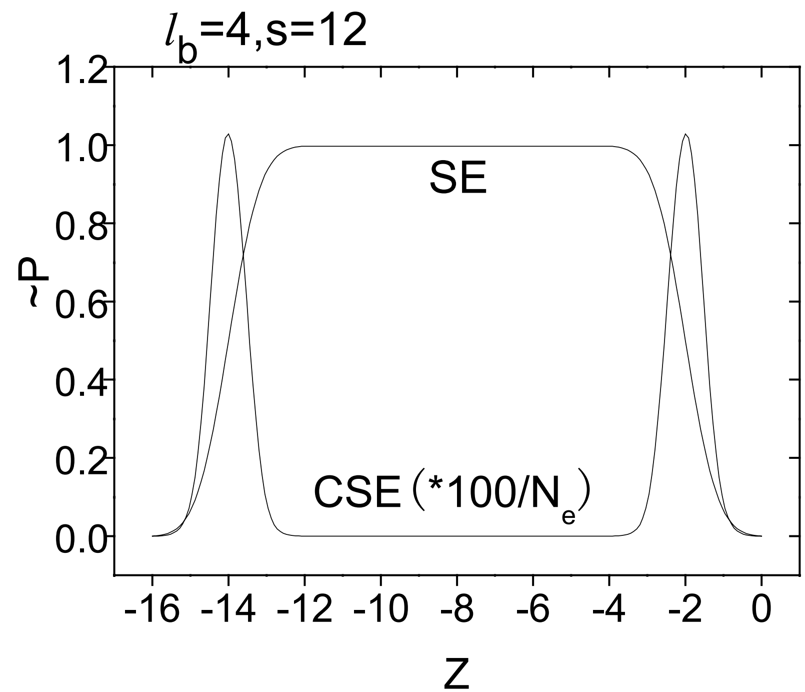


Incoherent and coherent spontaneous emission
for a rectangle profile e- pulse

The longitudinal quantities are scaled to the radiation wavelength long e- bunch

short e- bunch





Incoherent and coherent spontaneous emission
for a Gaussian profile e- pulse

The longitudinal quantities are scaled to the radiation wavelength

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Some instance

rectangle e- bunch distribution:

Power

$$P_{SE} = (2k_s l)^2 \rho^3 P_e / N_{e,l}$$

$$P_{CSE} = 16\rho^3 P_e \sin^2(k_s l/2)$$

$$\frac{P_{CSE}}{P_{SE}} = N_{e,l} \operatorname{sinc}^2(\frac{\pi}{\lambda_s} l) \xrightarrow[l/\lambda_s \ll 1]{} N_{e,l}$$

l: $l(Z)$, integral over Z

Energy

$$W_{SE} = 4k_s^2 l_b^2 \rho^3 P_e S / N_e$$

$$W_{CSE} = 16\rho^3 P_e \min[S, l_b]$$

$$\frac{W_{CSE}}{W_{SE}} = \left(\frac{\lambda_s}{\pi l_b}\right)^2 N_e \frac{\min[S, l_b]}{S}$$

for a long e- bunch ($l_b >> S$)

coasting beam

in the body of the radiation pulse it has $l=S$

$$a_{SE}^2 = 16\pi N \lambda_s r_e \gamma \rho^3 / \Sigma_e$$

$$P_{SE} = 8\pi\omega_s N \rho^3 \varepsilon_e$$

$$d\omega = \omega_s / N$$

$$\frac{dP}{d\omega} = 8\pi N^2 \rho^3 \varepsilon_e = \frac{1}{6\pi}\left(\frac{N}{N_g}\right)^2 \rho\varepsilon_e$$

$N = \sqrt{3} N_g$

$$\left(\frac{dP}{d\omega}\right)_{sn} = \frac{\rho\varepsilon_e}{2\pi}$$

Effective noise spectra.(L.H.Yu)

$$d\Omega = \lambda_s^2 / \Sigma_e$$

$$\frac{dP}{d\omega d\Omega} = \frac{eI}{2c}\frac{a_u^2 \delta_p^2 L^2}{\lambda_s^2 \gamma^2}$$

Undulator radiation on axis (from L.W.potential)

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For the ideal case

All e-s are modulated, the spontaneous emission is full coherent:

$$a_{CSE}^2 = (\frac{r_e \lambda_u a_u \delta_p}{\gamma \Sigma_e})^2 N_{e,l}^2 \qquad P_{CSE} = (2k_s l \rho)^2 \rho P_e \qquad \frac{P_{CSE}}{P_{SE}} = N_{e,l}$$

For long bunch (l_b >S): l=S

$$P_{CSE} = (4\pi N \rho)^2 \rho P_e$$

$$\frac{P_{CSE}}{P_{SE}} = N_{e,s}$$

number of e-s in the slippage distance

Coherent enhancement factor of CHG (coherent harmonic generation)

existing theory:

L.-W. potential ⟹ spontaneous emission coherent enhancement factor ⟹ coherent radiation

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FEL equations ⟹ coherent radiation *

$$\tilde{a}_s^2 = \left(\frac{r_e \lambda_s a_u \delta_p n_e L}{\gamma}\right)^2 J_n^2(n\Delta\xi) f_r^{\ 2} \qquad P = (4\pi N\rho)^2 J_n^2(n\Delta\xi) f_r^{\ 2} \rho P_e$$

(previous ⟹) incoherent spontaneous radiation

$$a_{SE}^2 = 16\pi N \lambda_s r_e \gamma \rho^3 / \Sigma_e \qquad P_{SE} = 8\pi \omega_s N \rho^3 \varepsilon_e$$

Coherent enhancement factor

$$R_n = N_{e,s} J_n^2(n\Delta\xi) f_r^{\ 2}$$

$N_{e,s}$: number of e-s in the slippage distance for radiator section undulator of CHG.

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*JiaQika,; Phys.Rev.ST.Accel.Beams, Vol.8, No.6(2005)

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Effective Start-up Power of SASE

When there exist an optical field interaction with the e-s,
e- distribution cannot be regarded as independent on the time.

$$\frac{\partial}{\partial \tau}\tilde{a}_s = \frac{\lambda_s r_e a_u \delta_p}{\gamma \Sigma_e} e^{-i\varphi} \int d\dot{\varphi} f(\tau, Z, \dot{\varphi})$$

$$\left[\frac{\partial}{\partial \tau} - (1-\beta)\frac{\partial}{\partial Z} + \ddot{\varphi}\frac{\partial}{\partial \dot{\varphi}}\right] f = 0$$

$f = f_0 + f_1$, mono-energetic e-:

$$f_0 = \sum_j \delta(S - (\zeta_{j0} + z_0 - Z)) = \frac{1}{1-\beta}\sum_j \delta(\tau - \frac{z_0 - Z + \zeta_{j0}}{1-\beta})$$ give spontaneous emission

$$f_1 = \frac{2k_s k_u a_u \delta_p}{\gamma^2} \mathrm{Re}(e^{i\phi} \int_0^\tau \frac{\partial f_0}{\partial \dot{\phi}} \tilde{a}_s e^{-i\dot{\phi}(\tau-\tau')} d\tau')$$ perturbing term $<< f_0$

$$\tilde{a}_s = \tilde{a}_{s0} + \tilde{a}_{sp} + \frac{(2k_u\rho)^3}{\chi_e}\int_0^{\tau} d\tau' \int d\dot{\phi} \int_0^{\tau'} d\tau'' \frac{\partial f_0}{\partial \dot{\phi}} \tilde{a}_s e^{-i\dot{\phi}(\tau'-\tau'')}$$

a_{s0} : input optical field; a_{sp} : spontaneous emission, (previous)

χ_e : average linear density of e-s

Consider the coasting beam, the mono-energetic e-
above eq. can be solved by *Laplace* transform

$$\tilde{a}_s(\tau) \doteq \sum \mathrm{Re}\, s \frac{(\tilde{a}_{s0} + \mu\tilde{a}_{sp}(\mu))(\mu + i\phi_0')^2 e^{\mu z}}{\mu(\mu + i\phi_0')^2 - i(2k_u\rho)^3} = (2k_u\rho)^3 \sum_{\substack{m=1 \\ m\neq l,k}}^{3} \frac{i(\tilde{a}_{s0} + \mu_m\tilde{a}_{sp}(\mu_m))e^{\mu_m z}}{\mu_m(\mu_m - \mu_l)(\mu_m - \mu_k)}$$

$$\tilde{a}_{sp}(\mu) = \left(\frac{r_e\lambda_s a_u\delta_p}{\gamma\Sigma_e}\right)\frac{1}{\mu}\exp\left[\frac{\mu + i\dot{\phi}}{1-\beta}Z\right]\sum_j \exp\left[-\left(\frac{\mu + ik_u}{1-\beta}\right)(\zeta_j + z_0)\right]H(\zeta_j + z_0 - Z)$$

$$\mu(\mu + i\phi_0')^2 - i(2k_u\rho)^3 = 0$$

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the leading role is the exponential growth term at the resonant energy $\dot{\phi}_0 = 0$, $\mu_1 = k_u \rho(\sqrt{3} + i)$

We obtain

$$\tilde{a}_s^2(\tau) = \frac{1}{9}(\tilde{a}_{s0}^2 + \tilde{a}_{ef}^2)\,\mathrm{e}^{\frac{\tau}{L_g}}$$

$\tilde{a}_{ef}^2$ effective input power of SASE

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Effective input power of SASE

$$a_{ef}^2 = (\frac{r_e \lambda_s a_u \delta_p}{\gamma \Sigma_e})^2 \exp[\frac{(Z-z_0)}{L_c}] \left\langle \left| \sum_j \exp[-(\rho\sqrt{3}+i)k_s \zeta_j] H(\zeta_j + z_0 - Z) \right|^2 \right\rangle$$

$$= (\frac{r_e \lambda_s a_u \delta_p}{\gamma \Sigma_e})^2 \exp(\frac{(Z-z_0)}{L_c}) \{ \left\langle \sum_j \exp(-\frac{\zeta_j}{L_c}) H(\zeta_j + z_0 - Z) \right\rangle$$

$$+ \left\langle \sum_{j,l(j\neq l)}^{N_e} \exp[-(\rho\sqrt{3}+i)k_s \zeta_j - (\rho\sqrt{3}-i)k_s \zeta_l] H(\zeta_j + z_0 - Z) H(\zeta_l + z_0 - Z) \right\rangle \}$$

$$a_{ef}^2 \cong (\frac{r_e \lambda_s a_u \delta_p}{\gamma \Sigma_e})^2 (\frac{N_e}{l_b} L_c + \frac{N_e^2}{l_b^2 k_s^2})$$

$L_c = (L_g / \lambda_u)\lambda_s$, slippage distance per L_g

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The firs term: effective shot noise power
(incoherent spontaneous emission contribution)

$$a_{sn}^2 = \frac{4\lambda_s r_e}{\sqrt{3}\Sigma_e}\gamma\rho^2 \qquad P_{sn} = \frac{2\omega_s \rho^2 \varepsilon_e}{\sqrt{3}} = \frac{1}{3N_{e,c}}\rho P_e$$

Comparing with previous($a_{SE}^2 = 16\pi N\lambda_s r_e \gamma\rho^3 / \Sigma_e$, $P_{SE} = 8\pi\omega_s N\rho^3\varepsilon_e$)
it is equal to the fraction of the spontaneous undulator radiation in one L_g

frequency domain approach:
the effective start-up noise power spectrum =
= spontaneous undulator radiation power spectrum in $\sqrt{3}\,L_g$
$\sim 2L_g$

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The second term: <u>effctive super-radiance power</u>

(coherent spontaneous emission contribution)

$$P_{sr} = 4\rho^3 P_e$$

$$\frac{P_{sr}}{P_{sn}} \simeq N_{e,\lambda} \frac{\sqrt{3}}{\pi} \rho = N_{e,c} (\frac{\lambda_s}{2\pi L_c})^2$$

$N_{e,\lambda}$ and $N_{e,c}$: number of e-s in one λ_s and in one L_c

SASE saturation estimate

Near the saturation one can expect

the e-s are approximately full modulated

and maintained in a distance αL_g before saturation

the radiation generated in this distance $\Delta P_f = \frac{\alpha^2}{3} \rho P_e$

(from previous formula $P_{CSE} = (4\pi N\rho)^2 \rho P_e$)

Saturation power

$$P_s = \frac{\alpha^2}{3(1-e^{-\alpha})} \rho P_e =$$

$1.542 \rho P_e$, $\alpha=2$, the distance is the last field gain length

$0.527 \rho P_e$, $\alpha=1$, the distance is the last power gain length

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Taking $P_s \approx \rho P_e$, and only consider shot noise effective start-up power

Saturation length

$$L_s \cong \ln[27 N_{e,c}] L_g = (3.252 + \ln[I(A)\lambda_s(nm)/\rho]) L_g$$

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e.g. VISA FEL: λ_s=842*nm*, I=250A, ρ=0.0081

L_s=20.3L_g=2.075*m* (L_g= 10.2*cm* for ideal condition);

L_s= 3.63*m* (if L_g= 17.9*cm* for non-ideal condition were used) agrees with the experiment

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Summary

With the time domain approach,

Spontaneous emission (incoherent and coherent)

for an arbitrary e- pulse profile.

The effective start-up power of SASE

Consist of:

the shot noise term, the incoherent spontaneous emission

= the usual spontaneous undulator radiation in the one L_g

the super radiant term, the coherent spontaneous emission.

An analytical estimation of saturation power and length

Thank you