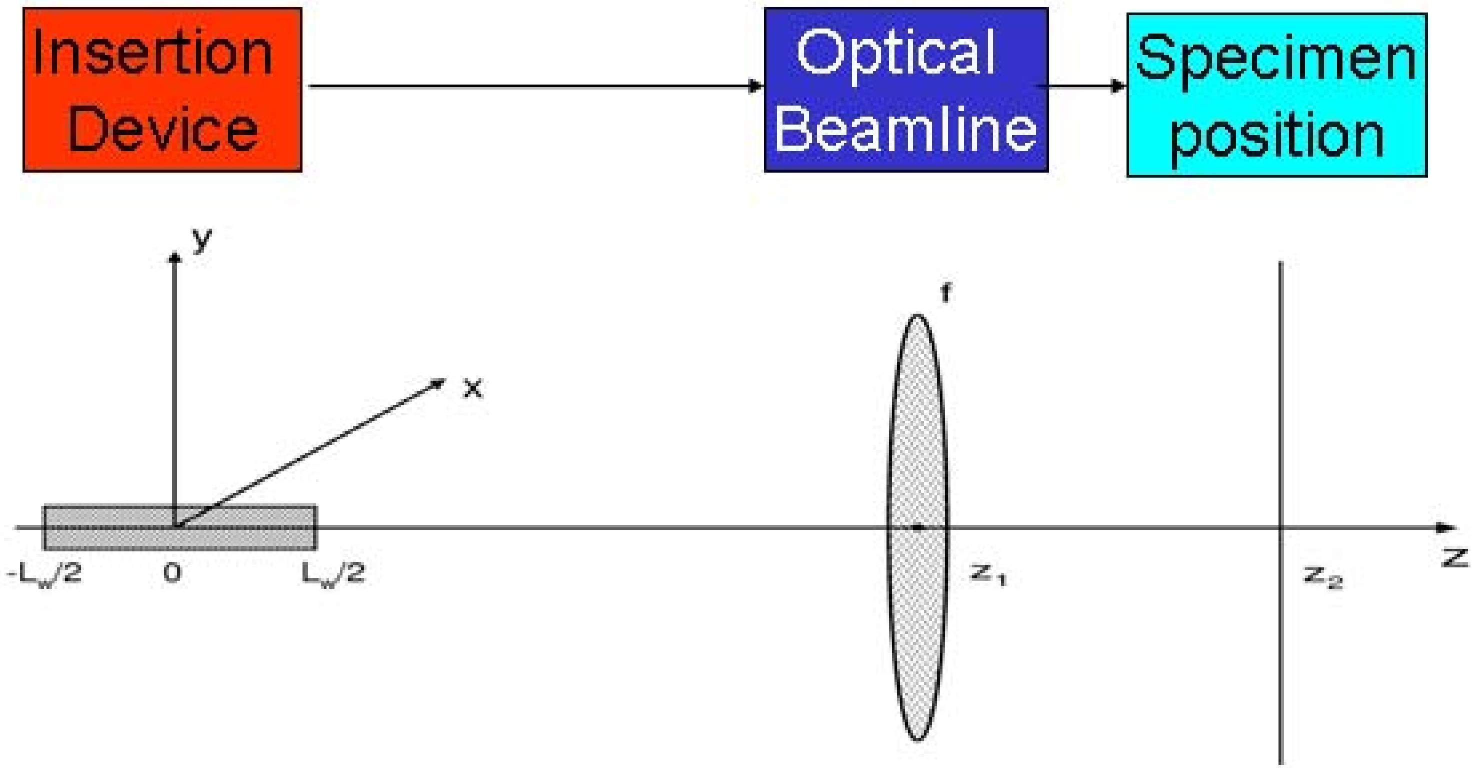
Statistical Optics and Partially Coherent X-ray Beams in Third Generation Light Sources

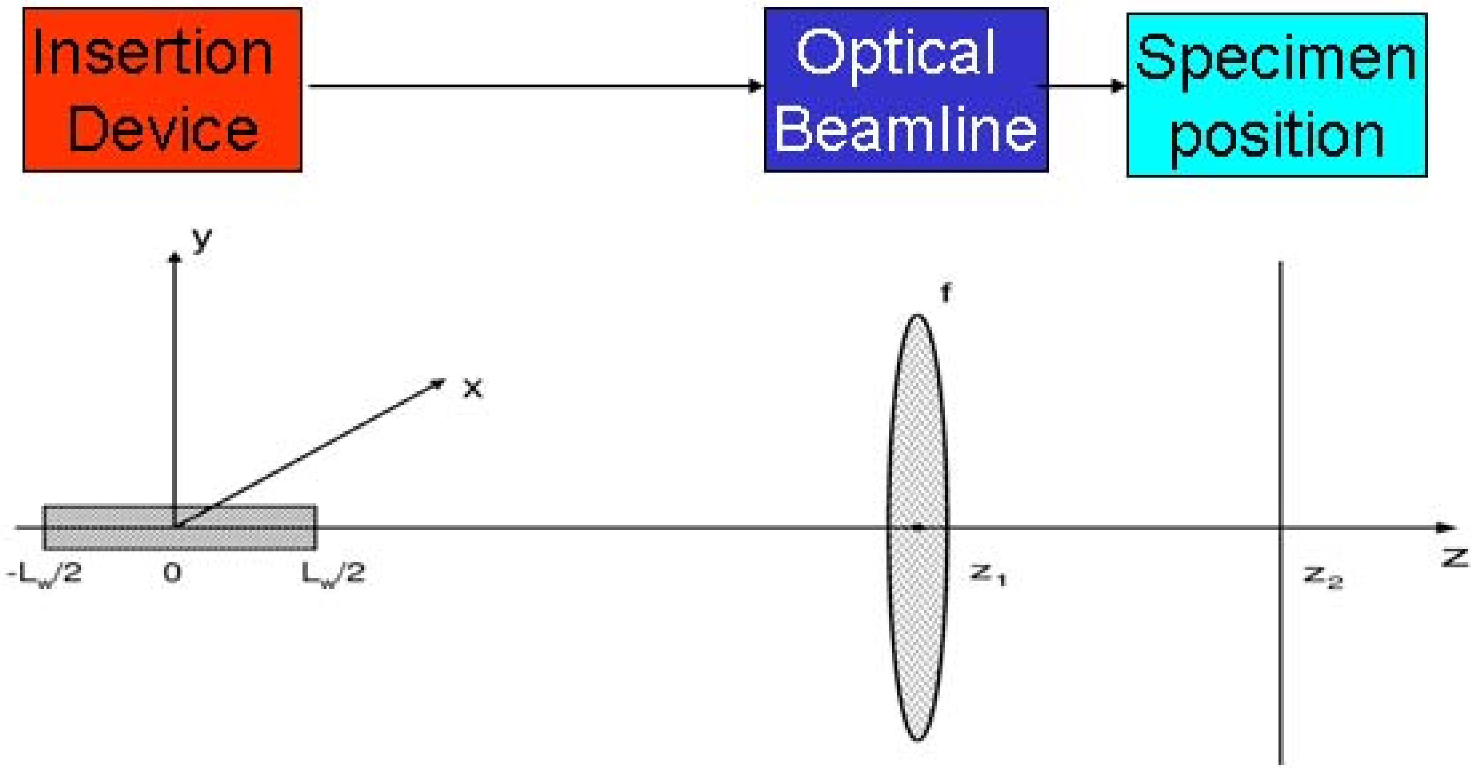
Gianluca Geloni, **Evgeni Saldin**,
Evgeni Schneidmiller and Mikhail Yurkov

Deutsches Elektronen-Synchrotron DESY, Hamburg

DESY 05-032 - DESY 05-109 - DESY 06-037

Statistical Optics and SR





$$\Gamma_{\omega}(z_o,\vec{r}_{\perp o1},\vec{r}_{\perp o2},\omega,\omega') = \left\langle \overline{E}(z_o,\vec{r}_{\perp o1},\omega)\overline{E}^{*}(z_o,\vec{r}_{\perp o2},\omega')\right\rangle$$

Statistical Optics and SR

Assumptions:

✓ $\lambda << \sigma_b$, σ_b bunch length

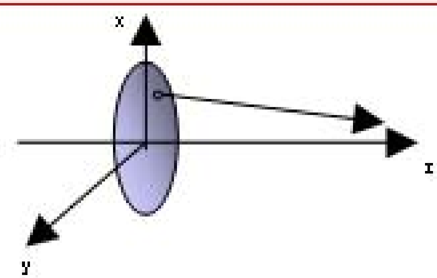
✓ $\Delta\omega_{undu} >> c/\sigma_b$

$$\Gamma_\omega(z_o, \vec{r}_{\perp o1}, \vec{r}_{\perp o2}, \omega, \omega') = N F_\omega(\omega - \omega') G_\omega(z_o, \vec{r}_{\perp o1}, \vec{r}_{\perp o2}, \omega)$$

Cross-spectral density

$$G_\omega(z_o, \vec{r}_{\perp o1}, \vec{r}_{\perp o2}, \omega) = \left\langle \bar{E}_{s\perp}(\vec{\eta}, \vec{l}, z_o, \vec{r}_{\perp o1}, \omega) \bar{E}^*_{s\perp}(\vec{\eta}, \vec{l}, z_o, \vec{r}_{\perp o2}, \omega) \right\rangle_{\vec{\eta}, \vec{l}}$$

Random variables:
e^- angle η +offset l



Statistical Optics and SR

STATISTICAL OPTICS

1) Source characterization

↓

2) Tracking of the radiation through optical elements

Source characterization

$$\varepsilon_x = 1 \div 3 \ \text{nmrad}$$
$$\varepsilon_y = 0.01 \div 0.03 \ \text{nmrad}$$
$$\lambda = 10^{-2} \div 10^{2} \text{nm}$$

→

$$\hat{\varepsilon} = \frac{2\pi \varepsilon}{\lambda}$$

Hard X-rays: λ=0.01 nm

$$\hat{\varepsilon}_x = 6.3 \cdot 10^2 \div 19 \cdot 10^2$$
$$\hat{\varepsilon}_y = 6.3 \div 19$$

Intermediate: λ=0.1 nm

$$\hat{\varepsilon}_x = 63 \div 190$$
$$\hat{\varepsilon}_y = 0.63 \div 1.9$$

Soft X-rays: λ=1 nm

$$\hat{\varepsilon}_x = 6.3 \div 19$$
$$\hat{\varepsilon}_y = 0.063 \div 0.19$$

VUV: λ=100 nm

$$\hat{\varepsilon}_x = 6.3 \cdot 10^{-2} \div 1.9 \cdot 10^{-1}$$
$$\hat{\varepsilon}_y = 6.3 \cdot 10^{-4} \div 1.9 \cdot 10^{-3}$$

Geometrical Optics ← **Full Statistical Optics Approach required !** → **Deterministic Wave Optics**

$$\hat{\varepsilon}_x >> 1$$

Source characterization

Source characterization

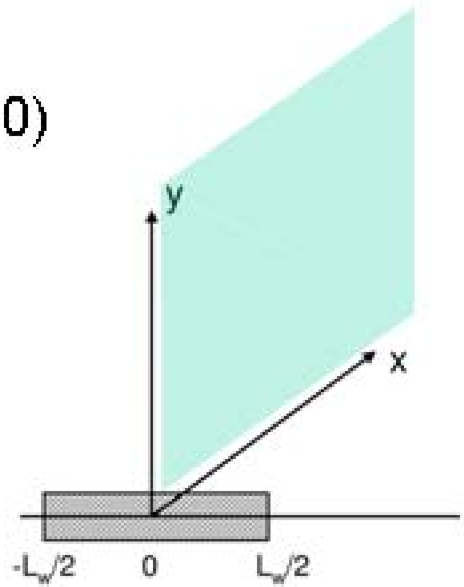
Cross-spectral density G at a certain z in free space + Maxwell Equations

→ source characterized

but useful to describe the source through a **VIRTUAL SOURCE**

1) located at position of minimal betatron functions (z=0)

2) at any distance: same field as undulator

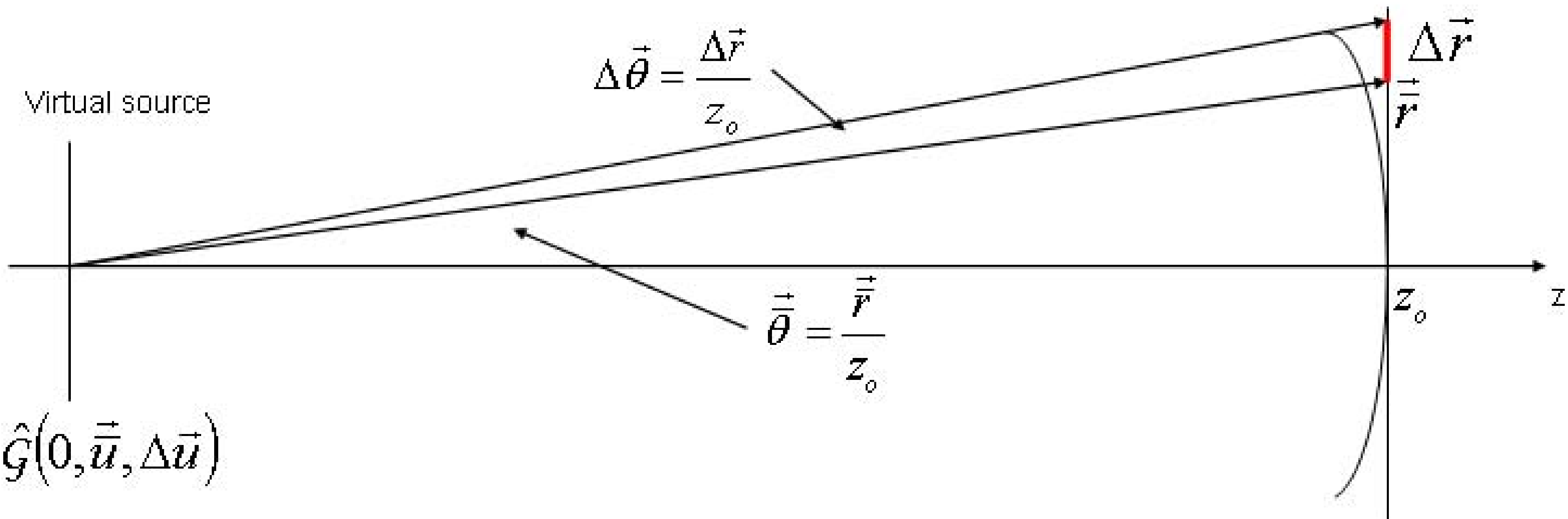


Virtual source is characterized when G at z=0 is given

Fourier Optics offers relation between G in the far zone and at the virtual source

$$\hat{\mathcal{G}}\left(z,\vec{\bar{u}},\Delta\vec{u}\right)=\int d\vec{\bar{r}}'\, d\Delta\vec{r}'\, G(z,\vec{\bar{r}}',\Delta\vec{r}')\exp[2i(\vec{\bar{u}}\cdot\Delta\vec{r}'+\Delta\vec{u}\cdot\vec{\bar{r}}')]$$

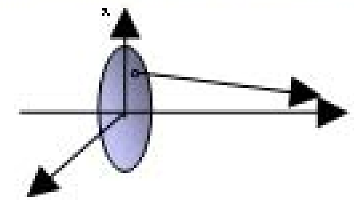
$$\bar{r}=\frac{r_1+r_2}{2}\qquad \Delta r=\frac{r_1-r_2}{2}$$



$$\hat{G}(z_o,\vec{\bar{\theta}},\Delta\vec{\theta})\sim\exp\left[2iz_o\vec{\bar{\theta}}\cdot\Delta\vec{\theta}\right]\hat{\mathcal{G}}\left(0,-\vec{\bar{\theta}},-\Delta\vec{\theta}\right)$$

Source characterization: undulator field in normalized units

$$G_\omega(z_o, \vec{r}_{\perp o1}, \vec{r}_{\perp o2}, \omega) = \left\langle \bar{E}_{s\perp}(\vec{\eta}, \vec{l}, z_o, \vec{r}_{\perp o1}, \omega) \bar{E}^*_{s\perp}(\vec{\eta}, \vec{l}, z_o, \vec{r}_{\perp o2}, \omega) \right\rangle_{\vec{\eta}, \vec{l}}$$



•Paraxial approximation:

$L_f >> \lambda$ always applicable for SR

•Resonance approximation:

$N_w >> 1$

freq. ~ fundamental freq. $\omega_o = \dfrac{4\pi c\gamma^2}{\lambda_w (1 + K^2/2)}$

also non-restrictive

Source characterization: undulator field in normalized units

Use of normalized units:

$$\hat{z} = \frac{z}{L_w}$$

Longitudinal length

$$\vec{\hat{l}} = \vec{l}\sqrt{\frac{\omega}{L_w c}}$$

Transverse length

$$\vec{\hat{\eta}} = \vec{\eta}\sqrt{\frac{\omega L_w}{c}}$$

All angles: transverse/longitudinal

$$\hat{C} = L_w C = 2\pi N_w \frac{\omega - \omega_o}{\omega_o}$$

Detuning from resonance

Source characterization: undulator field in normalized units

$$\text{New variables; from } \vec{r} \text{ to } \vec{\theta} = \frac{\vec{r}}{z_o}\text{; then } \hat{\vec{\theta}} = \vec{\theta}\sqrt{\frac{\omega L_w}{c}}$$

$$\hat{E}_{s\perp} = \exp(i\Phi_u)\int_{-1/2}^{1/2} \frac{\hat{z}_o d\hat{z}'}{\hat{z}_o - \hat{z}'} \exp\left\{ i\left[\hat{C}\hat{z}' + \frac{\hat{z}_o\hat{z}'}{2(\hat{z}_o - \hat{z}')}\left(\hat{\vec{\theta}} - \frac{\hat{\vec{l}}}{\hat{z}_o} - \hat{\vec{\eta}}\right)^2\right]\right\}$$

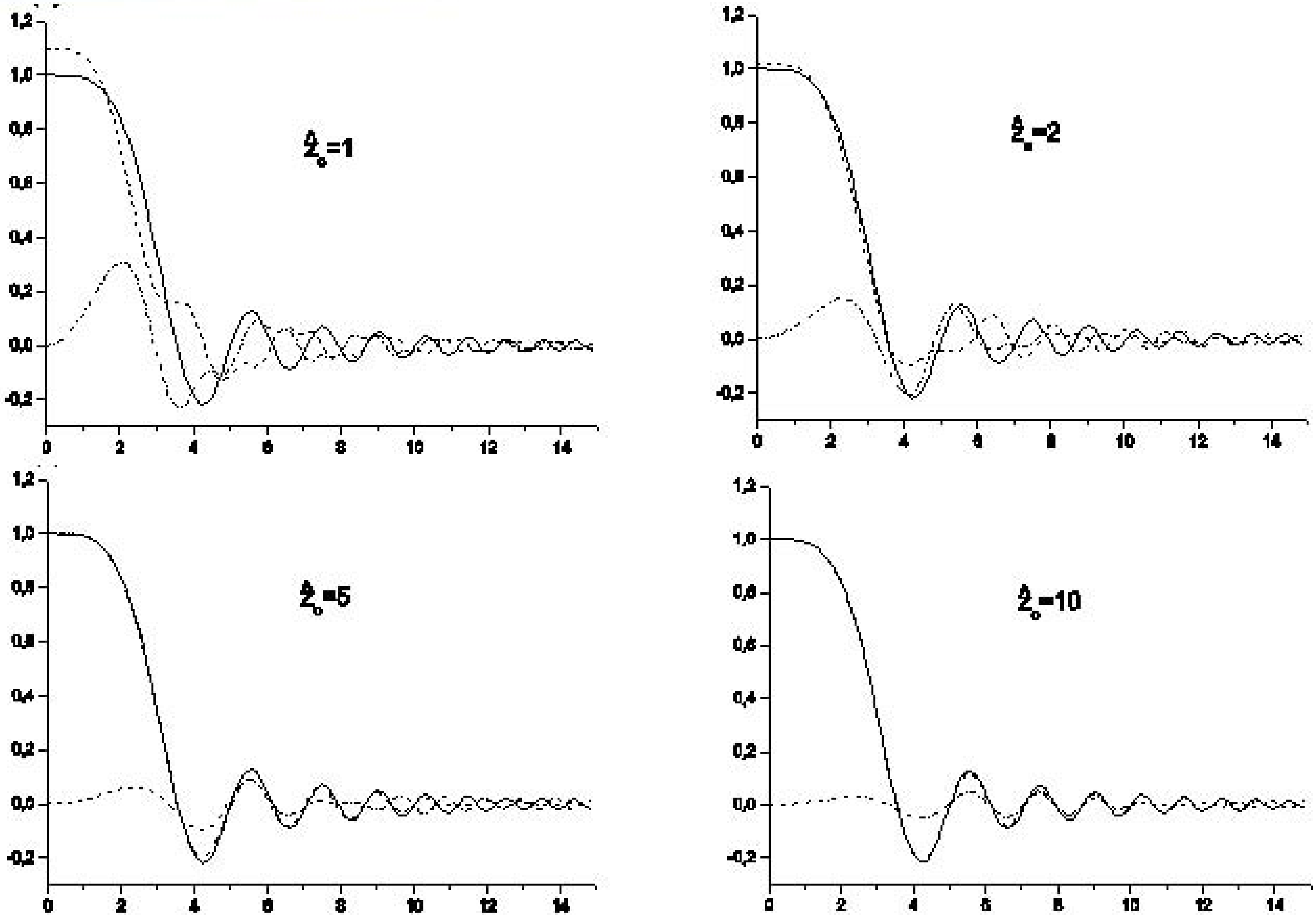
$$\Phi_U = \left(\hat{\vec{\theta}} - \frac{\hat{\vec{l}}}{\hat{z}_o}\right)^2 \frac{\hat{z}_o}{2}$$

$$\hat{E}_{s\perp}\left(\hat{C}, \hat{z}_o, \hat{\vec{\theta}} - \frac{\hat{\vec{l}}}{\hat{z}_o} - \hat{\vec{\eta}}\right) = \exp(i\Phi_U) S\left[\hat{C}, \hat{z}_o, \left(\hat{\vec{\theta}} - \frac{\hat{\vec{l}}}{\hat{z}_o} - \hat{\vec{\eta}}\right)^2\right]$$

And when $\hat{C}=0$ (perfect resonance)

$$S\left(0, \hat{z}_o, \zeta^2\right) = \exp(-i\hat{z}_o\zeta^2/2)\hat{z}_o\left[\mathbf{Ei}\left(\frac{i\hat{z}_o^2\zeta^2}{-1+2\hat{z}_o}\right) - \mathbf{Ei}\left(\frac{i\hat{z}_o^2\zeta^2}{1+2\hat{z}_o}\right)\right]$$

Source characterization: the undulator field



Source characterization: the cross-spectral density function

Angular coordinates: $\bar{\theta} = \frac{\hat{\theta}_1 + \hat{\theta}_2}{2} \qquad \Delta\hat{\theta} = \frac{\hat{\theta}_1 - \hat{\theta}_2}{2}$

The cross-spectral density function in free space at resonance ($\hat{C}=0$)

$$\hat{G}(\bar{\theta}_x, \bar{\theta}_y, \Delta\hat{\theta}_x, \Delta\hat{\theta}_y; \hat{z}_o, \hat{N}_x, \hat{N}_y, \hat{D}_x, \hat{D}_y)$$

Main transverse parameters: $\hat{D}_{x,y} = \sigma^2_{x',y'} \frac{\omega L_w}{c} \qquad \hat{N}_{x,y} = \sigma^2_{x,y} \frac{\omega}{c L_w}$

$$\hat{\varepsilon} = \frac{2\pi\,\varepsilon}{\lambda} = \sqrt{\hat{N}\hat{D}}$$

- Involves only 2 integrations
- still pretty generic
- useful for numerical techniques

Source characterization: cross-spectral density function

When $\hat{D}_x >> 1$ and $\hat{N}_x >> 1$ we obtain factorization of G:

$$\hat{G} = \hat{G}_x \hat{G}_y$$

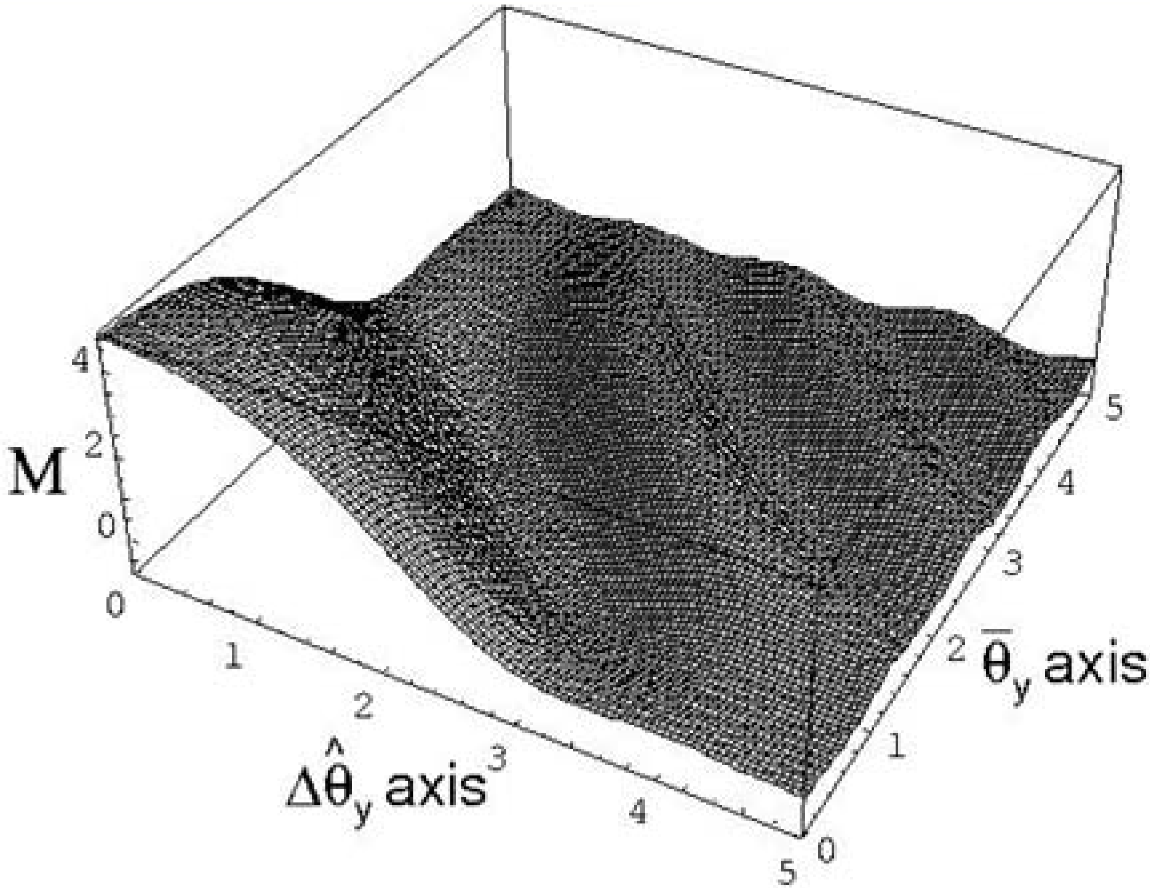
In the far zone limit with $\hat{N}_x >> 1$, $\hat{D}_x >> 1$:

$$\hat{G}_x = \exp[i2\bar{\theta}_x \hat{z}_o \Delta\hat{\theta}_x]\exp[-2\hat{N}_x \Delta\hat{\theta}_x^2]$$

$$\hat{G}_y = \exp[i2\bar{\theta}_y \hat{z}_o \Delta\hat{\theta}_y]\exp[-2\hat{N}_y \Delta\hat{\theta}_y^2] \cdot \int_{-\infty}^{\infty} d\hat{\varphi}_y \exp\left[-\frac{\left(\bar{\theta}_y + \hat{\varphi}_y\right)^2}{2\hat{D}_y}\right] M(\hat{\varphi}_y, \Delta\hat{\theta}_y)$$

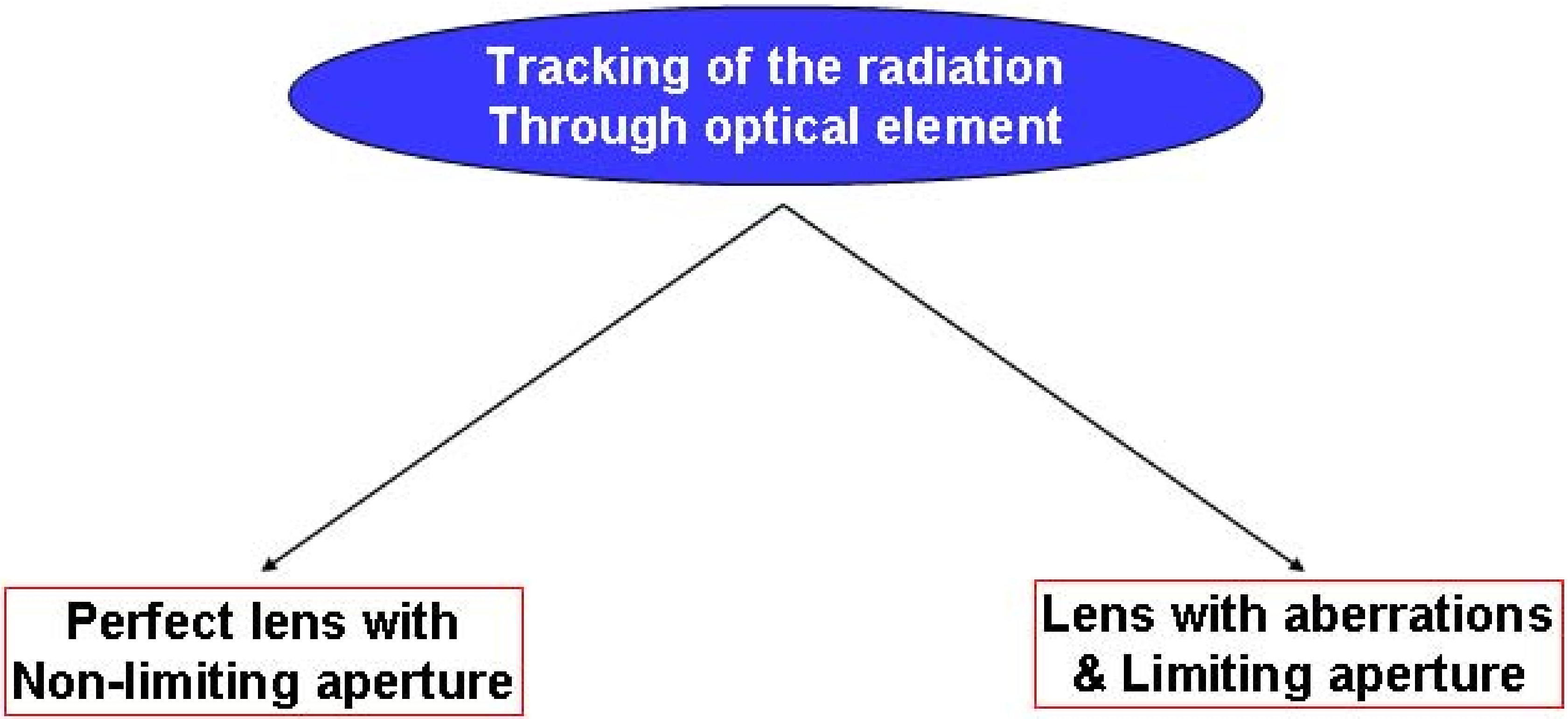
Source characterization: the M function

$$M(\bar{\theta}_y, \Delta\hat{\theta}_y) = \int_{-\infty}^{\infty} d\hat{\phi}_x \operatorname{sinc}\left[\frac{\hat{\phi}_x^2 + (\bar{\theta}_y - \Delta\hat{\theta}_y)^2}{4}\right] \operatorname{sinc}\left[\frac{\hat{\phi}_x^2 + (\bar{\theta}_y + \Delta\hat{\theta}_y)^2}{4}\right]$$



Tracking of G

$\hat{G}$ in the far field → $\hat{\mathcal{G}}\left(0,\vec{u},\Delta\vec{u}\right) \leftrightarrow \hat{G}\left(0,\vec{r},\Delta\vec{r}\right)$



Tracking of G: perfect lens case

For perfect lenses results are immediate both at the focal and the image plane

At the focal plane we reproduce the F.T. of the cross spectral density at the source:

$$\hat{G}\left(\hat{z}_f, \vec{\bar{r}}_f, \Delta\vec{r}_f\right) \sim \exp\left[\frac{2i}{\hat{f}}\left(1-\frac{\hat{z}_1}{\hat{f}}\right)\vec{\bar{r}}_f \cdot \Delta\vec{r}_f\right]\hat{\mathcal{G}}\left(0, \frac{\vec{\bar{r}}_f}{\hat{f}}, \frac{\Delta\vec{r}_f}{\hat{f}}\right)$$

At the image plane we reproduce the cross spectral density at the source:

$$\hat{G}\left(\hat{z}_i, \vec{\bar{r}}_i, \Delta\vec{r}_i\right) \sim \exp\left[\frac{2im}{\hat{z}_1}(1+m)\vec{\bar{r}}_i \cdot \Delta\vec{r}_i\right]\hat{G}\left(0, -m\vec{\bar{r}}_i, -m\Delta\vec{r}_i\right)$$

$$\text{where } m = \frac{\hat{z}_1}{\hat{z}_2 - \hat{z}_1}$$

Tracking of G: non-perfect lens case

Limiting aperture+aberrations

In the most general case, at the image plane in the far field the cross spectral density is obtained convolving the cross spectral density at the object plane (virtual source) with the Amplitude point-spread function of the system:

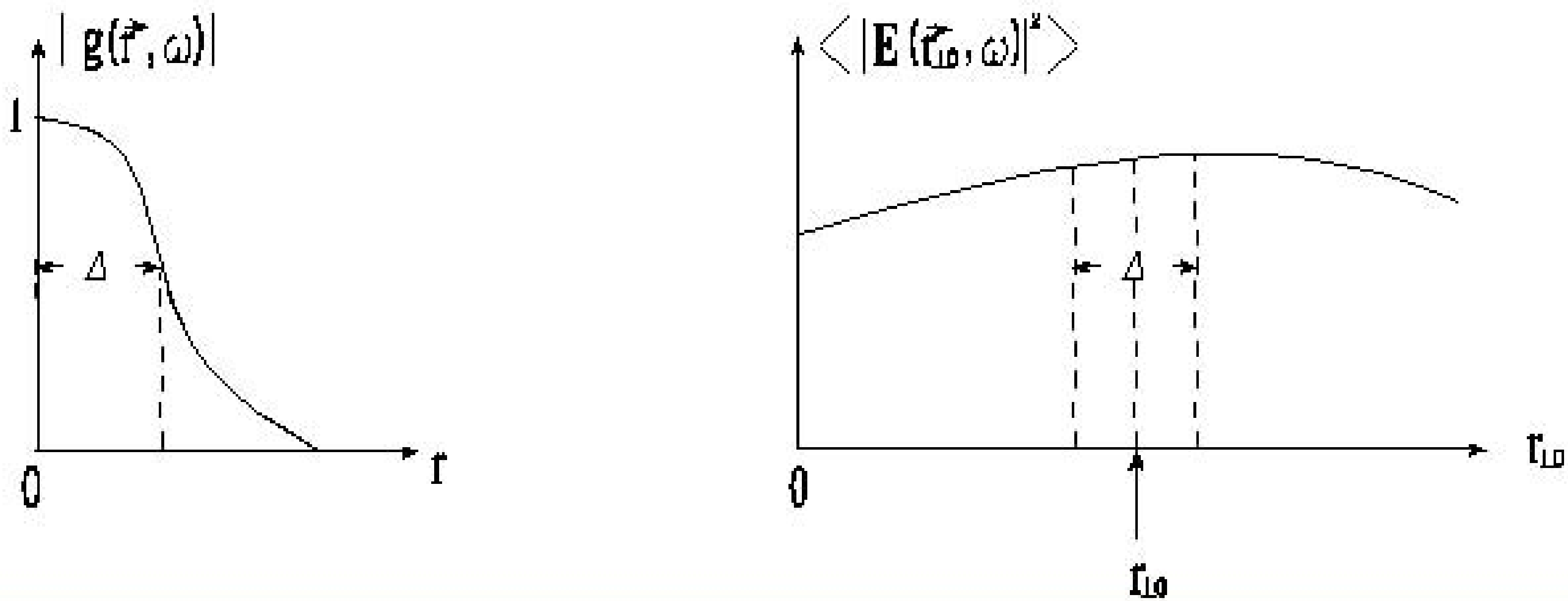
$$\hat{G}_P(\hat{z}_i, \vec{\hat{r}}_i, \Delta\vec{\hat{r}}_i) = \mathrm{m} \exp\left[\frac{2i\mathrm{m}}{\hat{z}_1}\vec{\hat{r}}_i \cdot \Delta\vec{\hat{r}}_i\right] \int d\vec{\hat{u}}\, d\Delta\vec{\hat{u}}\, \hat{G}\left(0, -\hat{z}_1\vec{\hat{u}}, -\hat{z}_1\Delta\vec{\hat{u}}\right) .$$
$$\times\ \hat{\mathcal{P}}\left[\frac{\mathrm{m}}{\hat{z}_1}\left(\vec{\hat{r}}_i + \Delta\vec{\hat{r}}_i\right) - \vec{\hat{u}} - \Delta\vec{\hat{u}}\right] \hat{\mathcal{P}}^*\left[\frac{\mathrm{m}}{\hat{z}_1}\left(\vec{\hat{r}}_i - \Delta\vec{\hat{r}}_i\right) - \vec{\hat{u}} + \Delta\vec{\hat{u}}\right]$$

$\hat{\mathcal{P}}$ **is the F.T. of the pupil function**

Source characterization

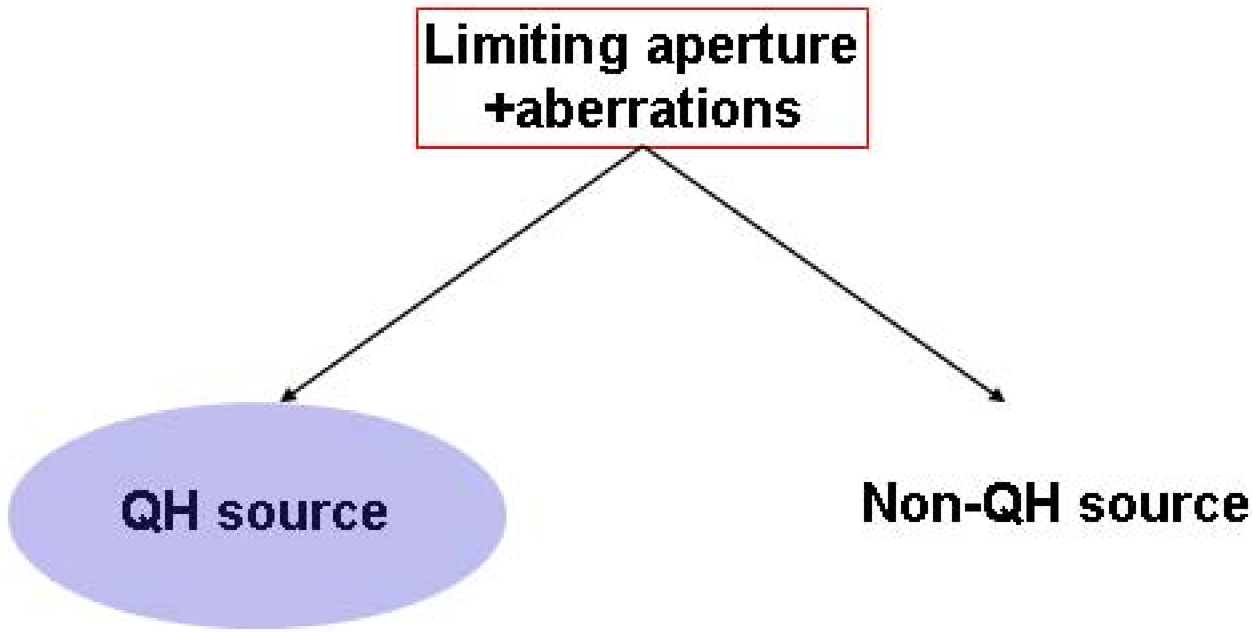
Particular kind of VIRTUAL SOURCE:

Quasi-homogeneous source



$$G = I(\vec{r}_{\perp 1},\omega)\, g(\vec{r}_{\perp 2} - \vec{r}_{\perp 1},\omega) \quad \text{where} \quad I = \left\langle \left| \overline{E}(\vec{r}_{\perp 1},\omega) \right|^2 \right\rangle$$

Tracking of G: non-perfect lens case



Simplification for the intensity at the image plane

Tracking of G: non-perfect lens case

Line spread function:

$$I_p = I_{ideal} * l$$

Diffraction effects are not important: the line spread function can be calculated with Geometrical Optics techniques

(ray tracing possible)

Diffraction effects are important: the line spread function can only be calculated with Physical Optics techniques

(no ray tracing possible)

Conclusions

- A theory describing spatial coherence from UR is proposed

- Step 1. Characterization of the source through cross-spectral density

  - FT of s.p. electric field from ID at any distance z_0, offset **l** + angle $\boldsymbol{\eta}$
  - General expression found for G (5+1 parameters)
  - $\hat{N}_x$, $\hat{D}_x$ >> 1 → Separability $\hat{G} = \hat{G}_x \hat{G}_y$
  - G in the far zone allows characterization of virtual source

- Step 2. Tracking of G through the beamline

  - Case of perfect lens is immediate: focal and image plane
  - Non-perfect lens: convolve G with amplitude line spread function
  - Imaging. QH case → line spread function approach