ISBN: 978-3-95450-210-3 doi:10.18429/JACoW-FEL2019-TUP054

AN INVESTIGATION OF POSSIBLE NON-STANDARD PHOTON STATISTICS IN A FREE-ELECTRON LASER II: THEORY

Jeong-Wan Park*, University of Hawaii at Manoa, Honolulu, U.S.
Kwang-Je Kim[1], Ryan Lindberg, Argonne National Laboratory, Lemont, U.S.
[1]also at University of Chicago, Chicago, U.S.

Abstract

In this paper we explore whether we can at present find a theoretical basis for non-standard, sub-Poissonian photon statistics in the coherent spontaneous harmonic radiation of an FEL as was claimed to have been measured with the Mark III FEL [1]. We develop a one dimensional quantum FEL oscillator model of the harmonic radiation in the linear gain regime to calculate the photon statistics. According to our study, it seems unlikely that the photon statistics for an FEL oscillator starting from the noise could be sub-Poissonian.

INTRODUCTION

This is the second, theoretical part of our study into possible non-standard, sub-Poissonian photon statistics for harmonic emission from an FEL. The motivation is an experiment by Chen and Madey [1], which claimed to have observed sub-Poissoinian statistics at the seventh harmonic during the linear gain regime of the MARK III FEL oscillator. In our first paper we take a critical look at the experiment [2]; here, we revisit the standard theory of FEL photon statistics of the fundamental mode starting from the noise, and then develop the simplest quantum extension of the classical theory of harmonic radiation production as driven by the fundamental mode. We include the cavity loss with a beam splitter model, and compute the photon statistics of the harmonic modes. We show that the statistics cannot be sub-Poissonian for any initial state of the electrons.

A MODEL OF LIGHT EMITTED BY RANDOMLY DISTRIBUTED ELECTRONS

First, we revisit the photon statistics of undulator radiation when there is no significant bunching in the electron beam. The variance of the number of photons for coherent wave trains emitted by randomly distributed electrons (Fig. 1) is given by [3,4]:

$$\langle(\delta n)^2\rangle = \langle(\delta n)^2\rangle_Q + \langle(\delta n)^2\rangle_C; \tag{1}$$

$\langle(\delta n)^2\rangle_Q$ is the portion of variance originating from quantum mechanics, whereas $\langle(\delta n)^2\rangle_C$ arises from classical fluctuation of the electrons [5]. Consequently, the Fano factor $F = \frac{\langle(\delta n)^2\rangle}{\langle n\rangle}$ of the photon statistics can also be decomposed into the quantum and classical portions:

$$F = F_Q + F_C. \tag{2}$$

When the mode number $M = t_p/\tau_{coh} \gg 1$ (t_p is the pulse length and τ_{coh} is the coherence time) one can write [4,5]

$$F_Q = \frac{\langle(\delta n)^2\rangle_Q}{\langle n\rangle} = 1,\ F_C = \frac{\langle n\rangle}{M}. \tag{3}$$

Thus, in this case $F > 1$ and the photon statistics is super-Poissonian.

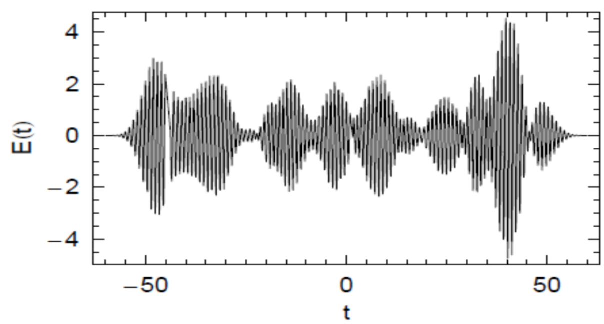

Figure 1: A sum of coherent wave trains emitted by randomly distributed electrons [5].

PHOTON STATISTICS OF THE FUNDAMENTAL MODE

A one Dimensional Quantum FEL Model

To investigate the photon statistics, we shall develop a quantum model of the FEL oscillator starting from the noise. In this model we will only consider one frequency mode of the field, since in the linear gain regime the FEL acts as a linear amplifier and all frequency components are independent [6]. When energy spread can be neglected the evolution of the electric field operator as the pass number n increases becomes [5,7]

$$\mathfrak{a}_{1,n+1} = g\mathfrak{a}_{1,n} + \mathcal{F}_n; \tag{4}$$

the first subscript of field operator $\mathfrak{a}$ represents the harmonic number, the second denotes the cavity round-trip number, and $|g|^2$ is the gain per cavity round-trip. An explicit expression for g is:

$$g \equiv \frac{1}{3}\sum_{\alpha} e^{-i\mu_\alpha 2k_u\rho L_u}, \tag{5}$$

where μ_α are the three roots of $\left(\mu - \frac{\Delta\nu}{2\rho}\right)\left(\mu^2 - \frac{q^2}{4}\right) = 1, \Delta\nu = \frac{\omega-\omega_1}{\omega_1}$, ω_1 is the fundamental mode's angular frequency, $q = \frac{\hbar\omega_1}{\rho\gamma_r mc^2}$ is the ratio of the characteristic photon energy to

* jwpark@hawaii.edu

ISBN: 978-3-95450-210-3 doi:10.18429/JACoW-FEL2019-TUP054

the energy bandwidth of the FEL (also called the quantum FEL parameter), and γ_r is the resonant Lorentz factor of the electron, L_u is the undulator length, k_u is the undulator's wave number, and ρ is the FEL Pierce parameter. We assume that the electron beam is fresh at the beginning of each pass, and its operator at the end of the n^{th} pass is

$$\begin{aligned}\mathcal{F}_n = &-i\sum_{\alpha}\Upsilon_\alpha e^{-i\mu_\alpha 2k_u\rho L_u}\mathcal{B}_{n_0}\\ &-i\sum_{\alpha}\frac{\Upsilon_\alpha}{\mu_\alpha}e^{-i\mu_\alpha 2k_u\rho L_u}\mathcal{P}_{n_0},\end{aligned}\tag{6}$$

where $\Upsilon_\alpha \equiv \frac{\mu_\alpha}{(\mu_\alpha-\mu_\beta)(\mu_\alpha-\mu_\gamma)}$, and $\mathcal{B}_{n_0}$ and $\mathcal{P}_{n_0}$ are the initial bunching and collective momentum of the electron beam, respectively.

Photon Statistics

We incorporate cavity loss using the beam splitter model [8, 9], in which the finite reflectivity $\sqrt{\eta_h}$ at harmonic h is modeled by the division of the signal into the "output" portion $\sqrt{\eta_h}e^{i\phi_1}\mathfrak{a}_h$ and the "unused" part $\sqrt{1-\eta_h}e^{i\phi_{1,u}}\mathfrak{a}_{h,u}$. Here, $\phi_{h,n}$ and $\phi_{h,u,n}$ are the phase shift for the output field and unused field, respectively, and each of these two fields separately satisfy the commutation relation $[\mathfrak{a},\mathfrak{a}^\dagger]$ but commute with each other. Then, we have the quantum iterative relation

$$\begin{aligned}\mathfrak{a}_{1,n+1} = &\, g\sqrt{\eta_1}e^{i\phi_{1,n}}\mathfrak{a}_{1,n} + \mathcal{F}_n\\ &+ g\sqrt{1-\eta_1}e^{i\phi_{1,u,n}}\mathfrak{a}_{1,u,n}.\end{aligned}\tag{7}$$

Note that the commutation relation $[\mathfrak{a}_{1,n},\mathfrak{a}^\dagger_{1,n}] = 1$ is preserved. Solving the recursion relation, we obtain

$$\begin{aligned}\mathfrak{a}_{1,n} = &\,(g\sqrt{\eta_1})^{n-1}e^{i\Phi_1}\mathfrak{a}_{1,1}\\ &+\sum_{j=1}^{n-1}(g\sqrt{\eta_1})^{n-j-1}e^{i\Phi_{j+1}}\\ &\times\left[\mathcal{F}_j + g\sqrt{1-\eta_1}e^{i\phi_{1,u,j}}\mathfrak{a}_{1,u,j}\right]\end{aligned}\tag{8}$$

where $\Phi_j = \sum_{k=j}^{n-1}\phi_{1,k}$. The first two moments of the photon number operator become:

$$\langle\mathfrak{n}\rangle_{1,n} = \sum_{j,k=1}^{n-1}(|g|\sqrt{\eta_1})^{2(n-1)-(j+k)}\langle\mathcal{F}'^\dagger_{n_j}\mathcal{F}'_{n_k}\rangle\tag{9}$$

$$\begin{aligned}\langle\mathfrak{n}^2\rangle_{1,n} = &\,\frac{(|g|^2\eta_1)^{n-1}(|g|^2-1)-|g|^2(1-\eta_1)}{|g|^2\eta_1-1}\langle\mathfrak{n}\rangle_{1,n}\\ &+\sum_{j,k,l,m=1}^{n-1}(|g|\sqrt{\eta_1})^{4(n-1)-(j+k+l+m)}\\ &\times\langle\mathcal{F}'^\dagger_{n_j}\mathcal{F}'_{n_k}\mathcal{F}'^\dagger_{n_l}\mathcal{F}'_{n_m}\rangle,\end{aligned}\tag{10}$$

where $\mathcal{F}'_{n_j} \equiv e^{i[\Phi_{j+1}-j\mathrm{Arg}(g)]}\mathcal{F}_j$. One can show that the Fano factor of the fundamental after the n^{th} cavity round-trip, $F_{1,n}$, satisfies

$$F_{1,n} = \frac{\{(|g|^2\eta_1)^{n-1}-1\}(|g|^2-1)}{|g|^2\eta_1-1} + \frac{\langle(\delta\mathfrak{A}_n)^2\rangle}{\langle\mathfrak{n}_1\rangle_n} + 1 \geq 1,\tag{11}$$

provided $|g|^2 \geq 1$. Therefore, the photon statistics is not sub-Poissonian for any initial state of the electrons. For completeness, we also include the expression for the Hermitian operator $\mathfrak{A}_n$:

$$\mathfrak{A}_n = \sum_{j,k=1}^{n-1}(|g|\sqrt{\eta_1})^{2(n-1)-(j+k)}\mathcal{F}'^\dagger_{n_j}\mathcal{F}'_{n_k}.\tag{12}$$

Statistics for the Minimum Noise Electron State

The Fano factor can be explicitly computed if the initial electrons are in the minimum noise state $|\Psi\rangle$, defined as that which is annihilated by the Hermitian conjugate of electron beam operator:

$$\mathcal{F}^\dagger_n|\Psi\rangle = 0.\tag{13}$$

Using Eqs. (9), (11), and (12), we compute the Fano factor to be

$$F_{1,n} = 1 + \langle\mathfrak{n}_1\rangle_n,\tag{14}$$

which is the same as that of chaotic light.

In the classical regime when $q \ll 1$, as applies to the Chen-Madey experiment, the phase operator of the jth electron prior to the FEL interaction can be decomposed as

$$\Theta_{j_0} = \theta^c_j + \tilde{\Theta}_j,\tag{15}$$

where θ^c_j is a c-number denoting the initial classical position while $\tilde{\Theta}_j$ is the quantum correction that will be treated as a small quantity.

The minimum noise state's wave function of the jth electron in $\tilde{\theta}$ space is a Gaussian function centered about its classical position θ^c_j with RMS width equal to $\sqrt{q}$ [5]. Although the corresponding wave function's width in the position space is much less than the radiation wavelength, as θ^c_j is randomly distributed, the electrons' radiation can still be regarded as coherent wave trains emitted by randomly distributed electrons. This may explain why Eqn. (14) is in accordance with Eqn. (3).

Comparison to Existing Literature

Banacloche [10] showed that the radiation is chaotic during the linear gain regime of the high-gain FEL starting from the noise, if the electrons are initially in momentum eigenstates prior to the FEL interaction. In his analysis, he attributed the random phase distribution of the initial electrons to the infinite width of the wave function in the position space, and found the same Fano factor as that for spontaneous emission, Eqn. (3). Interestingly, this result is the same as what was just derived for electrons initially being described by the minimum noise state.

Gjaja and Bhattacharjee [11] studied whether fluctuation of the field variable in phase with the output field can be reduced below the symmetrical vacuum fluctuation's quantum limit, in the linear gain regime of a FEL starting from the noise. A field with such reduced fluctuation is named amplitude-squeezed radiation. They found that in a high-gain FEL, regardless of the initial state of the electrons,

ISBN: 978-3-95450-210-3 doi:10.18429/JACoW-FEL2019-TUP054

amplitude-squeezed radiation cannot be emitted. For a low-gain FEL, if the electrons are not entangled initially, and the initial wave functions of all electrons have the same width in position and also in momentum, they found that the amplitude-squeezed radiation cannot be emitted either. However, they discovered that some initially entangled state of the electrons can result in the amplitude-squeezed radiation, which is interestingly not accompanied by the sub-Poissonian photon statistics according to Eqn. (11).

THE HARMONIC MODE'S PHOTON STATISTICS

Our quantum theory for the generation of non-linear higher harmonic modes is based upon the simplest quantum extention of the classical theory. For a harmonic mode h that is dominantly-driven by the fundamental mode we have

$$a_{h,n+1} = a_{h,n} + k_h a_{1,n}^h. \tag{16}$$

We promote the classical fields to quantum operators to obtain the corresponding iterative relation. Including the cavity loss with the beam splitter model we find that

$$\begin{aligned}\mathfrak{a}_{h,n+1} = \sqrt{\eta_h} e^{i\phi_{h,n}} \mathfrak{a}_{h,n} + k_h \mathfrak{a}_{1,n}^h \\ + \sqrt{1-\eta_h} e^{i\phi_{h,u,n}} \mathfrak{a}_{h,u,n}.\end{aligned} \tag{17}$$

In terms of the initial values $\mathfrak{a}_{h,n}$ therefore becomes

$$\begin{aligned}\mathfrak{a}_{h,n} = \sqrt{\eta_h}^{n-1} e^{i\Phi_{h,1}} \mathfrak{a}_{h,1} + \sum_{j=1}^{n-1} \sqrt{\eta_h}^{n-j-1} e^{i\Phi_{h,j+1}} \\ \times \left(k_h \mathfrak{a}_{1,j}^h + \sqrt{1-\eta_h} e^{i\phi_{h,u,j}} \mathfrak{a}_{h,u,j} \right),\end{aligned} \tag{18}$$

where, as before, $\Phi_{h,j} = \sum_{k=j}^{n-1} \phi_{h,k}$. Consequently, the first two moments of the photon statistics become:

$$\langle \mathfrak{n}_h \rangle_n = |k_h|^2 \sum_{j,k=1}^{n-1} \sqrt{\eta_h}^{2(n-1)-(j+k)} \langle \mathfrak{a}_{1,n_j}^{\prime\dagger h} \mathfrak{a}_{1,n_k}^{\prime h} \rangle, \tag{19}$$

$$\begin{aligned}\langle \mathfrak{n}_h^2 \rangle_n = \langle \mathfrak{n}_h \rangle_n + \sum_{j,k,l,m=1}^{n-1} \sqrt{\eta_h}^{4(n-1)-(j+k+l+m)} \\ \times |k_h|^4 \left\langle \mathfrak{a}_{1,n_j}^{\prime\dagger h} \mathfrak{a}_{1,n_k}^{\prime h} \mathfrak{a}_{1,n_l}^{\prime\dagger h} \mathfrak{a}_{1,n_m}^{\prime h} \right\rangle,\end{aligned} \tag{20}$$

where $\mathfrak{a}_{1,n_j}' \equiv \mathfrak{a}_{1,j} e^{i\Phi_{h,j+1}/h}$. In a similar way to what we did before, the Fano factor of the harmonic mode after the n^{th} cavity round-trip, $F_{h,n}$, satisfies

$$F_{h,n} = \frac{\langle (\delta \mathfrak{C}_{h,n})^2 \rangle}{\langle \mathfrak{n}_h \rangle_n} + 1 \geq 1, \tag{21}$$

and the photon statistics is not sub-Poissonian for any initial state of the electrons. The Hermitian operator $\mathfrak{C}_{h,n}$ is:

$$\mathfrak{C}_{h,n} = |k_h|^2 \sum_{j,k=1}^{n-1} \sqrt{\eta_h}^{2(n-1)-(j+k)} \mathfrak{a}_{1,n_j}^{\prime\dagger h} \mathfrak{a}_{1,n_k}^{\prime h}. \tag{22}$$

CONCLUSION

We have theoretically investigated the FEL photon statistics using the standard theory. It does not seem probable that the photon statistics in the linear gain regime of the FEL starting from the noise is sub-Poissonian, regardless of the harmonic number or the initial state of the electrons. Therefore if a sub-Poissonian FEL light is experimentally observed contradicting our finding, a theory beyond the standard one may be necessary to explain the observation.

ACKNOWLEDGEMENT

Work supported by U.S. DOE, Office of Science, Office of BES, under Award No. DE-SC0018428.

REFERENCES

[1] T. Chen and J. M. Madey, "Observation of Sub-Poisson Fluctuations in the Intensity of the Seventh Coherent Spontaneous Harmonic Emitted by a rf Linac Free-Electron Laser", *Phys. Rev. Lett.*, vol. 86, p. 5906, Jun 2001. doi:10.1103/PhysRevLett.86.5906

[2] J.-W. Park, K.-J. Kim, and R. R. Lindberg, "An Investigation of Possible Non-Standard Photon Statistics in a Free-Electron Laser I: Experiment", presented at the FEL'19, Hamburg, Germany, Aug. 2019, paper TUP053.

[3] J. Goodman, *Statistical Optics*, 1st ed. (Wiley-Interscience, 2000), ISBN:9780471399162.

[4] R. Loudon, *The Quantum Theory of Light*, 3rd ed. (Oxford University Press, 2000), ISBN:9780198566731.

[5] K.-J. Kim, Z. Huang, and R. R. Lindberg, *Synchrotron Radiation and Free-Electron Lasers*, 1st ed. (Cambridge University Press, 2017). doi:10.1017/9781316677377

[6] W. Becker and J. K. McIver, "Photon statistics of the free-electron-laser startup", *Phys. Rev. A*, vpl. 28, p. 1838, Sep 1983. doi:10.1103/PhysRevA.28.1838

[7] R. Bonifacio, N. Piovella, G. R. M. Robb, and A. Schiavi, "Quantum regime of free electron lasers starting from noise", *Phys. Rev. ST Accel. Beams*, vol. 9, p. 090701, Sep 2006. doi:10.1103/PhysRevSTAB.9.090701

[8] M. S. Fox, *Quantum Optics: An Introduction*, 1st ed. (Oxford University Press, 2006), ISBN:9780198566731.

[9] H. Bachor and T. Ralph, *A Guide to Experiments in Quantum Optics*, 2nd ed. (Wiley-VCH, 2004). doi:10.1002/9783527619238

[10] J. Gea-Banacloche, "Quantum theory of the free-electron laser: Large gain, saturation, and photon statistics", *Phys. Rev. A*, vol. 31, p. 1607, Mar 1985. doi:10.1103/PhysRevA.31.1607

[11] I. Gjaja and A. Bhattacharjee, "Generation of squeezed radiation from a free-electron laser", *Phys. Rev. A*, vol. 36, pp. 5486–5489, Dec 1987. doi:10.1103/PhysRevA.36.5486