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A NOVEL ONE-DIMENSIONAL MODEL FOR CSR WAKEFIELDS*

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Abstract

The existing 1D models of the coherent synchrotron radiation (CSR) wakefield in free space assume that the longitudinal bunch distribution remains constant when the beam propagates through a magnetic lattice. In this paper, we derive a formula for a 1D CSR wake that takes into account variation of the bunch length along the orbit. The formula is valid for arbitrary curvilinear beam trajectory. We analyze the validity of the 1D model in a typical implementation of an FEL bunch compressor.

INTRODUCTION

When the trajectory of a relativistic beam is bent by magnetic field, the beam radiates electromagnetic field and experiences a radiation reaction force. A popular 1D model for this force in the case of a circular motion, often called the coherent synchrotron radiation (CSR) wake, was first developed in Refs. [1–3]. A generalization of this model for the case of a bending magnet of finite length is described in Refs. [4, 5] and is implemented in several computer codes[1]. These models assume a constant bunch length, σ_z, along the orbit. Their applicability becomes questionable in a bunch compressor where $\sigma_z(s)$ is a function of the beam position s. In practice, one usually substitutes a local bunch length $\sigma_z(s)$ into the formulas derived with the assumption $\sigma_z = \text{const}$. Such a substitution, strictly speaking, is not justified because it ignores the fact that the wake is formed by the beam radiation emitted at previous times, when the bunch length is different from the value of σ_z at the moment when the wake interacts with the beam. More recently, in Ref. [6], a 1D CSR model was derived from Jefimenko's form for the electric field of a relativistic beam with an attempt to include a time dependent bunch length by heuristically introducing $\sigma_z(t)$ into equations derived with the assumption of constant bunch length.

In this paper, we derive a 1D CSR wake that takes into account the variation of the bunch length along the orbit. Our derivation is based on the 3D formulas from Ref. [7].

We use the CGS system of units throughout this paper.

FORMULA FOR THE LONGITUDINAL WAKE IN 3D

We begin from the formulation of general 3D wake from Ref. [7]. In 3D, the beam is represented by its charge density $\rho(\boldsymbol{r}, t)$ that depends on time t and the position vector $\boldsymbol{r}$, and its velocity $\boldsymbol{v}(\boldsymbol{r}, t)$, with the beam current density $\boldsymbol{j}$ given by the product $\boldsymbol{j} = \rho\boldsymbol{v}$. Note that in this model assigning a particular value of $\boldsymbol{v}$ at each point we neglect the uncorrelated velocity spread in the beam due the the angular and energy spread — an approximation that is typically well satisfied for relativistic beams. For given functions $\rho(\boldsymbol{r}, t)$ and $\boldsymbol{j}(\boldsymbol{r}, t)$, one can derive an equation for the electric field in the beam, $\boldsymbol{E}(\boldsymbol{r}, t)$, and calculate the instantaneous energy change per unit time and *per unit charge*, $\mathcal{P}$,

$$\mathcal{P}(\boldsymbol{r}, t) = \boldsymbol{v}(\boldsymbol{r}, t) \cdot \boldsymbol{E}(\boldsymbol{r}, t). \quad (1)$$

We will loosely call $\mathcal{P}$ the *longitudinal wake*, although the classical wake fields are typically associated with the energy loss integrated over the beam path and the transverse cross section of the beam.

A general expression for the quantity $\mathcal{P}$ is given in Ref. [7]. It consists of three terms two of which in many cases are much smaller that the third one. We neglect these terms in our analysis leaving only the dominant one:

$$\mathcal{P}(\boldsymbol{r}, t) = -c \int \frac{d^3 r'}{|\boldsymbol{r}' - \boldsymbol{r}|} [\boldsymbol{\beta}(\boldsymbol{r}, t) - (\boldsymbol{\beta}(\boldsymbol{r}, t) \cdot \boldsymbol{\beta}(\boldsymbol{r}', t_{\text{ret}}))\boldsymbol{\beta}(\boldsymbol{r}', t_{\text{ret}})] \cdot \partial_{\boldsymbol{r}'} \rho(\boldsymbol{r}', t_{\text{ret}}), \quad (2)$$

where $\boldsymbol{\beta} = \boldsymbol{v}/c$, $t_{\text{ret}}(\boldsymbol{r}, \boldsymbol{r}', t) = t - |\boldsymbol{r}' - \boldsymbol{r}|/c$, and we use the notation $\partial_{\boldsymbol{r}'} \rho(\boldsymbol{r}', t_{\text{ret}})$ to indicate differentiation with respect to the space coordinates in function $\rho(\boldsymbol{r}', t_{\text{ret}})$ with a fixed t_{ret} (in other words, the operator $\partial_{\boldsymbol{r}'}$ ignores the fact that t_{ret} also depends on $\boldsymbol{r}'$). Note that due to the factor $|\boldsymbol{r}' - \boldsymbol{r}|^{-1}$ the integrand has a singularity at $\boldsymbol{r}' \to \boldsymbol{r}$, however, this singularity is integrable[2]. The singularity disappears in the ultra-relativistic limit $|\boldsymbol{\beta}| = 1$ because the factor in square brackets vanishes when $\boldsymbol{r} \to \boldsymbol{r}'$. As one can see, the integral (2) is taken over the volume around the beam trajectory at preceding times $t_{\text{ret}} < t$.

1D MODEL

It is known that the transverse size of the beam, $\sigma_\perp$, does not affect the CSR wake if $\sigma_\perp \lesssim (\sigma_z^2 R)^{1/3}$, where R is the bending radius [3] (see, however, discussion below). Assuming that this is the case, we can simplify integral (2) and derive a 1D model for the wake $\mathcal{P}$. In 1D all particles in the beam move along a curve given by the radius-vector $\boldsymbol{r}_0(s)$, where s is the path length measured along the orbit. Positions in the vicinity of this orbit are represented in a Frenet–Serret coordinate system as $\boldsymbol{r}_0(s) + \hat{\boldsymbol{x}}(s)x + \hat{\boldsymbol{y}}(s)y$, where $\hat{\boldsymbol{x}}(s)$ and $\hat{\boldsymbol{y}}(s)$ are the unit vectors perpendicular to the tangential vector, $\boldsymbol{\tau}(s) \equiv d\boldsymbol{r}_0/ds$, and each other so that x, y and s constitute a local orthogonal coordinate system. In what follows we will assume an ultra-relativistic beam

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[1] Here, we only deal with the longitudinal part of the forces in the bunch that changes particles' energy in the beam; for the effect of the transverse force, see a recent study [8].

[2] It is also integrable in a 2D model when the three dimensional integration $\int d^3 r'$ is replaced by $\int d^2 r'$. It is not integrable in 1D.

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with $|\boldsymbol{\beta}| = 1$. The velocity is everywhere directed along the trajectory, so we can identify $\boldsymbol{\beta}$ with the tangential vector, $\boldsymbol{\beta} = \boldsymbol{\tau}(s)$. We then write Eq. (2) as

$$\mathcal{P}(s,t) = -c \int \frac{ds'\,dx'\,dy'}{|\boldsymbol{r}_0(s') - \boldsymbol{r}_0(s)|} [\boldsymbol{\tau}(s) - (\boldsymbol{\tau}(s)\cdot\boldsymbol{\tau}(s'))\boldsymbol{\tau}(s')] \cdot \partial_{\boldsymbol{r}'} \rho(x',y',s',t_{\text{ret}}), \quad (3)$$

where we have associated the observation vector $\boldsymbol{r}$ with $\boldsymbol{r}_0(s)$ (the wake is calculated on the beam orbit) and replaced $\boldsymbol{r}'$ by $\boldsymbol{r}_0(s')$[3]. We note that $\boldsymbol{\tau}(s)\cdot\boldsymbol{\tau}(s')$ is the projection of vector $\boldsymbol{\tau}(s)$ onto the vector $\boldsymbol{\tau}(s')$ and subtracting $(\boldsymbol{\tau}(s)\cdot\boldsymbol{\tau}(s'))\boldsymbol{\tau}(s')$ from $\boldsymbol{\tau}(s)$ gives a part of the vector $\boldsymbol{\tau}(s)$ that is perpendicular to $\boldsymbol{\tau}(s')$. Denoting this difference by $\boldsymbol{m}(s,s')$,

$$\boldsymbol{m}(s,s') = \boldsymbol{\tau}(s) - (\boldsymbol{\tau}(s)\cdot\boldsymbol{\tau}(s'))\boldsymbol{\tau}(s'), \quad (4)$$

we have $\boldsymbol{m}(s,s')\cdot\boldsymbol{\tau}(s') = 0$, and hence this vector has only x' and y' components at s'. Using this vector, we can re-write Eq. (3) as

$$\begin{aligned}\mathcal{P}(s,t) &= -c \int \frac{ds'\,dx'\,dy'}{|\boldsymbol{r}_0(s') - \boldsymbol{r}_0(s)|} \boldsymbol{m}(s,s') \cdot \partial_{\boldsymbol{r}'} \rho(x',y',s',t_{\text{ret}}) \\ &= -c \int \frac{ds'\,dx'\,dy'}{|\boldsymbol{r}_0(s') - \boldsymbol{r}_0(s)|} [m_{x'}(s,s')\partial_{x'}\rho(x',y',s',t_{\text{ret}}) \\ &+ m_{y'}(s,s')\partial_{y'}\rho(x',y',s',t_{\text{ret}})] . \end{aligned} \quad (5)$$

At a first glance, it seems that this integral is equal to zero because we integrate partial derivatives of ρ with respect to x' and y' over x' and y' from minus to plus infinity, and, of course, the distribution function goes to zero at $|x'|, |y'| \to \infty$. However, at this point we need to take into account the dependence of $t_{\text{ret}} = t - |\boldsymbol{r} - \boldsymbol{r}'|/c$ versus x' and y'. In our curvilinear coordinate system, vector $\boldsymbol{r} - \boldsymbol{r}'$ can be written in the following way:

$$\boldsymbol{r} - \boldsymbol{r}' = \boldsymbol{r}_0(s) + (\hat{\boldsymbol{x}}x + \hat{\boldsymbol{y}}y) - \boldsymbol{r}_0(s') - (\hat{\boldsymbol{x}}'x' + \hat{\boldsymbol{y}}'y'), \quad (6)$$

where we use the notations: $\hat{\boldsymbol{x}} = \hat{\boldsymbol{x}}(s)$, $\hat{\boldsymbol{x}}' = \hat{\boldsymbol{x}}(s')$, and similar abbreviations for the y components. Given the assumed smallness of the transverse size of the beam, we will use the Taylor expansion of t_{ret}, keeping only linear terms in x' and y' (the linear terms in x and y will be annihilated by the integration by parts below, so we ignore them),

$$t_{\text{ret}} \approx t - \frac{1}{c}|\boldsymbol{r}_0(s) - \boldsymbol{r}_0(s')| + \frac{1}{c}\frac{\boldsymbol{r}_0(s) - \boldsymbol{r}_0(s')}{|\boldsymbol{r}_0(s) - \boldsymbol{r}_0(s')|} \cdot (\hat{\boldsymbol{x}}'x' + \hat{\boldsymbol{y}}'y'). \quad (7)$$

We then expand function ρ,

$$\begin{aligned}\rho(x',y',s',t_{\text{ret}}) &\approx \rho\left(x',y',s',t - \frac{1}{c}|\boldsymbol{r}_0(s) - \boldsymbol{r}_0(s')|\right) \\ &+ \partial_t \rho\left(x',y',s',t - \frac{1}{c}|\boldsymbol{r}_0(s) - \boldsymbol{r}_0(s')|\right) \\ &\times \frac{1}{c}\frac{\boldsymbol{r}_0(s) - \boldsymbol{r}_0(s')}{|\boldsymbol{r}_0(s) - \boldsymbol{r}_0(s')|} \cdot (\hat{\boldsymbol{x}}'x' + \hat{\boldsymbol{y}}'y'). \end{aligned} \quad (8)$$

[3] Replacing d^3r by $ds'\,dx'\,dy'$ we ignore the Lamè coefficients in the curvilinear coordinate system x, y, s. The account of these coefficients would add only small corrections to our results.

Substituting this expression into (5) we note that the first term in ρ vanishes after the integration over x' and y', and the second term, after integration by parts, gives

$$\mathcal{P}(s,t) = \int ds' \frac{(\boldsymbol{r}_0(s) - \boldsymbol{r}_0(s'))\cdot\boldsymbol{m}(s,s')}{|\boldsymbol{r}_0(s') - \boldsymbol{r}_0(s)|^2} \times \partial_t \lambda\left(s', t - \frac{1}{c}|\boldsymbol{r}_0(s) - \boldsymbol{r}_0(s')|\right), \quad (9)$$

where λ is the longitudinal distribution function of the beam

$$\lambda(s,t) = \int dx'dy'\rho(x',y',s',t). \quad (10)$$

Note that the integrand in the 1D integral (9) does not have a singularity at $s' \to s$ because both terms in the numerator, $\boldsymbol{r}_0(s) - \boldsymbol{r}_0(s')$ and $\boldsymbol{m}(s,s')$, vanish when $s' = s$.

Using the method of images, Eq. (9) can be also generalized for the case when a plane beam orbit lies between two parallel conducting plates.

TRANSIENT CSR WAKE IN A BEND MAGNET

To benchmark Eq. (9) against known solutions of 1D CSR problems, we first calculated the steady-state wake for a short Gaussian bunch, $\sigma_z \ll R$, moving in a circular orbit of constant radius R. Our result (not shown here) agrees very well with the wake profile which can be found in the literature.

Another benchmark problem, that of a bending magnet of finite length L was studied in Refs. [4, 5]. In this problem, the beam travels on an arc of radius R inside the magnet, $-\frac{1}{2}L < s < \frac{1}{2}L$. Outside of the magnet, the beam moves with $v = c$ along straight lines tangential to the circular orbit at the points of entrance and exit, respectively. For the sake of comparison, we have chosen the same set of parameters as in Ref. [5]: $R = 1.5$ m, $\sigma_z = 50\,\mu$m, $Q = 1$ nC and $L = 25$ cm.

We first calculated the wake inside the bunch when it enters the bend from a straight line. The plot of this wake at various distances from the magnet entrance edge is shown in Fig. 1 by solid lines. For comparison, the dashed green lines show the result of 1D model computed in Ref. [5].

We have also calculated the wake in the bunch after it exits the bend magnet and continues to travel along a straight line, and found an excellent agreement of our theory with the 1D model of Ref. [5].

CSR WAKE IN A BUNCH COMPRESSOR

A much more difficult problem is presented by a chicane bunch compressor consisting of four dipole magnets. To illustrate how Eq. (9) can be used in a situation when σ_z varies with s, we calculated the CSR wake in a configuration studied at the CSR workshop at DESY-Zeuthen in 2002 [9]. The four magnets have the length $L = 0.5$ m with the bending radius $R = 10.35$ m resulting in the momentum compaction factor $R_{56} = 2.5$ cm. In our simulations, the beam with

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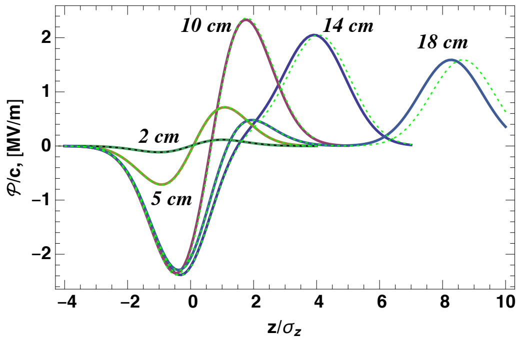

Figure 1: Longitudinal wake in the bunch as a function of distance from the entrance edge of the bend shown by a number near each curve. The longitudinal coordinate z in each case is measured from the center of the bunch.

the energy of 5.0 GeV and Gaussian distributions in energy and coordinates is compressed from the initial rms length of 200 μm to the final length of 150 μm, as shown in Fig. 2[4]. The beam charge is 1 nC, the slice energy spread is 10^{-4}, and the energy chirp is -10 m^{-1}. We calculated the CSR wake in the middle of the second and third magnets as well as at the center of the chicane, see Fig. 2.

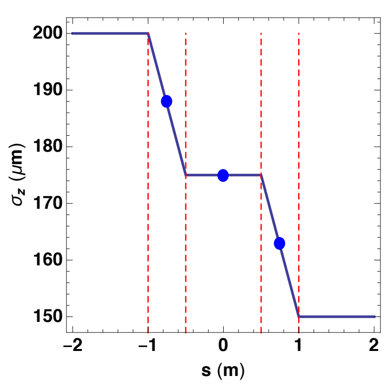

Figure 2: Variation of the bunch length $\sigma_z(s)$ through the second and third magnets of the chicane (the magnet edges are shown by red dashed lines). The coordinate $s = 0$ corresponds to the center of the chicane. The blue dots show three positions in the chicane where the CSR wake was calculated.

The wake calculated in the 1D model with the help of Eq. (9) is shown in Fig. 3 by solid lines. These wakes are compared with the wakes calculated with a 2D version of Eq. (2) (shown by dashed lines). The 2D wakes plotted as a function of z are actually calculated along the axis of the tilted beam, that is at $x = z \tan \alpha$, where α is the tilt angle ($\alpha = 72°$ at the center of the chicane). The plots show a considerable difference between the 1D and 2D wakes. We discuss the origin of this difference in the next section.

[4] We have intentionally chosen a small compression factor in an attempt to improve the applicability of the 1D model for the CSR wake.

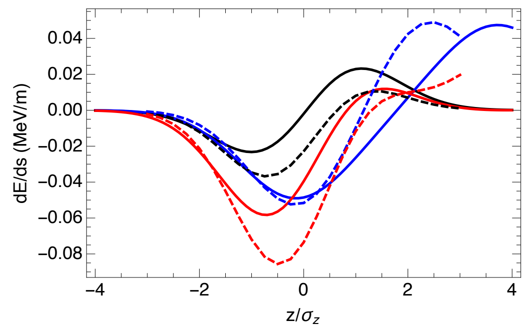

Figure 3: Wakes calculated in 1D model are shown by solid lines: black — in the middle of the second magnet ($s = -0.75$ m), blue — at the center of the chicane ($s = 0$), and red — in the middle of the third magnet ($s = 0.75$ m). Dashed lines (with the corresponding color) are calculated for the same locations using a 2D version of Eq. (2). The coordinate z is normalized by the rms bunch length, $\sigma_z(s)$.

DISCUSSION

The simple 1D models [2–4, 6] of the CSR wake have two important limitations when applied to bunch compressors. First, they assume a constant bunch length, and, second, they ignore the tilt of the bunch with an energy chirp when it passes through the region of large dispersion. In contrast to the previous theories, our Eq. (9), takes the variation of the bunch length into account but it still misses the bunch tilt. This is the reason of noticeable discrepancy between our 1D and 2D calculations in Fig. 3.

It is often assumed that a 1D model that ignores the transverse size of the beam is supposed to work when the following condition [3] is met: $\sigma_\perp \lesssim (\sigma_z^2 R)^{1/3}$. This is well satisfied for the parameters of the chicane studied in the preceding section: at the center of the chicane $\sigma_\perp = 0.54$ mm, $\sigma_z = 175$ μm, and $(\sigma_z^2 R)^{1/3} = 6.8$ mm; however, as Fig. 3 shows, our 1D model does not agree with the more accurate 2D one. One can argue that a more accurate estimate should use not the total bunch length σ_z, but the lengths of a longitudinal slice of the beam (say, for $x = 0$), $\sigma_{z,x=0} = 16$ μm, which makes $(\sigma_{z,x=0}^2 R)^{1/3} = 1.3$ mm, but even this estimate apparently underestimate the effect of the beam tilt, as our results demonstrate.

We note that our choice of a relatively small energy chirp in the beam, and hence a small compression, was motivated by the desire to minimize the effect of the beam tilt when it passes through the second and third magnets. For the compression factor of 10 originally studied in [9], the tilt is much stronger, and the 1D model even less likely to work near the center of the chicane. It should work, though, in the last magnet of the chicane where the bunch is already compressed longitudinally and the tilt gradually vanishes together with the dispersion. Our results indicate that one has to be very careful when simulating the beam dynamics in chicanes using a 1D model of the CSR wakefields.

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