Numerical solution of the FEL correlation function equation

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Outline

1. Introduction
2. Basic definitions and derivation of kinetic equations
3. Simplification of the equations for the case of narrow beam and stationary current
4. Algorithm of numerical solution
5. Simulation results
6. Conclusion

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Derivation of kinetic equations

Step 1: ***Equations of particle longitudinal motion***

Assumptions:

1. Transverse motion is not effected by FEL operation. Electron trajectories are given (and known) functions of longitudinal coordinate in undulator and electron initial transverse coordinates in 4-D phase space .
2. Interaction between two electrons in longitudinal direction is carried out trough radiation field. All other collective forces are neglected.
3. Radiation field of electrons has narrow bandwidth and obeys paraxial wave equation. Therefore interaction force can be averaged over several undulator periods.

$$\frac{dz_i}{dt} = 1 - \frac{1}{2\gamma_{//}^2} + \frac{\Delta_i}{\gamma_{//}^2} - \Delta\beta(z_i, X_i)$$

energy deviation $\delta\gamma/\gamma_0$ *of the* i*-th particle*

velocity shift due to betatron oscillations

$$\frac{d\Delta_i}{dt} = \sum_{l \neq i} \Phi\left[z_i, X_i, z_l(t'_l), X_l\right]$$

longitudinal interaction "force" between two particles

$$t - z_i = t'_l - z_l(t'_l)$$

retardation

$$\Phi(1,2) = 2\frac{r_e k_0}{\gamma_0} \frac{K(z_1)K(z_2)}{1+K^2(z_2)} \frac{\sin\left(k_w(z_1 - z_2) + k_0 \frac{\left(\vec{R}_1(X_1, z_1) - \vec{R}_2(X_2, z_2)\right)^2}{2(z_1 - z_2)} + \varphi_1 - \varphi_2\right)}{z_1 - z_2} \theta(z_1 - z_2)\theta(z_2)$$

transverse trajectory

$$\frac{dz_i}{dt} = 1 - \frac{1}{2\gamma_{//}^2} + \frac{\Delta_i}{\gamma_{//}^2} - \Delta\beta(z_i, X_i)$$

$$\frac{d\Delta_i}{dt} = \sum_{l \neq i} \Phi\left[z_i, X_i, z_l(t_l'), X_l\right]$$

$$t - z_i = t_l' - z_l(t_l')$$

System of ordinary differential equations

$$\xi = t - z$$

$$\frac{dz_i}{d\xi} \approx 2\gamma_{//}^2\left[1 + 2\Delta_i - 2\gamma_{//}^2 \Delta\beta(z_i, X_i)\right]$$

$$\frac{d\Delta_i}{d\xi} \approx 2\gamma_{//}^2 \sum_{l \neq i} \Phi(z_i, X_i, z_l, X_l)$$

Motion equations with retardation

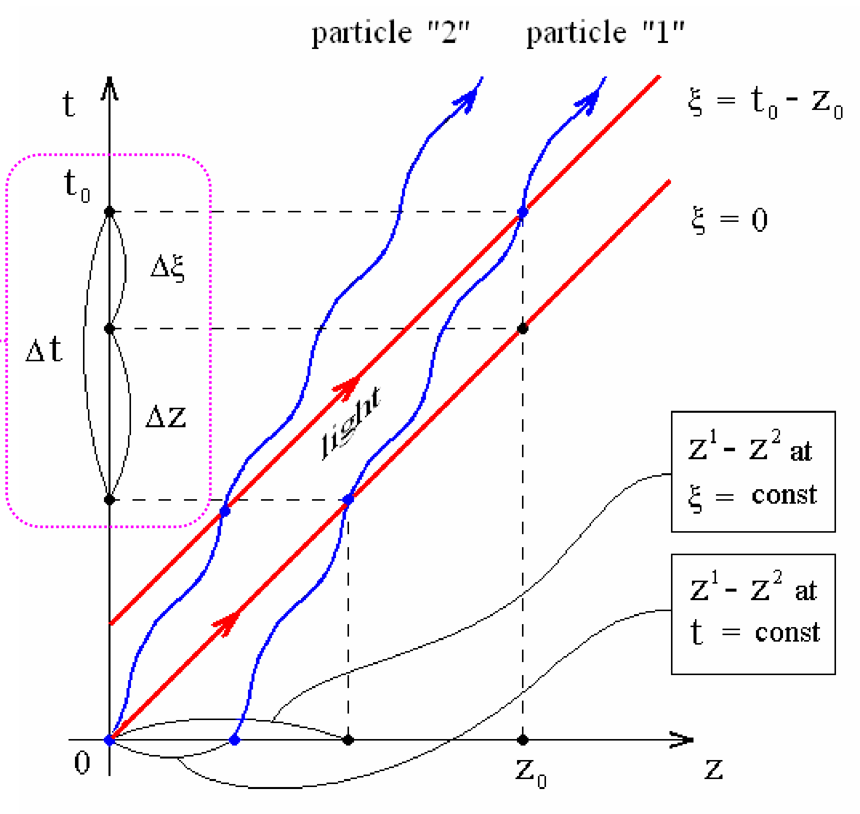
$$\theta = 2\gamma_{\|}^2 \xi = 2\gamma_{\|}^2 (t - z) \quad (c = 1)$$

New "time" variable

$$\Delta t = \Delta\xi + \Delta z$$

$$\Delta\xi = \Delta t(1 - V_z)$$

$$\frac{df}{d\xi} = \frac{df}{dt}\frac{1}{1 - V_z}$$



New independent variable is just a new parameterization of the electron world lines in the space-time continuum

Step 2: Continuity equation for the N-particles distribution function

$$L(i) = \left(1 + 2\Delta_i - 2\gamma_\parallel^2 \Delta\beta(z_i, X_i)\right)\frac{\partial}{\partial z_i}$$

$$V(i,j) = -N\Phi(z_i, X_i, z_j, X_j)\frac{\partial}{\partial \Delta_i}$$

$$\left[\frac{\partial}{\partial\theta} + \sum_{i=1}^{N} L(i) - \frac{1}{N}\sum_{i \neq j}^{N} V(i,j)\right] f_N(1,\ldots,N;\theta) = 0$$

distribution function in $6 \times N$ *- dimensional phase space*

Physical meaning of the distribution function

Probability to find the system of N particles in the $6\times N$ dimensional phase space volume $dX_1 \ldots dX_N$ at the "time" moment θ

$$f_N(1,\ldots,N;\theta)\,d\{1\}\ldots d\{N\} = f_N(X_1,\ldots,X_N;\theta)\,dX_1\ldots dX_N$$

Probability density

6D vector of one particle coordinates and momenta

m-particle distribution function

$$f_m = \int f_N \, dX_{m+1}\ldots dX_N$$

$$f_1(X_1;\theta)\,dX_1$$

Probability to find one particle in the 6D phase space volume dX_1

Probability to find one particle in the volume dX_1 and another particle in the volume dX_2

$$f_2(X_1,X_2;\theta)\,dX_1 dX_2$$

Step 3: BBGKY chain of equations

$$\left[\frac{\partial}{\partial\theta}+L(1)\right]f_1(1;\theta)=\int V(1,2)\,f_2(1,2;\theta)\,d\{2\},$$

$$\left[\frac{\partial}{\partial\theta}+L(1)+L(2)-\frac{1}{N}\left[V(1,2)+V(2,1)\right]\right]f_2(1,2;\theta)=$$

$$=\int\left[V(1,3)+V(2,3)\right]f_3(1,2,3;\theta)\,d\{3\},$$

....

$$\left[\frac{\partial}{\partial\theta}+\sum_{i=1}^{N}L(i)-\frac{1}{N}\sum_{i\neq j}^{N}V(i,j)\right]f_N(1,\ldots,N;\theta)=0,$$

$$f_{m-1}=\int f_m\,d\{m\}$$

See e.g. S. Ishimaru, "*Basic Principles of Plasma Physics*", Benjamin, London, 1973.

Step 4: Truncation of the BBGKY chain

Correlation functions decomposition

$$f_1(1,\theta) = F(1,\theta)$$

$$f_2(1,2,\theta) = F(1,\theta)F(2,\theta) + G(1,2,\theta)$$

$$f_3(1,2,3,\theta) = F(1,\theta)F(2,\theta)F(3,\theta) + F(1,\theta)G(2,3,\theta) + \\ + F(2,\theta)G(1,3,\theta) + F(3,\theta)G(1,2,\theta) + H(1,2,3,\theta)$$

$$G(1,2,0) = H(1,2,3,0) = \ldots = 0$$

Natural initial condition for the non-interacting particles

This assumption seems to be reasonable because the number of interacting particles (the number of particles on cooperation length) is large.

We assume that $H(1,2,3,\theta) \ll G(1,2,\theta)$

The similar assumption is valid in plasma physics where one has large number of particles in the Debye sphere.

Final system of kinetic equations

$$\frac{\partial}{\partial\theta}F(1,\theta)+v_1\frac{\partial}{\partial z_1}F(1,\theta)+N\frac{\partial F(1,\theta)}{\partial\Delta_1}\int\Phi(1,2)F(2,\theta)d2=-N\int\Phi(1,2)\frac{\partial G(1,2,\theta)}{\partial\Delta_1}d2$$

$$\frac{\partial}{\partial\theta}G(1,2,\theta)+v_1\frac{\partial}{\partial z_1}G(1,2,\theta)+v_2\frac{\partial}{\partial z_2}G(1,2,\theta)+N\frac{\partial F(1,\theta)}{\partial\Delta_1}\int\Phi(1,3)G(2,3,\theta)d3+$$

$$+N\frac{\partial F(2,\theta)}{\partial\Delta_2}\int\Phi(2,3)G(1,3,\theta)d3=-\Phi(1,2)F(2)\frac{\partial F(1)}{\partial\Delta_1}-\Phi(2,1)F(1)\frac{\partial F(2)}{\partial\Delta_2}$$

shot noise induced source term

longitudinal "velocity" $dz/d\theta$

$$v(i)=1+2\Delta_i-2\gamma_{\parallel}^2\left(\frac{\dot{x}_1^2(z_i,X_i)}{2}+\frac{\dot{y}_1^2(z_i,X_i)}{2}+\frac{x_1^{\,2}(z_i,X_i)}{2\beta_u^2(z_i)}+\frac{y_1^{\,2}(z_i,X_i)}{2\beta_u^2(z_i)}\right)$$

$$F(1,0)=F_0(z_1,\Delta_1),\; G(1,2,0)=0$$

Initial conditions

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Stationary current and narrow beam case

1. All time derivatives are equal to zero

$$\frac{\partial}{\partial \theta} = 0$$

2. Transverse beam size is constant and small

$$2\pi \frac{\sigma^2}{\lambda_0 L_g} << 1$$

3. Interaction force can be averaged over transverse distribution

$$\left\langle \Phi(1,2) \right\rangle_{\perp} = -\frac{r_e}{2\sigma^2 k_w \gamma} \frac{K^2}{1+K^2} \left(\frac{e^{ik_w(z_1 - z_2)}}{1 + i\alpha k_w (z_1 - z_2)} + c.c. \right)$$

Resulting system of equations for numerical solution

$$\nu_1 \frac{\partial}{\partial z_1} F(1) = -2\,\mathrm{Re}\left(\frac{\partial}{\partial \Delta_1} I(z_1, \Delta_1; z_1) \right)$$

$$\frac{1}{2}\left[(\nu_1 + \nu_2)\left(\frac{\partial}{\partial z_1} + \frac{\partial}{\partial z_2} \right) + (\nu_1 - \nu_2)\left(2i + \frac{\partial}{\partial z_1} - \frac{\partial}{\partial z_2} \right) \right] \tilde{G}(1;2) =$$

$$= \quad -\frac{\partial F(1)}{\partial \Delta_1} I^*(z_1; z_2, \Delta_2) \quad - \quad \frac{\partial F(2)}{\partial \Delta_2} I(z_2; z_1, \Delta_1) \quad -$$

$$-\frac{2\pi}{N_{\lambda_0}}\left(\tilde{\Phi}^*(z_1 - z_2)\frac{\partial}{\partial \Delta_1} + \tilde{\Phi}(z_2 - z_1)\frac{\partial}{\partial \Delta_2} \right) F(1) F(2)$$

$$I(z_1; z_2, \Delta_2) = \int_0^{z_1} \int_{-\infty}^{\infty} \tilde{\Phi}(z_1 - z_3) \tilde{G}(2;3)\, d\{3\}$$

$$G(z_1, \Delta_1; z_2, \Delta_2) = 2\,\mathrm{Re}\left(\tilde{G}(z_1, \Delta_1; z_2, \Delta_2) e^{i(z_1 - z_2)} \right)$$

slow varying complex amplitude

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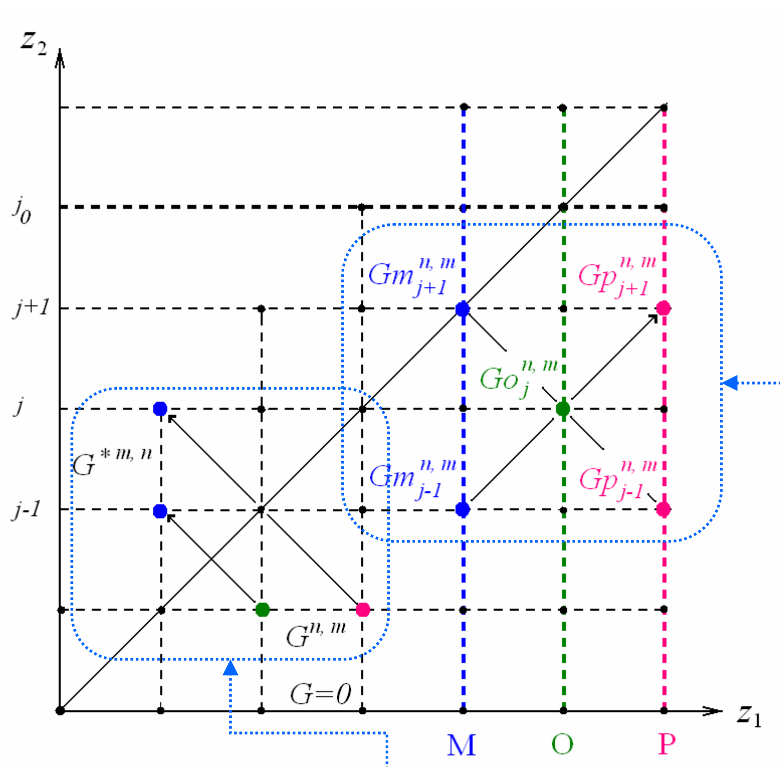
Algorithm of numerical solution



$$\tilde{G}(1,2) = \tilde{G}^{*}(2,1)$$

Explicit difference scheme

$$Gp^{n,m}_{j+1} = Gm^{n,m}_{j-1} - \frac{\Delta_n - \Delta_m}{1+\Delta_n+\Delta_m}\left(Gp^{n,m}_{j-1} - Gm^{n,m}_{j+1} + 4iH_z Go^{n,m}_j\right) -$$

$$-\frac{2H_z}{H_\Delta}\frac{1}{1+\Delta_n+\Delta_m}\Bigg(\left(F^n_{jo} - F^{n-1}_{jo}\right)I1^{*m}_j + \left(F^m_j - F^{m-1}_j\right)I2^n_j +$$

$$-\frac{2\pi}{N_{\lambda_0}}\frac{F^m_j + F^{m-1}_j}{2}\left(F^n_{jo} - F^{n-1}_{jo}\right)\tilde{\Phi}^*_{jo-j}\Bigg)$$

$$F^n_{jo+1} = F^n_{jo-1} - \frac{2}{1+2\Delta^F_n}\frac{2H_z}{H_\Delta}\mathrm{Re}\left(I2^{n+1}_{jo} - I2^n_{jo}\right)$$

$$I1^m_j = \frac{H_z}{2}\sum_{k=1}^{jo-1}\left(IGo^m_{j,k}\tilde{\Phi}_{jo-k+1} + IGo^m_{j,k+1}\tilde{\Phi}_{jo-k}\right)$$

$$I2^n_j = \frac{H_z}{2}\sum_{k=1}^{j-1}\left(IGo^n_{jo,k}\tilde{\Phi}_{j-k+1} + IGo^n_{jo,k+1}\tilde{\Phi}_{j-k}\right)$$

$$IGo^n_{jo,j} = H_\Delta \sum_{m=1}^{N_\Delta} Go^{n,m}_j$$

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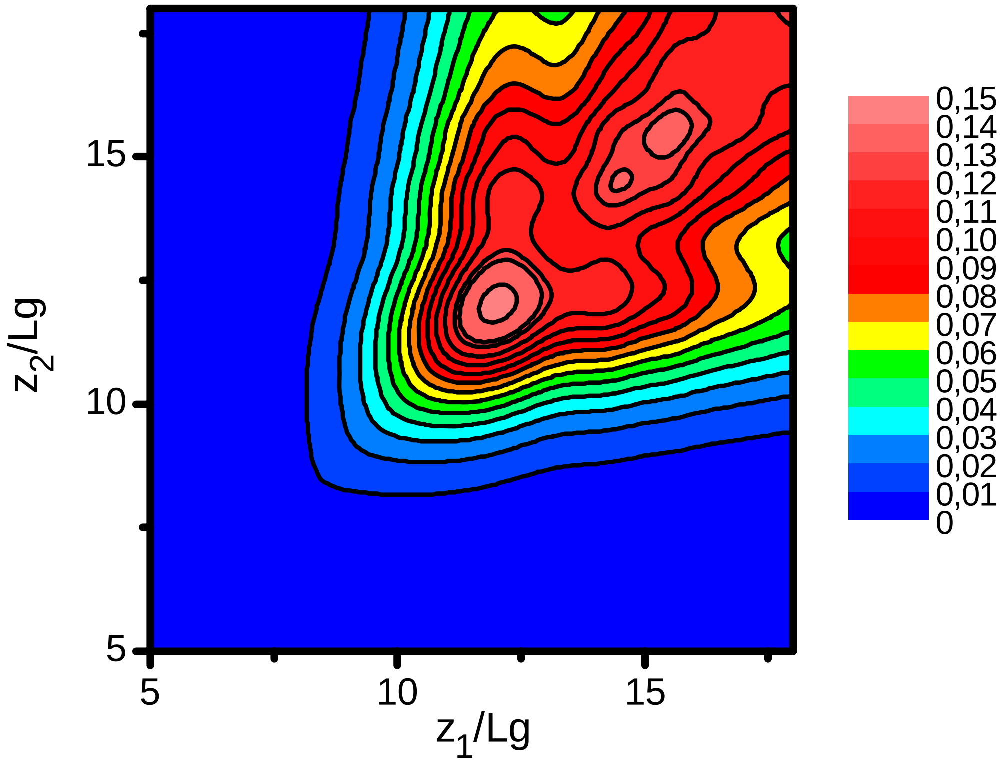
Results of simulations

Two coordinate distribution of the correlation function amplitude integrated over energy



Lg – *the gain length at linear stage*

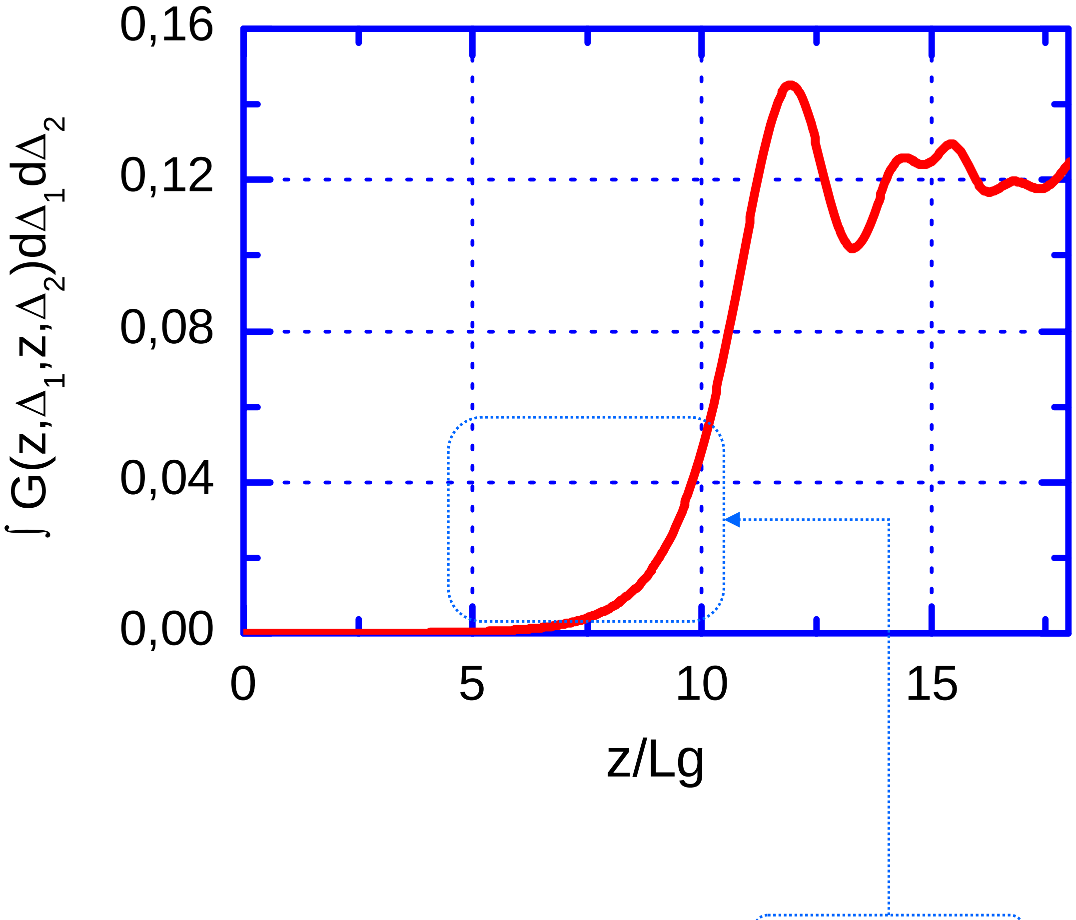
This function characterize spectral properties of the signal in FEL

$$J_\nu(z) = \int \tilde{G}\left(z - \frac{\tau}{2}, \Delta_1; z + \frac{\tau}{2}, \Delta_2 \right) e^{i\nu\tau} d\Delta_1 d\Delta_2 d\tau$$

This function represents the square of beam microbunching at coordinate z *in undulator*

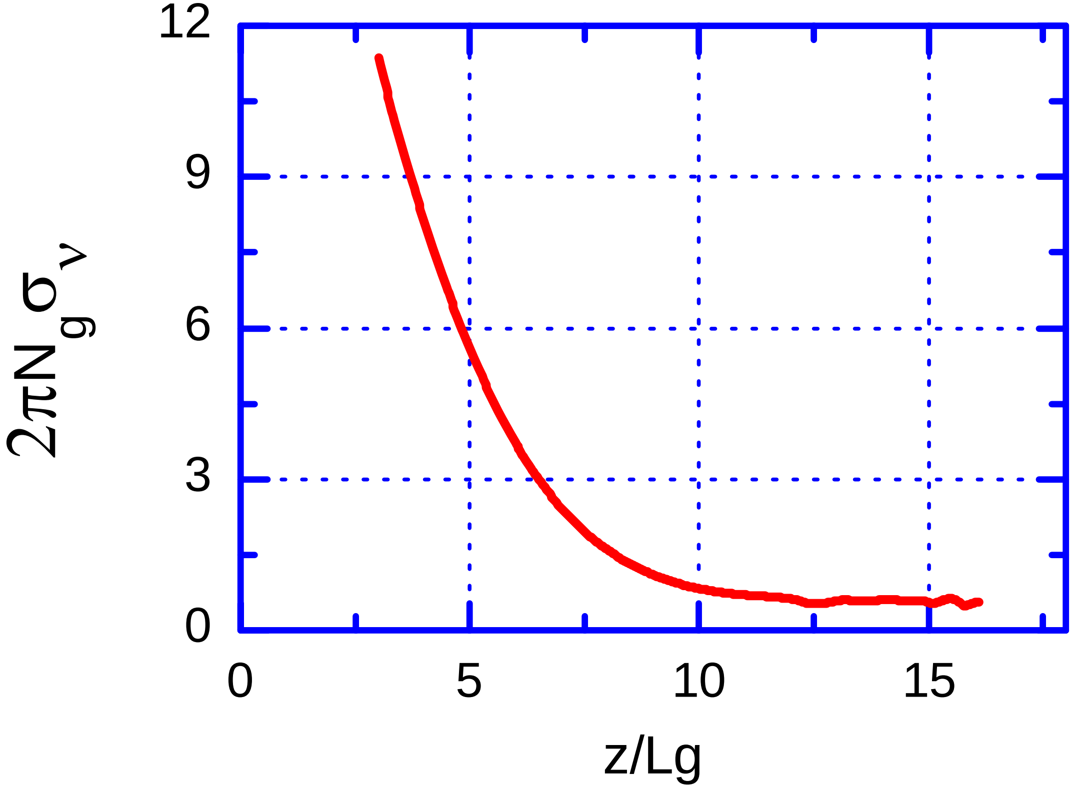
$$g(z) = \frac{1}{2\pi} \int J_\nu(z) d\nu = \int \tilde{G}(z, \Delta_1, z, \Delta_2) d\Delta_1 d\Delta_2$$

Growth of microbunching square in undulator



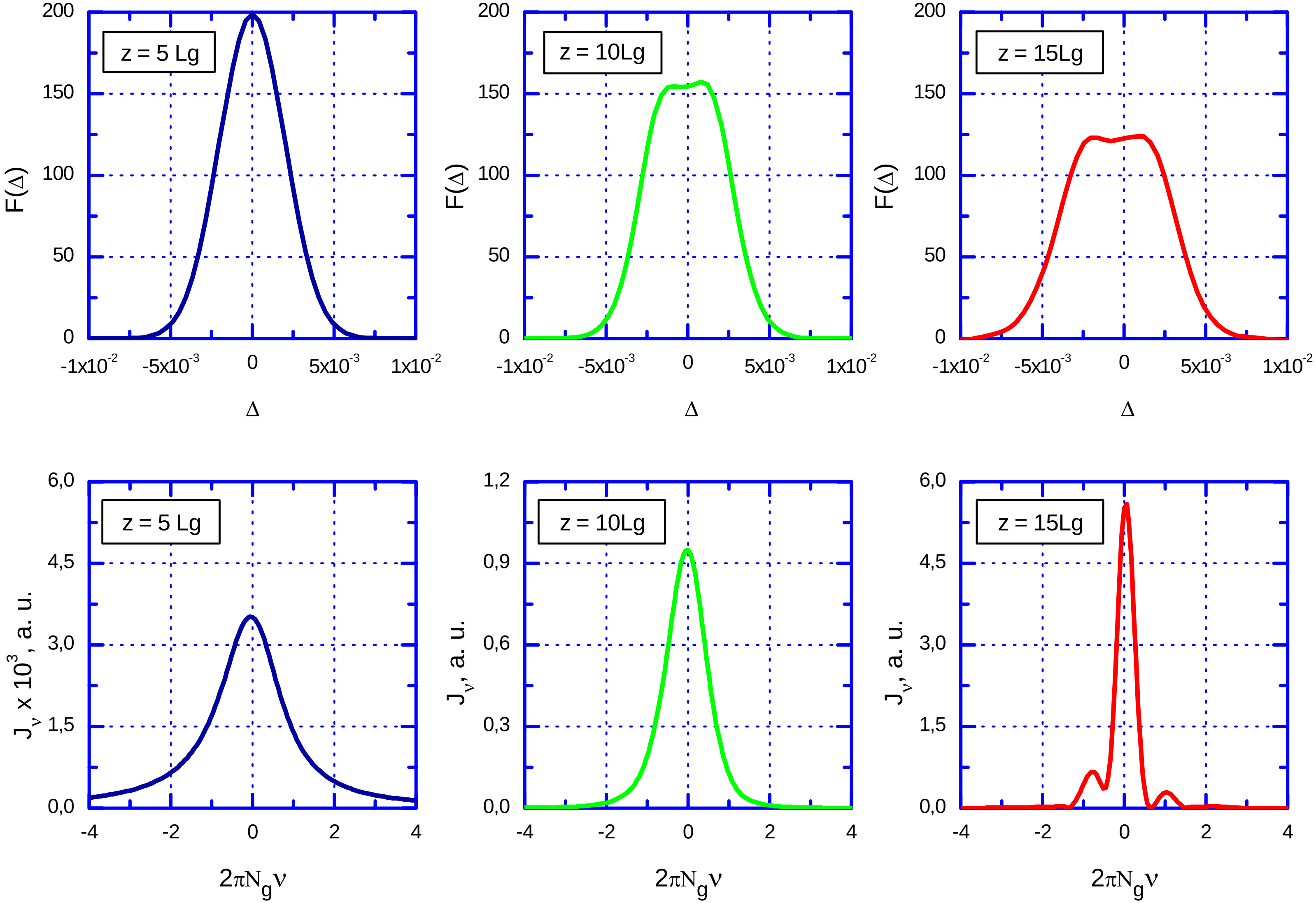
Lg – *the gain length at linear stage*

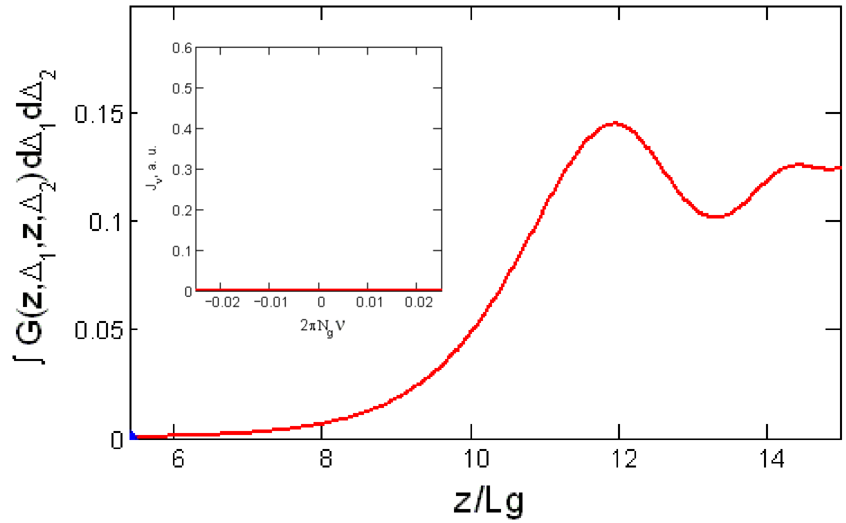
Decrease of the r.m.s. spectral bandwidth



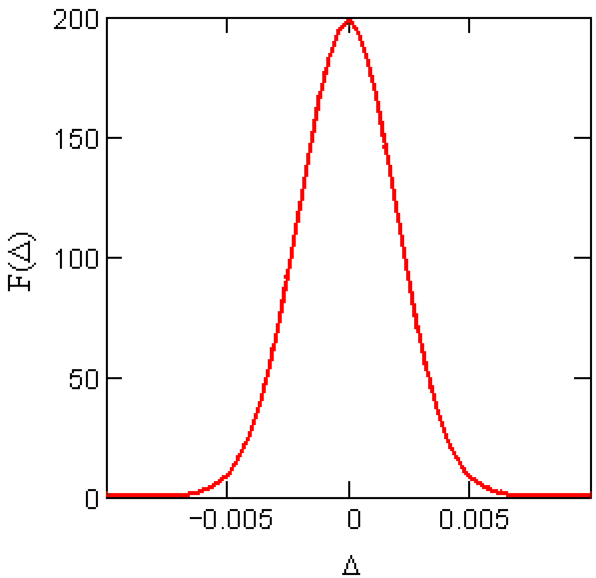
Lg – *the gain length at linear stage,* N_g – *number of undulator periods at one gain length*

Energy and spectral distributions at different points in undulator

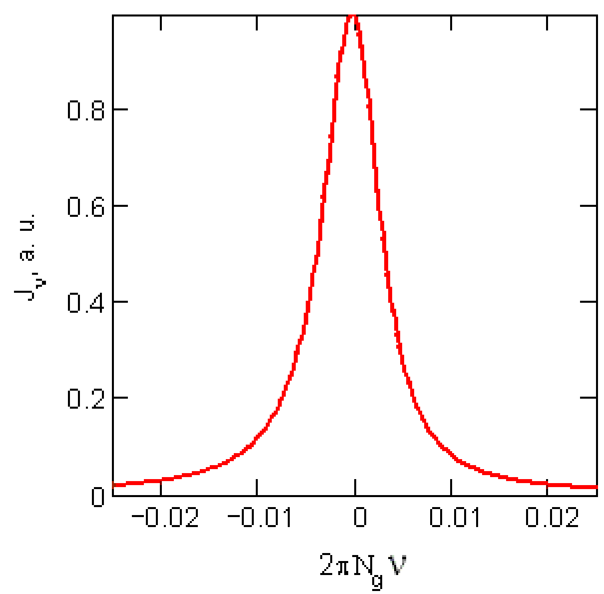




Correlation function amplitude



Energy distribution



Normalized spectral distribution

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Conclusion

- ✓ *We developed the description for saturation in SASE FEL based on rigorous statistical approach.*

- ✓ *For the simplest case of narrow electron beam we first obtained non-trivial solution for the correlation and single particle distribution functions nonlinear behaviour.*

Thank you for your attention !

The end.