LONGITUDINAL WAKE FIELD FOR AN ELECTRON BEAM ACCELERATED THROUGH A ULTRA-HIGH FIELD GRADIENT

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Abstract

The impact of longitudinal wake fields on an electron beam accelerated through a ultra-high field gradient is discussed, based on solution of Maxwell's equations for the longitudinal field. We consider an acceleration distance much smaller than the overtaking length, as for laser-plasma devices. We give expressions for impedance and wake function that may be evaluated numerically. A limiting expression is found for a large distance of the electron beam from the accelerator compared with the overtaking length. We derive analytical solutions for a Gaussian transverse and longitudinal bunch shape. We apply our analytical asymptote by studying the feasibility of a Table-Top VUV FEL (TT-VUV-FEL) based on laser-plasma driver. Numerical estimations indicate a serious threat to the operation of this device. **A detailed report with relevant references is given in [1]**.

INTRODUCTION

In this paper we study longitudinal wake fields produced within electron beams accelerated with high-gradient fields. We assume that the acceleration distance d_a is much smaller than the the overtaking length. This is the distance travelled by the electrons as a light signal from the tail of the bunch overtakes the head of the bunch. Given a bunch of rms length σ_z, the overtaking length is $2\gamma^2\sigma_z$, and corresponds to the radiation formation length $2\gamma^2\bar{\lambda}$ calculated at $\bar{\lambda} = \sigma_z$, $\bar{\lambda} = \lambda/(2\pi)$ being the reduced radiation wavelength. When $d_a \ll 2\gamma^2\sigma_z$, electrons can be assumed to be accelerated at position z_A down the beamline. This is the case for laser-plasma devices, since acceleration in the GeV range takes place within a few millimeter, but not for conventional accelerators. The assumption $d_a \ll 2\gamma^2\sigma_z$ simplifies wake calculations. When this condition is verified, the wake generated along the part of the trajectory following the acceleration point z_A is independent of any detail of the accelerator. Thus our study is valid independently of the particle accelerator technology chosen, provided that $d_a \ll 2\gamma^2\sigma_z$. One may also have contributions to the wake generated before z_A. These depend on the physical nature of the accelerator, can be separately calculated, and will be neglected here because they do not affect the bunch in the case of a laser-plasma accelerator. We base our study on solution of Maxwell's equations for the longitudinal field. Our consideration is valid in free-space. This approach gives accurate results if the typical dimension of the vacuum pipe a is larger than $\gamma\bar{\lambda} = \gamma\sigma_z$, that is the typical transverse size related with the longitudinal electric field. Paraxial approximation can be applied to describe electromagnetic sources up to the observation point, while sources after the observation point in the beam propagation direction can be neglected. We make use of the paraxial approximation to calculate the Fourier transform of the longitudinal field produced by fixed electromagnetic sources (current and charge densities) at a certain observation plane down the beamline. This constitutes the basis of our treatment, because it allows to calculate the longitudinal impedance and the wake function. Impedance and wake function yield an analytical expression in the asymptotic limit of a large distance of the electron beam from the accelerator compared with the overtaking length, i.e. in the far-field zone for all wavelengths of interest (up to $\bar{\lambda} \sim \sigma_z$). This asymptotic limit allows simple estimations of the impact of the longitudinal wake on the electron beam energy change. We apply our theory by studying the feasibility of TT-VUV-FELs based on parameters described in [2], and finding a serious threat.

FIELD IN SPACE-FREQUENCY

Let z_A be the exit of the accelerator. After z_A, the nominal Lorentz factor of electrons is γ. Acceleration happens on a scale $d_a \ll 2\gamma^2\bar{\lambda}$. Significant emission is present for wavelengths longer than the longitudinal bunch length σ_z, up to transverse beam sizes $\sigma_\perp \lesssim \gamma\sigma_z$. For typical ultra-relativistic beams, condition $d_a \ll 2\gamma^2\bar{\lambda}$ can be read as $d_a \ll 2\gamma^2\sigma_z$. We picture the fast acceleration process as a sudden "switch-on" of both harmonics of the electromagnetic sources and of the field. When the switching distance (the acceleration distance) $d_a \ll 2\gamma^2\sigma_z$, the description of the wake generated after the switching point (the acceleration point) z_A, is independent of the nature of the switcher. As long as sources are rapidly switched-on, it does not matter whether we are considering a fast longitudinal acceleration or other switching mechanisms. We first calculate $\bar{E}_z(z_o, \vec{r}_{\perp o}, \omega)$, that is the longitudinal electric field at frequency ω from given electromagnetic sources in vacuum. It is detected at longitudinal position $z_o > z_A$ and transverse position $\vec{r}_{\perp o}$. We define the Fourier transform of a function $f(t)$ as $\bar{f}(\omega) = \int_{-\infty}^{\infty} dt\, f(t) \exp[i\omega t]$, and we follow analogous convention in the definition of the Fourier transform in two dimensions. The field in the space-frequency domain $\vec{E}(z_o, \vec{r}_{\perp o}, \omega)$ is the Fourier transform of the field in the space-time domain, $\vec{E}(z_o, \vec{r}_{\perp o}, t)$. As is well-known, one may use the field in the space-frequency domain for a given frequency and yet think of the amplitude of a monochromatic field or viceversa. We should solve the paraxial equation for sources located in $[z_A, z_o]$: $c^2 \exp[i\omega z/c] \cdot [\nabla_\perp^2 + 2i\omega/c\partial/(\partial z)]\bar{\vec{E}} = 4\pi c^2 \vec{\nabla}\bar{\rho} - 4\pi i\omega \bar{\vec{j}}$. Since electrons are moving along the z-axis, we write the harmonic components of the charge density as $\bar{\rho}(z', \vec{r'}_\perp, \omega) = \rho_o(\vec{r'}_\perp)\bar{f}(\omega)\exp[i\omega z'/v]u(z'-z_A)$, while $\bar{j}_z(z', \vec{r'}_\perp, \omega) = \beta c\bar{\rho}$. Notation $u(z'-z_A)$ indicates a Heaviside step function centered at position z_A, whose presence signifies that there are no sources before the plasma accelerator. Functions ρ_o and $\bar{f}$ are specified once a model for the beam is chosen. The quantity ρ_o has the meaning of transverse electron beam distribution. It may depend on the harmonic ω, but we will assume it does not. Thus, all information about the longitudinal electron density distribution $f(t)$ is included in its Fourier transform $\bar{f}(\omega)$. A typical Gaussian beam model is defined by $\rho_o(\vec{r'}_\perp) = 1/(2\pi\sigma_\perp^2 c)\exp[-r_\perp^{'2}/(2\sigma_\perp^2)]$, and $f(t) = (-e)N/(\sqrt{2\pi}\sigma_t)\exp[-t^2/(2\sigma_t^2)]$, whose Fourier transform is given by $\bar{f}(\omega) = (-e)N\exp[-\omega^2\sigma_t^2/2]$, N being the number of electrons in the beam and $(-e)$ the electron beam charge. Here σ_t is the *rms* bunch duration, connected σ_z by $\sigma_z = \beta c\sigma_t$, so that in terms of lengths $f(s) = (-e)N/(\sqrt{2\pi}\sigma_z)\exp[-s^2/(2\sigma_z^2)]$. The wake generated along the part of the trajectory following z_A, is independent of the nature of the switcher. However, the above expressions for $\bar{\rho}$ and $\bar{j}_z$ violate the continuity equation, and should be completed by extra-contributions depending on the switcher. For example, if one thinks of an acceleration process where a low energy bunch with Lorentz factor γ_o is accelerated on a distance d_a up to a Lorentz factor γ, one should add to $\bar{\rho}$ and $\bar{j}_z$ the contribution of the harmonic at frequency ω associated with the beam with Lorentz factor γ_o. In that way, the continuity equation can be satisfied. However, these extra-contributions depend on the switching process. Here we will not consider them. We will focus, instead, on the switch-independent part of the problem. Moreover, in the case of a plasma accelerator, the switch-dependent contributions have no effect on the longitudinal wake field acting on the electron beam. For $z_A \to -\infty$, corresponding to the steady state solution, we obtain the following expression:

$$\widetilde{E}_z = -\frac{2i\omega}{\gamma^2 c}\exp\left[\frac{i\omega z_o}{2\gamma^2 c}\right]\bar{f}(\omega) \times \int d\vec{r'}_\perp\, \rho_o(\vec{r'}_\perp) K_o\left(\frac{|\omega|\,|\vec{r}_{\perp o} - \vec{r'}_\perp|}{\gamma c}\right). \quad (1)$$

$\widetilde{E}_z$ depends on z_o through a phase factor only. When $z_A = 0$, corresponding without loss of generality to the switch-on case, we have:

$$\widetilde{E}_z(z_o, \vec{r}_{\perp o}, \omega) = -\frac{\omega\bar{f}(\omega)}{\gamma^2 c}\int d\vec{r'}_\perp \rho_o\left(\vec{r'}_\perp\right) \times\left\{i\int_0^{z_o}\frac{dz'}{(z_o - z')}\exp\left[\frac{i\omega z'}{2\gamma^2 c} + i\omega\frac{|\vec{r}_{\perp o} - \vec{r'}_\perp|^2}{2c(z_o - z')}\right] + \frac{\gamma^2 c}{\omega z_o}\exp\left[i\omega\frac{|\vec{r}_{\perp o} - \vec{r'}_\perp|^2}{2cz_o}\right]\right\}. \quad (2)$$

Eq. (1) and Eq. (2) can be used to estimate the impact of the field generated by the electron beam on any particle in the beam. We do so with the help of the concepts of wake fields and impedances. By definition of impedance $Z(\omega, z)$ we have $Z(\omega, z) = 1/|\bar{f}(\omega)|^2 \int_V \bar{j}_z^* \bar{E}_z\, dV$, where $|\bar{f}(\omega)|^{-2}$ accounts for the fact that test and source disks have total charge (-e), while $\bar{j}_z^* \propto \bar{f}^*$ and $\bar{E}_z \propto \bar{f}$. Here the volume V encloses the beam up to position z. Relation between impedance Z and wake G is given by $Z = \int_{-\infty}^{\infty} d(\Delta s)/(\beta c)\; G(\Delta s)\exp[i\omega\Delta s/(\beta c)]$. An explicit expression for $Z(\omega, z)$ is

$$Z = -\frac{\omega}{\gamma^2}\int d\vec{r'}_\perp \int d\vec{r''}_\perp \rho_o^*(\vec{r'}_\perp)\rho_o(\vec{r''}_\perp)\int_0^z dz' \left\{i\int_0^{z'}\frac{dz''}{(z'-z'')}\exp\left[\frac{i\omega}{2c}\left(\frac{z''-z'}{\gamma^2} + \frac{|\vec{r}_{\perp o} - \vec{r'}_\perp|^2}{z'-z''}\right)\right] + \frac{\gamma^2 c}{\omega z'}\exp\left[i\omega\frac{|\vec{r}_{\perp o} - \vec{r'}_\perp|^2}{2cz'}\right]\exp\left[-\frac{i\omega z'}{2\gamma^2 c}\right]\right\}. \quad (3)$$

Eq. (3) constitutes a practical algorithm to compute both real and imaginary parts of the impedance (and, subsequently, of the wake).

Since ρ_o is a real quantity, $Z(\omega)$ has the property $Z(\omega) = Z^*(-\omega)$. Then, $\mathrm{Re}[Z](\omega)$ is an even function of ω, while $\mathrm{Im}[Z](\omega)$ is odd. The property $Z(\omega) = Z^*(-\omega)$ follows, more in general, from the fact that $G(\Delta s, z)$ is a real function. If we split the wake function in the sum $G = G_S + G_A$ of a symmetric and antisymmetric part, the Fourier transform of G_S gives back $\mathrm{Re}[Z]$, while the Fourier Transform of G_A yields $\mathrm{Im}[Z]$. The symmetric part of the wake, G_S, is also known as active part and is related with the energy lost by the bunch through radiation. The antisymmetric part G_A is known as reactive part and is related with energy redistribution within the bunch.

ANALYTICAL ASYMPTOTES

Impedance

With the help of Eq. (3) we calculate asymptotes of the real and imaginary parts of the impedance for $z \gg 2\gamma^2\lambda\!\!\!^{-}$ (see [1]). The real part is:

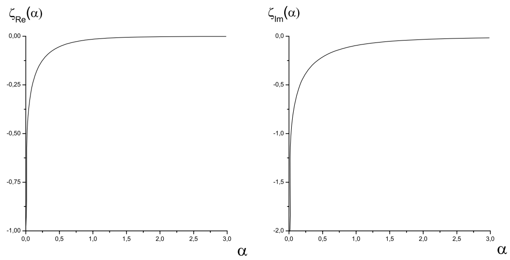

Figure 1: Plot of $\zeta_{\rm Re}$ (left) and plot of $\zeta_{\rm Im}$ (right).

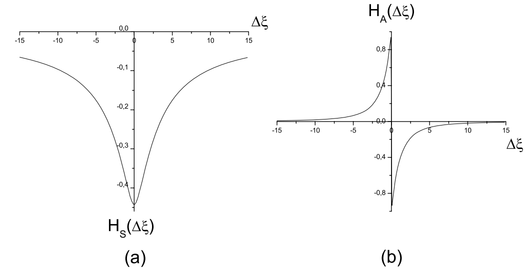

Figure 2: (a) Plot of H_S and (b) plot of H_A.

$$\mathrm{Re}[Z](\omega) = -\frac{c}{\pi}\int d\vec{\theta}\left|\tilde{\rho}_o(\vec{\theta},\omega)\right|^2 \frac{\gamma^4\theta^2}{(1+\gamma^2\theta^2)^2}\ . \qquad (4)$$

We now choose a Gaussian model for ρ_o, substituting $\tilde{\rho}_o(\vec{\theta},\omega) = 1/c\cdot\exp[-\theta^2\omega^2\sigma_\perp^2/(2c^2)]$ in Eq. (4). This gives $\mathrm{Re}[Z] = Z_o\zeta_{\rm Re}(\alpha)$, where

$$\zeta_{\rm Re}(\alpha) = \frac{1}{4\pi}\left\{1-(1+\alpha^2)\exp[\alpha^2]\Gamma(0,\alpha^2)\right\}\ , \qquad (5)$$

$\Gamma(0,x)$ being the incomplete gamma function, $Z_o = 4\pi/c$ the free-space impedance and $\alpha = \omega\sigma_\perp/(\gamma c)$. A plot of the universal function $\zeta_{\rm Re}(\alpha)$ is given in Fig. 1 (left plot). $\mathrm{Re}[Z]$ exhibits a singular behavior for $\sigma_\perp \longrightarrow 0$ (i.e. for $\alpha \longrightarrow 0$ in Fig. 1 (left plot)). This is linked with our particular model, where we chose $d_a \longrightarrow 0$. The imaginary part is:

$$\mathrm{Im}[Z](\omega) = -\frac{1}{\pi}\omega z\int d\vec{\theta}\left|\tilde{\rho}_o(\vec{\theta},\omega)\right|^2 \frac{1}{(1+\gamma^2\theta^2)}\ . \qquad (6)$$

Similarly as for the real part, use of a Gaussian beam model allows to give an explicit expression for $\mathrm{Im}[Z]$. Namely, $\mathrm{Im}[Z] = (Z_o\hat{z})\,\zeta_{\rm Im}(\alpha)$, where

$$\zeta_{\rm Im}(\alpha) = -\frac{1}{2\pi}\exp[\alpha^2]\,\Gamma(0,\alpha^2)\ , \qquad (7)$$

where and $\hat{z} = z/(2\gamma^2\lambda\!\!\!^{-})$. A plot of the universal function $\zeta_{\rm Im}(\alpha)$ is given in Fig. 1 (right plot).

Wake function

Analytical results given for $z \gg \gamma^2\lambda\!\!\!^{-}$. $G_S(\Delta s)$ can be found calculating the inverse Fourier transform of $\mathrm{Re}[Z](\omega)$ with respect to $\Delta s/\beta c$. Using notation $\Delta\xi = \gamma(\Delta s)/\sigma_\perp$, one obtains $G_S(\Delta\xi) = \gamma/\sigma_\perp\cdot H_S(\Delta\xi)$ where $H_S(\Delta\xi) = -1/(4\sqrt{\pi})\{\sqrt{\pi}\ |\Delta\xi| + \pi[1-(\Delta\xi)^2/2]\exp[(\Delta\xi)^2/4]\cdot\mathrm{erfc}[|\Delta\xi|/2]\}$. A plot of the universal function H_S as a function of $\Delta\xi$ is given in the left plot of Fig. 2 (a). The energy gained or lost by a single particle at position s within the bunch due to the active (symmetric) part of the wake, averaged over transverse coordinates is given by the convolution $\Delta\mathcal{E}_S(s) = (-e)\int_{-\infty}^{\infty} G_S(\Delta s)f(s-\Delta s)d(\Delta s)$. An explicit expression for $\Delta\mathcal{E}_S/\mathcal{E}_o$ as a function of $\xi = \gamma s/\sigma_\perp$, $\mathcal{E}_o = \gamma m_e c^2$ being the nominal energy of an electron (with rest mass m_e) is

$$\frac{\Delta\mathcal{E}_S}{\mathcal{E}_o} = \frac{I_{\rm max}}{\gamma I_A}\int_{-\infty}^{\infty} d(\Delta\xi)H_S(\xi-\Delta\xi)\exp\left[-\frac{(\Delta\xi)^2}{2\eta^2}\right]\ , \qquad (8)$$

where we introduced the new parameter $\eta = \gamma\sigma_z/\sigma_\perp$, and we called the Alfven current $I_A = e/(mc^3) \simeq 17$ kA and the beam current $I_{\rm max} = eNc/(\sqrt{2\pi}\,\sigma_z)$.

G_A can be found calculating the inverse Fourier transform of $i\mathrm{Im}[Z](\omega)$ with respect to $\Delta s/\beta c$. One obtains $G_A(\Delta\xi) = \gamma\eta\hat{z}/\sigma_\perp\cdot H_A(\Delta\xi)$, where $H_A(\Delta\xi) = -\Delta\xi/(2\sqrt{\pi})\ \{2\sqrt{\pi}/|\Delta\xi| - \pi\exp[(\Delta\xi)^2/4]\cdot\mathrm{erfc}[|\Delta\xi|/2]\}$ and we redefined $\hat{z} = z/(2\gamma^2\sigma_z)$. A plot of the universal function H_A is reported in Fig. 2 (b). The energy gained or lost by a single particle at position s within the bunch due to the reactive part of the wake (averaged over transverse coordinates) is given by the convolution $\Delta\mathcal{E}_A(s) = (-e)\int_{-\infty}^{\infty} G_A(\Delta s)f(s-\Delta s)d(\Delta s)$. $\Delta\mathcal{E}_A/\mathcal{E}_o$ as a function of $\xi = \gamma s/\sigma_\perp$ is given by :

$$\frac{\Delta\mathcal{E}_A}{\mathcal{E}_o}\left(\frac{s}{\sigma_z};\eta\right) = \frac{I_{\rm max}\hat{z}}{\gamma I_A}F\left(\frac{s}{\sigma_z};\eta\right)\ . \qquad (9)$$

where $\mathcal{E}_o = \gamma m_e c^2$, and

$$F\left(\frac{s}{\sigma_z};\eta\right) = \int_{-\infty}^{\infty} d(\Delta\xi)\ \eta\ H_A\left(\eta\frac{s}{\sigma_z}-\Delta\xi\right)\exp\left[-\frac{(\Delta\xi)^2}{2\eta^2}\right]\ . \qquad (10)$$

A plot of F is given as a function of s/σ_z in Fig. 3 for different values of η.

Plots for $\Delta\mathcal{E}_S/\mathcal{E}_o$ and $\Delta\mathcal{E}_A/\mathcal{E}_o$ as a function of ξ can be computed with the help of numerical techniques.

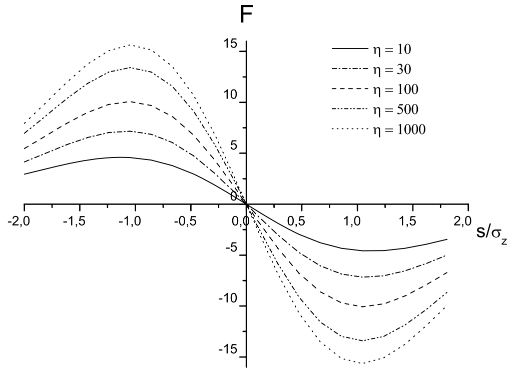

Figure 3: Plot of F at different η.

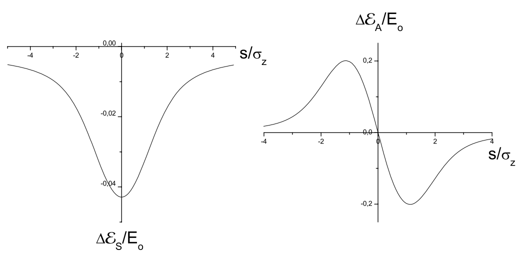

Figure 4: $\Delta\mathcal{E}_s/\mathcal{E}_o$ (left) and $\Delta\mathcal{E}_A/\mathcal{E}_o$ (right) for the TT-VUV-FEL at $z = 0.8$ m.

IMPACT ON TT-VUV-FEL

One of the applications envisaged for next-generation plasma accelerators is as drivers for table-top FELs [2]. Here we will apply our theory to the VUV case in Table I of reference [2]. The undulator parameter K is smaller than unity. Thus, wake field calculations in vacuum can still be used inside the undulator. In the TT-VUV-FEL case $2\gamma^2\lambdabar \simeq 18$ cm, several times smaller than the size of the planned undulator, that is about 0.8 m. Therefore we are well-within the applicability region of the our asymptotic expressions. Needed parameters are $\gamma = 300$, $\eta = 10$ and $I = 50$ kA. Also, we need to fix a position along the longitudinal axis. Here we set $z \simeq 0.8$ m, corresponding to the foreseen undulator length. The relative energy change $\Delta\mathcal{E}_{S,A}/\mathcal{E}_o$ are presented respectively on the left and on the right plot in Fig. 4 as a function of the longitudinal coordinate inside the bunch. While travelling down the undulator, a correlated energy change develops along the electron beam. For estimations of the effect on the FEL gain we can use the linear energy chirp parameter $\hat{\alpha} = -(\gamma\omega\rho_{1D}^2)^{-1} \cdot d\gamma/dt$. The effect of linear energy chirp starts to play a significant role on the FEL gain when $\hat{\alpha} \gtrsim 1$. One finds $\hat{\alpha} \simeq 10$, that indicates a very large effect. Note that the energy chirp in Fig. 4 (right plot) depends on the electron beam profile but also on time, because it develops along the undulator. This effect is fundamental, and is directly linked with the feasibility of the proposed FEL scheme. Radiation generated in one part of the undulator cannot interact in resonance in another part of the undulator, and the amplification process is destroyed.

CONCLUSIONS

We studied the impact of longitudinal wake fields on the quality of electron beams produced with high field-gradient accelerators. Our consideration does not depend on the particular realization of the accelerator. However, given present technological developments, one of the most relevant applications of our study should come from the realm of laser-plasma accelerators. We calculated longitudinally symmetric and anti-symmetric parts of the wake function as well as real and imaginary part of the impedance with the help of paraxial approximation within a space-frequency domain formulation of Maxwell's equation. While the general expressions for wakes and impedances need numerical techniques to be evaluated, it is possible to present analytical expressions for the asymptotic limit when the electron bunch has reached a position, down the beamline, that is far from the acceleration point with respect to the overtaking distance. Then, the wake and the impedance are proportional to universal functions. Our results can be used as analytical benchmarks for computer codes. Taking advantage of similarity techniques we presented such result in terms of a complex dependent on the problem parameters $(\gamma\eta\hat{z}/\sigma_\perp)$ multiplied by a universal function H_A. Our expression for G_A can further be used to calculate the relative energy change of a particle within a given bunch. In case of a Gaussian longitudinal profile the relative energy spread reduces to Eq. (9). Plots presented in Fig. 3 will help the reader to perform a fast estimation of the influence of the space-charge wake at the stage of planning of experiments. As a particular example, we used our results to estimate the impact of the longitudinal wake fields on the energy change of the electron beam for the TT-VUV-FEL setup in [2]. The total energy deviation is found to be critical.

For details on our work we refer the interested reader to [1].

REFERENCES

[1] G. Geloni, E. Saldin, E. Schneidmiller and M. Yurkov, Nucl. Instrum. and Methods in Phys. Research A 578 (2007) 3446, preprint version DESY 06-222 at http://arxiv.org/abs/physics/0612077

[2] F. Gruener et al., Appl. Phys. B 86, 431 (2007)