

SPACE CHARGE EFFECT IN AN ACCELERATED BEAM *

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Abstract

It is usually assumed that the space charge effects in relativistic beams scale with the energy of the beam as γ^{-2} , where γ is the relativistic factor. We show that for a beam accelerated in the longitudinal direction there is an additional space charge effect in free space that scales as E/γ , where E is the accelerating field. This field has the same origin as the “electromagnetic mass of the electron” discussed in textbooks on electrodynamics. We then consider the effect of this field on a beam generated in an RF gun and calculate the energy spread produced by this field in the beam.

INTRODUCTION

Modern light sources such as free electron lasers and energy recovery linacs require high-peak current, small-emittance beams. One of the important characteristics of such a beam is its energy spread. It determines the limits of a possible bunch compression, the stability against microbunching, and properties of the beam as a radiator of photons. There are several mechanisms that contribute to the energy spread in radio frequency electron guns with the dominant one, for nanocoulomb bunches, being the space charge effect.

Traditionally in accelerator physics the space charge effect is computed as a self field of a beam moving with constant velocity along a straight line. The longitudinal field in such a beam causes the energy exchange between the particles; it typically scales with the beam energy as γ^{-2} [1, p. 128], where γ is the relativistic factor, and usually becomes small for highly relativistic beams. In a broader sense, the space charge effect might be understood as a self field of the beam, even when it moves with acceleration. With this understanding, acceleration adds to the beam self field. One such contribution, that attracted a lot of attention lately, is due to the coherent radiation of the beam and is called the coherent synchrotron radiation wake (or CSR wake) [2]. The CSR wake is the radiation reaction force that keeps balance between the electromagnetic energy that is carried away by the radiation and the kinetic energy of the beam particles. It occurs when the beam is being accelerated in the direction perpendicular to the beam velocity in bending magnets or undulators.

Another type of radiation reaction force has been considered in recent papers [3, 4]—it is a self field that arises inside the beam during a violent longitudinal acceleration. Such a field can play a role in plasma acceleration experi-

ments, where the pace of acceleration is much larger than in a conventional RF cavities.

In this paper we point out to a new component of the space charge field that arises during longitudinal acceleration of the beam. We assume that the acceleration is not strong enough to cause a noticeable radiation. What it does, however, it changes the velocity of the beam. Because the beam electromagnetic field depends on how fast it moves, the electromagnetic energy of the beam field changes during acceleration. Similar to the case of a converging beam, one should expect an additional component of the self field that keeps balance between the beam and the field energy. We call this field the *acceleration field*. Being proportional to the acceleration, on average, it is equivalent to a renormalization of the mass; it is discussed in textbooks on electrodynamics in connection with a so called electromagnetic “mass” of a point charge [5, 6]. In this paper we are interested in the spatial distribution of the field and, more specifically, the energy spread in the beam induced by the acceleration field.

The model that we consider in this paper assumes that the beam does not change its shape during the acceleration. We neglect a component of self field that is associated with the converging (or diverging) beams (see [7] and references therein).

ENERGY OF ELECTROMAGNETIC FIELD OF A MOVING GAUSSIAN BUNCH

Consider a Gaussian bunch of charged particles moving with velocity v in the z direction with the particle distribution function given by

$$n(x, y, \zeta) = \frac{N}{(2\pi)^{3/2}\sigma_z\sigma_\perp^2} \exp\left(-\frac{r^2}{2\sigma_\perp^2} - \frac{\zeta^2}{2\sigma_z^2}\right), \quad (1)$$

where N is the number of particles in the bunch, $r = \sqrt{x^2 + y^2}$, $\zeta = z - vt$, $\sigma_x = \sigma_y = \sigma_\perp$ is the rms bunch size in the transverse direction, and σ_z is the rms bunch length in the longitudinal direction. The electromagnetic field of such a bunch can be calculated using the Lorentz transformation from the beam frame, see, e.g., [8]. The integrated electromagnetic energy over the whole space

$$W = \int \frac{E^2 + H^2}{8\pi} dV, \quad (2)$$

grows with γ , as shown in Fig. 1. Note that this energy tends to infinity when $\gamma \rightarrow \infty$. As a detailed analysis shows, at $\gamma \gg 1$ the asymptotic expression for W is

$$W = \frac{Q^2}{\sqrt{\pi}\sigma_z} \log \gamma, \quad \gamma \gg 1. \quad (3)$$

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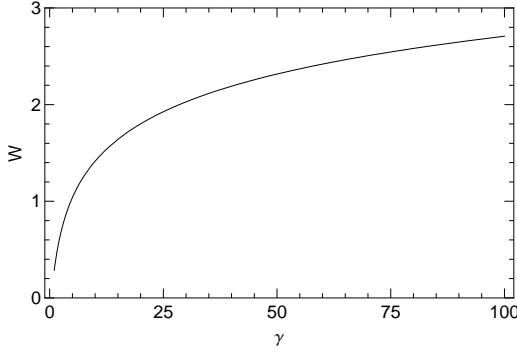


Figure 1: Integrated electromagnetic energy density in units Q^2/σ for a spherical Gaussian bunch ($\sigma_\perp = \sigma_z = \sigma$) as a function of γ .

Imagine now that the beam is being accelerated from rest to velocity v corresponding to some value of γ . The increased electromagnetic energy is taken from the kinetic energy of the beam via a longitudinal electric field E_z induced by the acceleration. Such a field is known from the theory of the radiation reaction force [6, p. 386], where it is responsible for the electromagnetic field contribution to the mass of a charged particle. This force is linear in acceleration, and changes sign when the acceleration is reversed. It is not related to the radiation; in addition to transferring energy from the beam to the electromagnetic field during acceleration (and transferring it in the opposite direction during deceleration) it introduced an energy spread in the beam. In the next section we derive an expression for E_z using the retarded potentials.

SPACE CHARGE AND ACCELERATION FIELDS

Consider a beam moving along the z axis with velocity $v(t)$ that varies with time and is the same for all particles of the beam. If $n(x, y, z)$ is the particle density at initial time, then at time t the charge density ρ and the current density j_z are: $\rho = en(x, y, z - z_0(t))$, $j_z = ev(t)n(x, y, z - z_0(t))$, with $v = dz_0/dt$, and e the elementary charge. The scalar and vector potentials of the beam are given by the following equations [5]

$$\begin{aligned}\phi(\mathbf{r}, t) &= \int \frac{\rho(\mathbf{r}', t - \tau)}{|\mathbf{r} - \mathbf{r}'|} d^3r', \\ \mathbf{A}(\mathbf{r}, t) &= \frac{1}{c} \int \frac{\mathbf{j}(\mathbf{r}', t - \tau)}{|\mathbf{r} - \mathbf{r}'|} d^3r',\end{aligned}\quad (4)$$

where the retarded time $t - \tau$ is defined by $c\tau = |\mathbf{r} - \mathbf{r}'|$. We will assume that the acceleration $a(t) = dv/dt$ is small and expand the potentials in Taylor series keeping only linear terms in acceleration. We have approximately

$$\begin{aligned}z_0(t - \tau) &\approx z_0(t) - v(t)\tau + \frac{1}{2}a(t)\tau^2 \\ v(t - \tau) &\approx v(t) - a(t)\tau.\end{aligned}\quad (5)$$

This gives for the scalar potential

$$\begin{aligned}\phi(\mathbf{r}, t) &= e \int \frac{d^3r'}{|\mathbf{r} - \mathbf{r}'|} \\ &\times n\left(x', y', z' - z_0(t) + v(t)\tau - \frac{1}{2}a(t)\tau^2\right).\end{aligned}\quad (6)$$

We now expand the function n to obtain $\phi \approx \phi_{sc} + \tilde{\phi}$, where ϕ_{sc} is the space charge potential that does not depend on acceleration, and $\tilde{\phi}$ is a part of the potential proportional to the acceleration:

$$\begin{aligned}\phi_{sc}(\mathbf{r}, t) &= e \int \frac{n(x', y', z' - z_0(t) + v(t)\tau)}{|\mathbf{r} - \mathbf{r}'|} d^3r', \\ \tilde{\phi}(\mathbf{r}, t) &= -\frac{e}{2}a(t) \int \frac{\tau^2}{|\mathbf{r} - \mathbf{r}'|} d^3r' \\ &\times \partial_z n(x', y', z' - z_0(t) + v(t)\tau),\end{aligned}\quad (7)$$

with $\partial_z n = \partial n(x, y, z)/\partial z$. Similarly, we expand the vector potential \mathbf{A} which has the z component only,

$$\begin{aligned}A_z(\mathbf{r}, t) &= \frac{e}{c} \int \frac{(v(t) - a(t)\tau)}{|\mathbf{r} - \mathbf{r}'|} d^3r' \\ &\times n\left(x', y', z' - z_0(t) + v(t)\tau - \frac{1}{2}a(t)\tau^2\right),\end{aligned}\quad (8)$$

to obtain $\mathbf{A} \approx \mathbf{A}_{sc} + \tilde{\mathbf{A}}$, with

$$\begin{aligned}A_{z,sc}(\mathbf{r}, t) &= \frac{e}{c} \int \frac{v(t)n(x', y', z' - z_0(t) + v(t)\tau)}{|\mathbf{r} - \mathbf{r}'|} d^3r' \\ \tilde{A}_z(\mathbf{r}, t) &= -\frac{e}{2c^3}a(t)v(t) \int \frac{|\mathbf{r} - \mathbf{r}'|}{|\mathbf{r} - \mathbf{r}'|^3} d^3r' \\ &\times \partial_z n(x', y', z' - z_0(t) + v(t)\tau) \\ &- \frac{ea(t)}{c^2} \int n(x', y', z' - z_0(t) + v(t)\tau) d^3r'.\end{aligned}\quad (9)$$

One can formulate conditions of applicability of the approximations used above by requiring that the terms discarded in the Taylor expansions are small compared to those left. There are two such conditions

$$a \ll \frac{c^2}{l}, \quad \left| \frac{\dot{a}}{a} \right| \ll \frac{c}{l},\quad (10)$$

where l is the characteristic size of the bunch. These conditions mean that the acceleration is not large and does not change fast.

The electric field of the beam is a sum of the space charge field and a component that vanishes in the limit when $a = 0$, $\mathbf{E} \approx \mathbf{E}_{sc} + \tilde{\mathbf{E}}$, where $\mathbf{E}_{sc} = -\nabla\phi_{sc} - c^{-1}\partial\mathbf{A}_{sc}/\partial t$ and $\tilde{\mathbf{E}} = -\nabla\tilde{\phi} - c^{-1}\partial\tilde{\mathbf{A}}/\partial t$. The electric field \mathbf{E}_{sc} (and related to it the magnetic field $\mathbf{B} = \mathbf{v} \times \mathbf{E}_{sc}/c$) is traditionally associated in accelerator physics with the the space charge effect. The total energy of this field is plotted in Fig. 1.

In this paper, we are interested in the electric field $\tilde{\mathbf{E}}$, and more specifically in the longitudinal component \tilde{E}_z

$$\tilde{E}_z = -\frac{\partial\tilde{\phi}}{\partial z} - \frac{1}{c} \frac{\partial\tilde{A}_z}{\partial t},\quad (11)$$

that changes the kinetic energy of the beam particles. Using Eqs. (7) and (8) for calculation of \tilde{E}_z we find that in addition to terms proportional to a it also contains terms that involve \dot{a} and a^2 . We discard the latter as being small because of the conditions (10). This approximation gives the following expression for \tilde{E}_z ,

$$\begin{aligned} \tilde{E}_z = & -\frac{e}{c^2}a \int \frac{n(x', y', z' + v\tau)}{|\mathbf{r} - \mathbf{r}'|} d^3r' \quad (12) \\ & -\frac{e}{2c^2}a\beta \int (\beta|\mathbf{r} - \mathbf{r}'| - (z - z')) d^3r' \\ & \times \partial_{zz}n(x', y', z' + v\tau) \\ & +\frac{e}{2c^2}a \int \left(\frac{z - z'}{|\mathbf{r} - \mathbf{r}'|} - 4\beta \right) \partial_z n(x', y', z' + v\tau) d^3r'. \end{aligned}$$

In this equation we set $t = z_0 = 0$ (which means that the density distribution of the beam at the observation time is now $n(x, y, z)$), and suppressed the argument t in a and v . In what follows, for brevity, we call \tilde{E}_z the *acceleration field*.

ACCELERATION FIELD FOR A GAUSSIAN BUNCH

For a Gaussian bunch with the charge distribution function given by Eq. (1) the calculation of the acceleration field can be reduced to a one dimensional integral. The expression for this field is derived in Ref. [9] and is given by the following equations

$$\tilde{E}_z(x, y, z) = -\frac{eNa}{\sqrt{2\pi}c^2\sigma_z}G\left(\frac{r}{\sigma_\perp}, \frac{z}{\sigma_z}\right), \quad (13)$$

with

$$\begin{aligned} G(R, Z) = & \int_0^\infty dv F(v, Z) \quad (14) \\ & \times \exp\left[-\frac{R^2 A^2}{2(A^2 + v\gamma^2)} - \frac{Z^2}{2(v+1)}\right], \end{aligned}$$

where

$$\begin{aligned} F(v, Z) = & \frac{\gamma^2}{2(v+1)^{9/2}(A^2 + \gamma^2 v)} \times \quad (15) \\ & [v(5Z^2 + (2(v-2)(v+1)Z^2 + v(v+3)^2 - vZ^4)\gamma^2) \\ & + v(4\gamma^2 + v(Z^4 - (v-4)Z^2 - v) + 3)]. \end{aligned}$$

Using the relation between the acceleration and the rate of change of the gamma factor, $a = (c/\gamma^3\beta)d\gamma/dt$, we can write the energy change of a particle in the beam due to the acceleration field as

$$\Delta\mathcal{E}(r, z) = \int ve\tilde{E}_z dt = -\frac{I_0}{I_A}mc^2 \int_{\gamma_i}^{\gamma_f} \frac{d\gamma}{\gamma^3} G, \quad (16)$$

where $I_0 = Nec/\sqrt{2\pi}\sigma_z$ is the peak current in the bunch, $I_A = mc^3/e$ is the Alfvén current, γ_i and γ_f are the initial and final values of the gamma factor, and the function G is given by Eq. (14).

FEL Theory

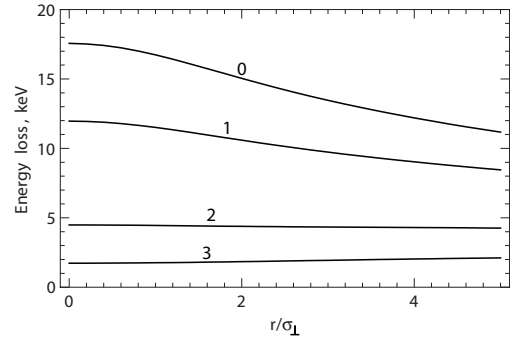


Figure 2: Energy loss induced by the acceleration field for four different slices in the bunch ($z/\sigma_z = 0, 1, 2$, and 3 ; this number is indicated near the curves) as a function of electron radial position.

In the ultrarelativistic limit $\gamma \gg 1$, one can find from Eqs. (14) and (15) that $G \approx 2\gamma^2 e^{-z^2/2\sigma_z^2}$. The longitudinal acceleration field in this limit does not depend on radius. Taking into account that in the external field E_{ext} the acceleration of a relativistic particle is $a = eE_{\text{ext}}/m\gamma^3$ we arrive at the following expression for \tilde{E}_z

$$\tilde{E}_z = -\sqrt{\frac{2}{\pi}} \frac{r_e N}{\sigma_z \gamma} E_{\text{ext}} e^{-z^2/2\sigma_z^2}, \quad \gamma \gg 1, \quad (17)$$

with $r_e = e^2/mc^2$. We see that the acceleration field is directed against the external field E_{ext} and scales as E_{ext}/γ . This contrasts to the usual scaling $\propto \gamma^{-2}$ of the longitudinal space charge forces. Note that due to the scaling $G \propto \gamma^2$ the integral (16) diverges logarithmically when $\gamma_f \rightarrow \infty$. This is related to the fact that the electromagnetic energy of the bunch logarithmically tends to infinity when $\gamma \rightarrow \infty$, as indicated by Eq. (3). In reality, the beam is being accelerated inside a vacuum volume that has a characteristic transverse size b . The electromagnetic energy of a relativistic beam propagating in a pipe of radius b does not change with γ when $\gamma \gtrsim b/\sigma_z$. To take into account this shielding effect of the metallic pipe, for rough estimates, we will assume that $\gamma_f = b/\sigma_z$. Note that typically $b/\sigma_z \gg 1$, and because our result has only a logarithmic dependence on γ , it is rather insensitive to the exact value of γ_f .

For a numerical example we consider now parameters of the LCLS rf-gun beam. Because our model assumes Gaussian distribution and the LCLS beam has a flat longitudinal profile, we choose the model parameters in such a way that σ_x and σ_z are equal to the corresponding rms values for the LCLS beam. We have $\sigma_z = 0.86$ mm, $\sigma_x = 0.6$ mm, and $Q = 0.72$ nC (corresponding to the peak current of $I_0 = 100$ A). We also choose $\gamma_i = 1$ and $\gamma_f = 20$, corresponding to the beam pipe radius of about 1.2 cm. Using Eq. (16) we calculated the energy loss of each particle in the beam. Note that for the nominal LCLS rf gun accelerating field of 120 MV/m the applicability conditions (10) are reasonably well satisfied. The plot of the radial dependence of the function $\Delta\mathcal{E}(r, z)$ for several values of z is shown in Fig. 2. The energy loss for the same beam aver-

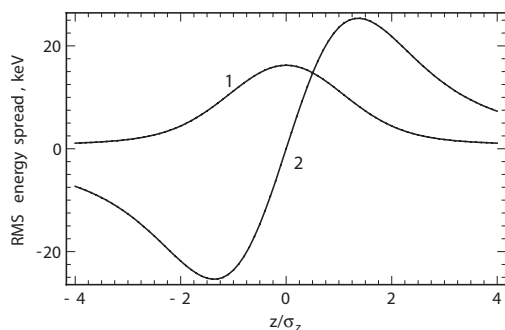


Figure 3: Energy loss induced by the acceleration field (averaged over transverse dimensions) as a function of electron longitudinal position, curve 1. For comparison, we also show the energy loss introduced by the space charge effect, curve 2.

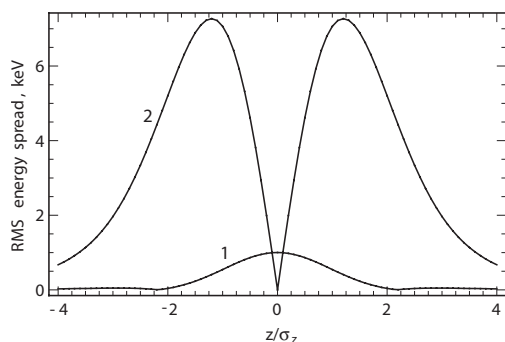


Figure 4: RMS energy spread induced by the acceleration field as a function of electron longitudinal position, curve 1. For comparison, we also show the energy spread introduced by the space charge effect, curve 2.

aged over the transverse coordinate is shown in Fig. 3 as a function of the position z . Finally, Fig. 4 shows the rms energy spread in slices as a function of z . In Figs. 2 and 3 we also show the energy loss and the rms energy spread introduced by the space charge forces (due to the longitudinal component of the field \vec{E}_{sc}). The space charge effects were calculated using the theory of Ref. [10].

DISCUSSION

As was mentioned in the introduction, the acceleration field keeps balance between the electromagnetic energy of the beam and the kinetic energy of the particles. Mathematically, this property is formulated as the equality between the rate of change of the electromagnetic energy W and the work of the field \vec{E}_z on the moving particles

$$\frac{dW}{dt} = -e \int \tilde{E}_z v n d^3r. \quad (18)$$

Although we do not prove this statement here (see [9]), we demonstrate its validity with the numerical example from the previous Section. We calculated the electromagnetic energy difference for the beam at the final state with $\gamma_f = 20$ and the initial state with $\gamma_i = 1$, which gave us $\Delta W =$

$22.5 \mu\text{J}$. When we integrated the right side of Eq. (18) over time from the initial to the final state, we found that the work of the acceleration numerically is equal to ΔW , in perfect agreement with the energy balance equation.

Computation of beam self-fields is a critical aspect in numerical simulations of high-brightness electron beam generation. Many simulation codes (ASTRA, IMPACT, PARMELA) employ the quasi-static approximation and compute only the space charge fields. There exist a few codes that calculate the beam fields from the exact solutions of Maxwell's equations (i.e., the retarded potentials). A review of different computational approaches can be found in Ref. [11]. In this paper, starting with the retarded potentials and separating explicitly the self-fields into space charge and acceleration ones, we give an analytical expression for the longitudinal component of the acceleration field. Using a Gaussian bunch model and typical (LCLS) RF gun parameters, we calculate the energy spread introduced by the acceleration field and show that it gives rise to a small correction of the energy spread introduced by the space charge field. These results may be useful in guiding the simulation studies of high-brightness beams.

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