NUMERICAL SOLUTION OF THE FEL CORRELATION FUNCTION EQUATION

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Abstract

The equation for two-particle correlation function in FEL was derived recently to provide a new way of noise calculations in SASE FELs [1]. In this paper this equation is solved numerically for the simplest case of narrow electron beam. Time independent solution with saturation is obtained. It is compared with the results of quasilinear theory and results of previous SASE linewidth estimates.

INTRODUCTION

High gain FELs operated in SASE mode are considered now as one of the most perspective high-brightness radiation source in the x-ray region. Therefore it is very important to know the radiation properties of such FELs. Parameters of radiation in a single shot are determined by the shot noise in the beam current which has stochastic nature. Because of that these parameters fluctuate from shot to shot and they can not be determined without exact solution of the particle motion and Maxwell equations. On the other hand the parameters averaged over many shots can be found by the methods of statistical mechanics.

The statistical approach has been treated by many authors but usually it was limited to the linear case when one can introduce the Green function and the averaging becomes straightforward [2,3]. Some authors considered the averaged results of simulations obtained by macroparticle based codes [4]. But in this case it is not evident that artificially constructed initial distribution of macroparticles leads to correct results at saturation stage.

The regular nonlinear approach to the start-up from noise has been proposed in [1]. It is based on the BBGKY set of equations which is truncated to two equations for single-particle distribution function and two-particle correlation function. In this paper we obtain the numerical solution of these equations for the simplest model of narrow beam comprised of charged disks with Gaussian transversal charge distribution.

BASIC EQUATIONS

In the case of the charged disks model the equations (4-5) of [1] for the single-particle distribution function and two-particle correlation function have the following form:

$$\left(\frac{\partial}{\partial\theta}+\nu_1\frac{\partial}{\partial z_1}\right)F(1,\theta)=-N\int d\{2\}\Phi(1,2)\frac{\partial}{\partial\Delta_1}G(1,2;\theta) \qquad (1)$$

$$\left(\frac{\partial}{\partial\theta}+\nu_1\frac{\partial}{\partial z_1}+\nu_2\frac{\partial}{\partial z_2}\right)G(1,2,\theta)=-N\frac{\partial F(1)}{\partial\Delta_1}\int\Phi(1,3)G(2,3,\theta)d\{3\}-$$
$$-N\frac{\partial F(2)}{\partial\Delta_2}\int\Phi(2,3)G(1,3,\theta)d\{3\}-\left(\Phi(1,2)\frac{\partial}{\partial\Delta_1}+\Phi(2,1)\frac{\partial}{\partial\Delta_2}\right)F(1)F(2) \qquad (2)$$

where $(i)=(z_i,\Delta_i)$, $d\{i\}=dz_i d\Delta_i$, $\nu_i=(1+2\Delta_i)$, Δ_i - relative electron energy deviation, z_i - electron longitudinal coordinate in undulator, N - number of electrons in the beam, $\theta=2\gamma_\parallel^2(t-z)$, $(c=1)$ - "time" variable and $\gamma_\parallel$ - relativistic factor of electron longitudinal motion. The longitudinal interaction force $\Phi(1,2)$ can be determined from eq. (6) of [1]. In the considered model it should be averaged over transversal distribution:

$$\langle\Phi(1,2)\rangle_\perp=-\frac{r_e}{2\sigma^2k_w\gamma}\frac{K^2}{1+K^2}\left(\frac{e^{ik_w(z_1-z_2)}}{1+i\alpha k_w(z_1-z_2)}+c.c.\right) \qquad (3)$$

here σ is r.m.s transversal beam size, r_e - classical electron radius, $\alpha=1/2k_0k_w\sigma^2$ - small dimensionless parameter which characterizes the beam "thickness", k_0 - radiation wave number and k_w - undulator wave number. We assume that undulator has constant deflection parameter K and helical symmetry.

From this point we shell consider stationary case, therefore the time derivative in Eq. (1-2) can be omitted and N should be replaced by the number of electrons per unit of length. In this case the single-particle distribution function has to be renormalized the following way:

$$\int\nu_1F(z_1,\Delta_1)d\Delta_1=1$$

To eliminate the fast oscillating terms it is convenient to introduce the slow varying complex amplitude $\tilde{G}$ of the correlation function:

$$G(z_1,\Delta_1;z_2,\Delta_2)=2\,\mathrm{Re}\left(\tilde{G}(z_1,\Delta_1;z_2,\Delta_2)e^{i(z_1-z_2)}\right) \qquad (4)$$

Here we have replaced k_wz_i by dimensionless variable z_i. Substituting (4) into (1-2) and neglecting fast oscillating terms we obtain the final system of equations:

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$$\nu_1 \frac{\partial}{\partial z_1} F(1) = -2\,\mathrm{Re}\left(\frac{\partial}{\partial \Delta_1} I(z_1,\Delta_1;z_1)\right) \qquad (5)$$

$$\frac{1}{2}\left[(\nu_1+\nu_2)\left(\frac{\partial}{\partial z_1}+\frac{\partial}{\partial z_2}\right)+(\nu_1-\nu_2)\left(2i+\frac{\partial}{\partial z_1}-\frac{\partial}{\partial z_2}\right)\right]\tilde{G}(1;2)=$$
$$= -\frac{\partial F(1)}{\partial \Delta_1} I^*(z_1;z_2,\Delta_2) - \frac{\partial F(2)}{\partial \Delta_2} I(z_2;z_1,\Delta_1) -$$
$$-\frac{2\pi}{N_{\lambda_0}}\left(\tilde{\Phi}^*(z_1-z_2)\frac{\partial}{\partial \Delta_1}+\tilde{\Phi}(z_2-z_1)\frac{\partial}{\partial \Delta_2}\right)F(1)F(2) \qquad (6)$$

where function $I(z_1;z_2,\Delta_2)$ is determined as

$$I(z_1;z_2,\Delta_2)=\int_0^{z_1}\int_{-\infty}^{\infty}\tilde{\Phi}(z_1-z_3)\tilde{G}(2;3)d\{3\}, \qquad (7)$$

$\tilde{\Phi}(z)=-2\rho^3\frac{1}{1-i\alpha z}$ is renormalized complex amplitude of the longitudinal force (3) , $\rho=\left(\frac{1}{(2\gamma)^3}\frac{I}{I_A}\frac{K^2}{k_w^2\sigma^2}\right)^{\frac{1}{3}}$ - Pierce parameter (I - beam peak current, $I_A = mc^3/e$ - Alfven current), N_{λ_0} - the number of electrons in the beam on one wavelength.

The solution of eq. (5-6) can be obtained by numerical methods which are described in the next section. It seems natural that initial electron distributions in different shots are not correlated. Then the boundary conditions for eq. (5-6) are $\tilde{G}|_{z_1=0}=\tilde{G}|_{z_2=0}=0$, $F|_{z_1=0}=F_0(\Delta_1)$, where $F_0(\Delta_1)$ is the electron energy distribution at the entrance to undulator. It is sufficient to find the solution only for $z_1 \geq z_2$ as the symmetry of the correlation function imposes additional condition

$$\tilde{G}(1,2)=\tilde{G}^*(2,1) \qquad (8)$$

It should be noted that one-time two-particle correlation function allows to find only the radiation peak power averaged over different shots. The averaged spectral distribution is determined by two-time two-particle correlation function which obeys the following equation:

$$\left(\frac{\partial}{\partial \theta_1}+\nu_1\frac{\partial}{\partial z_1}\right)G_2(1,\theta_1;2,\theta_2)=-N\frac{\partial F(1)}{\partial \Delta_1}\int \Phi(1,3)G_2(3,\theta_1;2,\theta_2)d\{3\} \qquad (9)$$

In the stationary case $G_2(1,\theta_1;2,\theta_2)=G_2(1,2;\theta_1-\theta_2)$ and this equation has to be solved with the initial condition $G_2(1,2;\theta_1-\theta_2)|_{\theta_1=\theta_2}=G(1,2)$.

The beam current correlation function at given longitudinal coordinate in undulator z can be found from the following expression:

$$\langle \delta I(z,t_1)\delta I(z,t_2)\rangle = A\int \nu_1\nu_2 G_2(z,\Delta_1,\theta_1;z,\Delta_1,\theta_2)d\Delta_1 d\Delta_2$$

where A is some constant and $\theta_i = 2\gamma_\parallel^2(t_i - z)$. It can be shown that if we neglect the dependence of longitudinal “velocities” ν_i on energy coordinates Δ_i then

$$\langle \delta I(z,t_1)\delta I(z,t_2)\rangle \equiv J(z,\theta_1-\theta_2)=$$
$$=A\int G\left(z-\frac{1}{2}(\theta_1-\theta_2),\Delta_1;z+\frac{1}{2}(\theta_1-\theta_2),\Delta_2\right)d\Delta_1 d\Delta_2$$

The current spectral density is determined by the expression:

$$J_\nu(z)=\int J(z,\tau)e^{i(1+\nu)\tau}d\tau$$

NUMERICAL SOLUTION

The system of equations (5-6) has been solved numerically using finite difference method. The difference scheme was obtained by replacing of partial derivatives by centered differences. The integral in Eq. (7) was approximated using method of central rectangles. The numerical algorithm is illustrated in Fig. 1. Indexes (n,m) correspond to energy coordinates, index (j) is related to the coordinate z_2.

The letters “M”, “O” and “P” denote three layers in z_1 direction. To find the solution at the layer “P” one has to know the values of correlation function at two preceding layers. Therefore it is required to keep in the computer memory only these three layers. The solution above the line $z_1 = z_2$ is obtained from the symmetry condition (8).

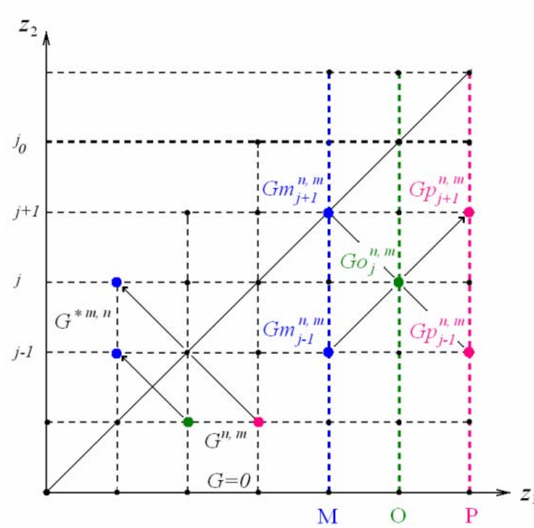

Figure 1. The finite difference scheme.

SIMULATION RESULTS

In this section we present an example of simulation results obtained for the following set of parameters: Pierce parameter $\rho = 0.01$, beam energy spread $\sigma_e = 0.2\rho$, $\alpha = 50\rho$.

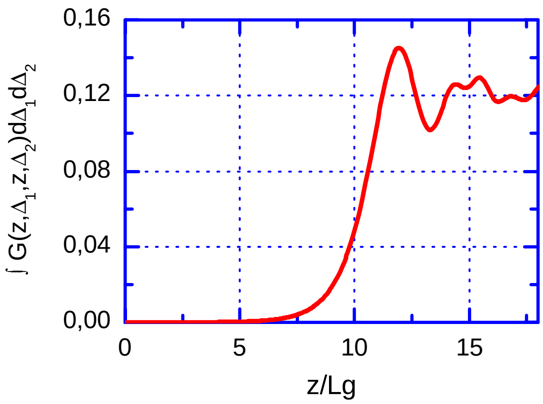

Figure 2. Dependence of the microbunching square on the longitudinal coordinate in undulator. L_g - the gain length at the linear stage.

At Fig. 2 one can see the dependence of the beam microbunching on the longitudinal coordinate in undulator. The amplitude gain length at linear stage $L_g \approx 40\lambda_w$, where λ_w is undulator period. Therefore the Fresnel number $1/2\alpha k_w L_g$ [1] is small and the narrow beam approximation is valid. The exponential growth starts at $z \approx 4L_g$ and comes to saturation at $z \approx 10L_g$.

Fig. 3 shows variation of the r.m.s. spectral bandwidth along undulator. The final value of the bandwidth is in very good agreement with the asymptotic formula (21) obtained in [1].

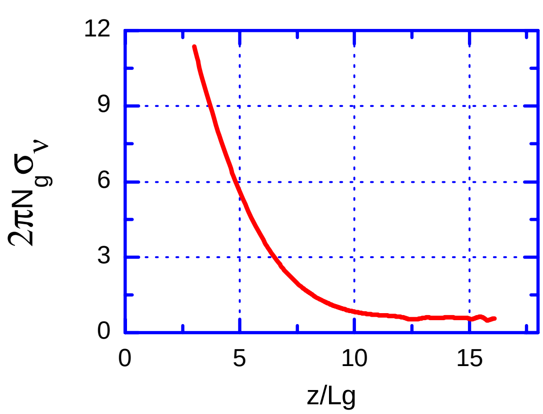

Figure 3. Dependence of the r.m.s. spectral bandwidth on the longitudinal coordinate in undulator. N_g - number of undulator periods per one gain length.

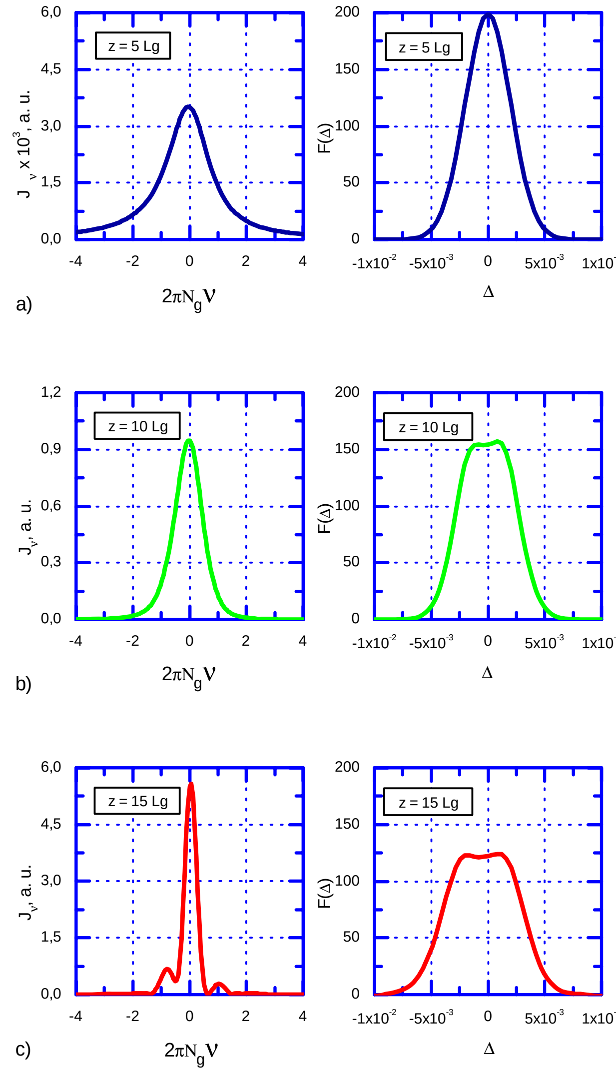

Figure 4. The spectral and energy distributions at different longitudinal positions: a) – start of the exponential growth, b) – start of saturation process, c) – saturation stage.

The saturation process is illustrated in Fig. 4. At the beginning of the exponential growth the spectrum is wide and the energy distribution function is unperturbed (a). Before saturation the bandwidth reduces rapidly and the energy spread starts growing (b). At the saturation stage the energy distribution becomes wide and amplification stops (c). The saturation mechanism is very similar to the quasilinear one [5].

Fig. 5 shows distribution of the correlation function amplitude integrated over energy in the (z_1, z_2) plane.

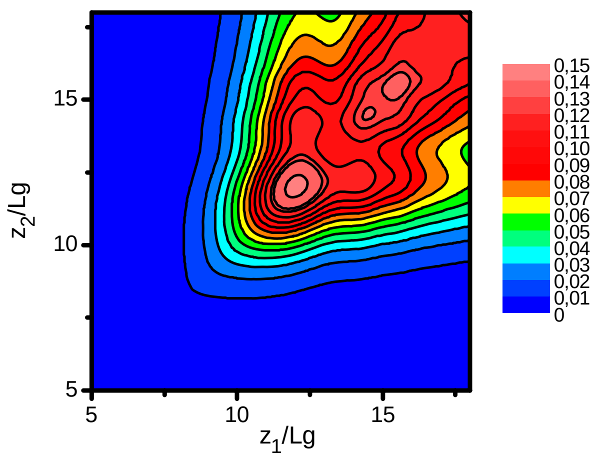

Figure 5. Two coordinate distribution of the correlation function amplitude integrated over energy.

CONCLUSION

In this paper we developed the description for saturation in SASE FEL based on rigorous statistical approach. By our knowledge it is the only existing method to consider nonlinear stage of noise amplification in FEL now. For the simplest case of narrow electron beam we first obtained non-trivial solution for the correlation function nonlinear behaviour.

From the other hand, the SASE FEL is an explicit illustration of fundamental ideas of statistical physics. Indeed, the averaging over the abstract assembly of macroscopically equivalent systems is simply the averaging over different electron bunches, which pass through undulator. It is also the clear example of ergodicity, as the averaging over bunches is a kind of time-averaging.

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