ANALYTICAL SOLUTION FOR FEL AND CARL NON LINEAR REGIME *

Rodolfo Bonifacio, INFN-MI, Milano, Italy and Departamento de Fisica, Universidade Federal de Alagoas, Maceiò, Brazil.

Abstract

We derive a simple analytical solution for the non linear steady state regime of the high gain Free Electron Laser (FEL) and Collective Atomic Recoil Lasing (CARL) model which up to now have been described only numerically.

INTRODUCTION

In this paper we show that the Free Electron Laser (FEL) [1] and the Collective Recoil Atomic Laser (CARL) [2] can be described by an exact reduced Hamiltonian which does not contain the field explicitly. We give simple analytical expressions for the field amplitude, frequency shift, bunching factor, particle average momentum and momentum spread, as well as the period of oscillations around the quasi steady state solution, in very good agreement with the numerical values.

THE MODEL

As it is well known very different systems as a high gain FEL and CARL can be described by the same classical 1D model. In the steady state approximation, neglecting propagation effects, the model equation can be written:

$$\dot{\theta}_j = p_j \tag{1}$$

$$\dot{p}_j = -\left(Ae^{i\theta_j} + cc\right) \tag{2}$$

$$\dot{A} = \left\langle e^{-i\theta}\right\rangle = \frac{1}{N}\sum_{j=1}^{N} e^{-i\theta_j} \tag{3}$$

where θ is the particle (electron or atom) phase in the ponderomotive potential, A is the radiated field and p is the momentum recoil in proper adimensional units [1,2]. The derivative are taken to respect to the coordinate along the wiggler for the FEL and respect to time for the CARL in adimensional units. For simplicity, we have assumed resonance condition.

Separating amplitude and field phase one has:

$$A = ae^{i\varphi} \tag{4}$$

Changing the sign of p_i and θ_i, and defining the "position"

$$q_i = \theta_i - \varphi + \pi/2 \tag{5}$$

Eqs. (1) - (3) can be written in the compact form:

$$\dot{q}_i = p_i - \left\langle \cos q\right\rangle / \sqrt{\left\langle p\right\rangle} = \partial H / \partial p_i \tag{6}$$

$$\dot{p}_i = -2\sqrt{\left\langle p\right\rangle}\sin q_i = -\partial H / \partial q_i \tag{7}$$

where

$$H = \sum_{i=1}^{N} \frac{p_i^2}{2} - 2\sqrt{\left\langle p\right\rangle}\cos q_i \tag{8}$$

with $\left\langle p\right\rangle = (1/N)\sum_{j=1}^{N} p_j$.

Eqs.(6)-(8) are equivalent to Eqs.(1)-(3). Keeping in mind that Eqs.(1-3) admit a constant of motion, the total momentum, $|A|^2 + \left\langle p\right\rangle = const = 0$, i.e.,

$$a = \sqrt{\left\langle p\right\rangle} . \tag{9}$$

We have assumed, for simplicity, that the constant is zero. The positive recoil is counterpropagating to the electron beam in a FEL and in the direction of the pump field for CARL.

Inserting (4) in Eq.(3) one obtains

$$\dot{\varphi} = \frac{\left\langle \cos q\right\rangle}{\sqrt{\left\langle p\right\rangle}} . \tag{10}$$

We stress that Eqs.(6,7) describe a pendulum with a self consistent frequency formally eliminated, with no approximation. They are completely equivalent to Eqs. (1-3) even if the field does not appear explicitly. However, its amplitude and phase can be exactly calculated from the solution of Eqs.(6) and (7) using Eqs.(9) and (10). The reduced Hamiltonian (8), which originates Eq.(6) and (7), represent the system in which the field has been exactly eliminated using the constant of motion and redefining the total phase q.

As it is well known, Eqs. (1)-(3) or (6)-(7) admit the unbunched equilibrium solution with $a = \sqrt{\left\langle p\right\rangle} = 0$ and $\left\langle \cos q\right\rangle = 0$. However, this is an unstable solution. In fact, if one linearize Eq.(1)-(3) around this equilibrium situation one gets the well known 3rd order equation for the field A [1,2]:

$$\dddot{A} - iA = 0 .$$

*Work supported by INFN, Italy
#E-mail : rodolfo.bonifacio@tin.it

This equation admits the run away solution $e^{i\lambda t}$, where λ are the roots of the characteristic equation $\lambda^3+1=0$, i.e., $\lambda=-1,1/2\pm i\sqrt{3}/2$. Therefore, the system evolves exponentially with rate $\sqrt{3}/2$ to a non linear regime in which it oscillates around a quasi state situation. This non linear regime has been describes only numerically [1,2].

In this paper we give a simple analytical derivation of the quasi steady state solution.

Note that Hamiltonian (8) contains the potential

$$V=-2\sqrt{\langle p\rangle}\cos q \tag{11}$$

that has a minimum at

$$q=0. \tag{12}$$

This is clearly a STABLE equilibrium point.

With this in mind, let us assume bunching around this equilibrium point, i.e.,

$$\langle \sin q\rangle \approx 0 \tag{13}$$

$$\langle \cos q\rangle \approx b \tag{14}$$

Here b is by definition the bunching factor, which is a real number such that $0<b<1$. Equation (13) and (14) are the basic assumption of our treatment. Note that (13) implies only a symmetrical distribution around the equilibrium (12).

EQUILIBRIUM SITUATION

We now describe a equilibrium situation in which $\langle p\rangle$ and $\langle q\rangle$ are constant. Therefore, using (13) and (14) in Eqs.(6)-(7), one obtains:

$$\langle p\rangle = b/\sqrt{\langle p\rangle} \tag{15}$$

Therefore, one has

$$\langle p\rangle = b^{2/3}=a^2=\dot{\varphi}>0 \tag{16}$$

where Eq.(10) and (14) have been used.

Eq. (16) establish a very useful identity between the average momentum and $\dot{\varphi}$, and their relation with the bunching. Note that the assumption $\langle q\rangle$ equal a constant, looking at Eq.(5), is equivalent to assume that the average electron phase is locked to the field phase.

The fact that the frequency shift, $\dot{\varphi}$, is always positive is in agreement with the numerical solution and demonstrate analytically that FEL and CARL in the high gain steady state regime behave as an optical fibre, i.e., a medium with a refraction index larger than one [3].

Using the fact that the Hamiltonian (8) is a constant of motion, which for simplicity we take to be zero, we can write $\langle p^2\rangle = 4\sqrt{\langle p\rangle}\langle\cos q\rangle$, using Eqs.(14) and (16), and $\langle p^2\rangle = 4b^{4/3}$. Hence, using (16), we have

$$\sigma(p)=\sqrt{\langle p^2\rangle - \langle p\rangle^2}=\sqrt{3}b^{2/3}. \tag{17}$$

Eq. (17) gives a relation between the energy spread and the bunching.

Equation (16) and (17) are the basic result of this paper, because it gives the analytical expression for field amplitude and frequency shift and for the momentum spread as a function of the bunching in the non linear quasi steady state situation.

The problem is: what is b? Looking at the numerical solution, one sees that in the non linear regime, the various quantities are oscillating around some mean values, as it must be for a system of non linear undamped pendulum. The equilibrium mean values are in excellent agreement with the above estimate, if one takes

$$b=0.5 \tag{18}$$

A rigorous justification of this choice will be given in the following. In fact, with this choice, using Eq.(16) and (17), one obtains

$$\langle p\rangle = a^2=\dot{\varphi}=b^{2/3}=0.6 \tag{19}$$

and

$$c_p \approx 1. \tag{20}$$

These values are in excellent agreement with the average values of the oscillating numerical solution, as shown in Fig.1b,c.

Summarizing, the physics we have demonstrate above is that of an undamped pendulum which oscillates around the stable equilibrium value in a self consistent potential (11). The self consistent pendulum frequency is given by

$$\omega^2=2\sqrt{\langle p\rangle}=2b^{1/3}.$$

Hence, the period of the equilibrium "synchrotron" oscillation, using Eq.(18), is given by

$$T=\frac{2\pi}{\omega}=\frac{\sqrt{2}\pi}{b^{1/6}}\approx 5. \tag{21}$$

This value is in very good agreement with the numerical simulations, shown in Fig.1a.

The value of $b=0.5$, which is the unique assumption of this paper, can be justified as follows:

Using Hamiltonian (8) it is simple to see that at the inversion point of the pendulum motion one has the relation $\langle p^2 \rangle = 4\sqrt{\langle p \rangle}\left(1-\langle \cos q \rangle\right)$. Using Eq.(16), and $\langle p^2 \rangle = 4b^{4/3}$, one obtains $b = 1-b$ i.e.,

$$b = 0.5 \tag{22}$$

We stress that this is the average value of the bunching, which is an oscillating quantity. The analysis can be easily extended to the detuned case.

CONCLUSIONS

In conclusion, classically the FEL and the CARL can be described by a reduced Hamiltonian from which the field amplitude and phase have been formally eliminated. This Hamiltonian system presents two equilibrium situations. The first one (the physical initial condition) is that of an unbunched system with uniformly distributed θ, with zero field and zero average recoil. However, this situation, at resonance, is unstable and the system evolves with an exponential instability to a “quasi equilibrium state” in which the system oscillates as an undamped pendulum around some average values. This non linear quasi “equilibrium”, to our knowledge, has been only numerically described up to now.

In this paper we have given explicitly analytical expressions of all the average values of the physical quantities, including the period of oscillations, in excellent agreement with the numerical simulations as shown in Fig. 1.

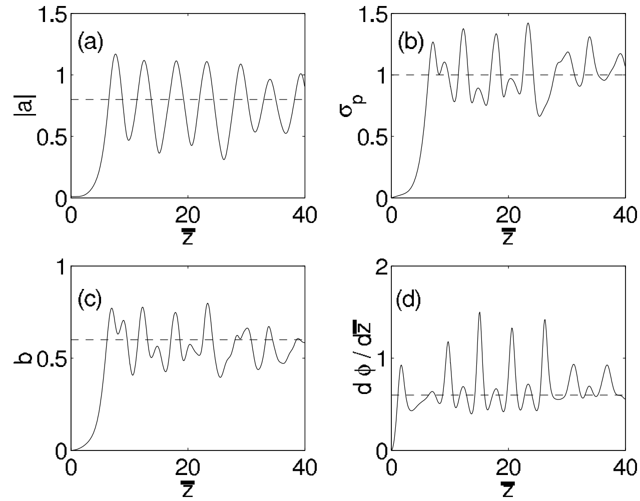

Figure 1: Numerical solutions of equations (1-3) for (a) field amplitude $a = \sqrt{\langle p \rangle}$, (b) momentum spread σ_p, (c) bunching b, and (d) phase derivative $\dot{\varphi}$. The calculated mean values (dotted lines) are, respectively *0.8, 1, 0.6* and *0.6*.

REFERENCES

[1] R. Bonifacio, C.Pellegrini, L.M.Narducci, Optics Comm., 50 (1984) 373.

[2] R. Bonifacio, L. De Salvo, L.M.Narducci, and E.J.D’Angelo, Phys. Rev. A, 50 (1994) 1716, and R. Bonifacio, L.De Salvo, Opt. Comm., 115 (1995) 505.

[3] R. Bonifacio, F. Casagrande, L. De Salvo, Optics Comm. 58, (1986) 259.