NUMERICAL SIMULATION OF SMITH-PURCELL FREE ELECTRON LASERS *

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Abstract

We present a one-dimensional time-dependent analysis and simulation of Smith-Purcell (SP) free-electron lasers (FELs). The coupled Maxwell-Lorentz equations are set up, and the details of numerical simulation are presented. At low electron beam energy, a SP-FEL is a backward wave oscillator (BWO), and oscillations can be achieved without the need for feedback mirrors. In the linear regime, we show that the optical power grows exponentially if the current is larger than a certain value, the start current. Results of our numerical calculation compare well with the analytic calculation in the linear regime and show saturation behavior in the nonlinear regime.

INTRODUCTION

An electron beam traveling close and parallel to a metallic grating, with grating rulings perpendicular to the electron motion, gives off polarized electromagnetic radiation known as Smith-Purcell radiation [1]. A Smith-Purcell free-electron laser (SP-FEL) based on this effect is interesting as a possible compact source of tunable, coherent THz radiation [2,3]. Analytic theory of SP-FELs in the linear regime has been discussed by several authors [4-6] under different approximations. Here, we present a fully self-consistent nonlinear analysis, which can be used to understand the saturation behavior and simulate the realistic effects; coupled Maxwell-Lorentz equations for an SP-FEL driven by sheet beam are derived and solved numerically to perform detailed analysis.

BASIC THEORY

Figure 1 shows the schematic of the SP-FEL setup. We assume the system to have translational invariance in the y-direction. The electron beam travels with a speed v along the z-axis, at a height b above the grating of length L, having grooves of depth d, width w and period λ_g. For our calculation, we use the parameters corresponding to the Dartmouth experiment [2], which are β = 0.35, λ_g = 173 μm, d = 100 μm, w = 62 μm, L = 12.7 mm and b = 10 μm. Here, β is the electron velocity in units of c, the speed of light.

As shown in Refs. 6 and 7, the grating supports a surface mode at a resonant frequency ω, consisting of various order space harmonics, each one decaying along the x-axis. The zeroth order space harmonic co-propagates with the electron beam. The dispersion relation of the surface mode is plotted in Refs. 6 and 7. As was first noted in Ref. 6, for the above parameters, the group velocity is negative and the system is a BWO. The magnitude v_g of the group velocity was calculated to be 0.53 times c for the above parameters. This is in contrast with conventional FELs, which work like a traveling wave amplifier (TWA). For a BWO, the energy emitted by the electron beam flows backward and bunches the incoming electrons. This feedback mechanism gives an oscillator-like action, and the optical power builds up more around the frequency of the surface mode. The calculation of the eigenfrequency of the surface mode is discussed in Refs. 6 and 7, and the free-space wavelength λ of the surface mode defined as $2\pi c/\omega$ was obtained to be 690 μm for the above parameters.

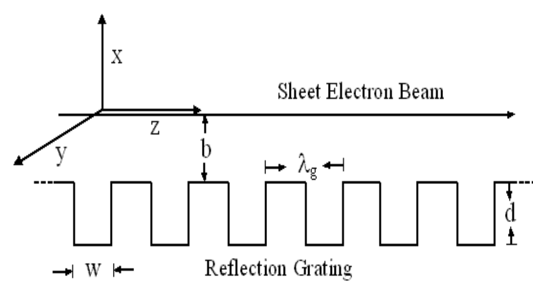

Figure 1: Schematic of an SP-FEL using a sheet electron beam. The sheet electron beam is in the plane x = 0.

Due to the interaction with the surface mode, the electron beam current develops the stongest Fourier component around the frequency ω. Expanding the surface current density in the Fourier series and keeping the term at ω, which will have the strongest interaction with the surface mode, we can write the surface current density as $K(z,t)e^{i(k_s z-\omega t)} + c.\ c$ and $K(z,t) = (I/\Delta y)\langle e^{-i\psi}\rangle$, where $\langle\cdots\rangle$ implies averaging over all the electrons over one wavelength. Here, $k_s = \omega/v$ and Δy is the width of the sheet electron beam in the y direction, and ψ_i is the phase of the i^{th} electron, which is $\omega(z/v - t_i)$, where t_i is the time at which i^{th} electron reaches z.

Let us first determine the electromagnetic field due to this current sheet in the absence of the grating. We assume a steady-state behavior and a slow variation in the surface current density of the type $e^{\mu z}$. We can then write the surface current density as $K_0 e^{i(\alpha_0 z-\omega t)} + c.\ c.$, where K_0 is independent of z and t, and $\alpha_0 = k_s - i\mu$. The Maxwell equation can then be exactly solved with this surface current density and the outgoing wave boundary condition to give the following expression for the amplitude of the elec-

* Work supported by U.S. Department of Energy, Office of Basic Energy Sciences, under Contract No. W-31-109-ENG-38.
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tromagnetic field

$$H_y^{inc}(x,z) = \frac{1}{2}\varepsilon(x)K(z)\exp[-\varepsilon(x)\Gamma_0 x], \tag{1}$$

where $\Gamma_0 = (\alpha_0^2 - \omega^2/c^2)^{1/2}$, $\varepsilon(x) = -1$ for $x < 0$ and $\varepsilon(x) = 1$ for $x > 0$. The electromagnetic field has H-polarization, which means $H_x^{inc} = H_z^{inc} = E_y^{inc} = 0$. Using the Maxwell equation, E_z can be obtained in terms of H_y by the expression $E_z = (i/\epsilon_0\omega)(\partial H_y/\partial x - \delta(x)K)$.

When this electromagnetic wave is incident at the grating surface, it gets reflected at various orders with amplitudes expressed in terms of reflection matrix elements of the grating [8]. The electron beam interacts only with the zeroth order wave which co-propagates with it. The amplitude of the axial electric field experienced by the electron is the sum of the incident and reflected zeroth order wave, and is given by

$$E_z = \frac{iIZ_0}{2\beta\gamma\Delta y}(e_{00}e^{-2\Gamma_0 b} - 1)\langle e^{-i\psi}\rangle, \tag{2}$$

where Z_0 = $1/\epsilon_0 c$ = 377 Ω is the characteristic impedence of free space and ϵ_0 is the permitivity of free space. Note that the total axial electric field experience by the electron is given by $E_z e^{(ik_s z - \omega t)} + c.\ c.$. The matrix element e_{00} in the above expression is a function of frequency and the behavior of e_{00} around the frequency of the surface mode has been studied in detail [7,8], and it is shown that one can write $e_{00} = -i\chi/\mu + \chi_1$. For our parmeters, we obtain χ = 10 per cm and χ_1 = 1.35 [7,8]. Using this parametrization, we can rewrite the above expression as

$$E_z = \frac{iIZ_0}{2\beta\gamma\Delta y}(\underbrace{\frac{-i\chi}{\mu}e^{-2\Gamma_0 b}} + \chi_1 e^{-2\Gamma_0 b} - 1)\langle e^{-i\psi}\rangle. \tag{3}$$

The above expression has two parts. The first part corresponding to the underbraced term is the outgoing evanescent wave, which is a component of surface mode. The remaining terms are independent of growth rate and are proportional to the beam current and are identified as space-charge terms. Let us denote the surface mode component as E and the space charge component as E_{sc}. In the expression for E, we can replace μ with the operator d/dz and obtain the steady-state differential equation for E. Incorporating the group velocity, this equation is further generalized to the following time-dependent differential equation for E:

$$\frac{\partial E}{\partial t} - v_g\frac{\partial E}{\partial z} = -\frac{IZ_0\chi v_g}{2\beta\gamma\Delta y}e^{-2\Gamma_0 b}\langle e^{-i\psi}\rangle. \tag{4}$$

The expression for the space charge field is given by

$$E_{sc} = -\frac{iIZ_0}{2\beta\gamma\Delta y}(1 - \chi_1 e^{-2\Gamma_0 b})\langle e^{-i\psi}\rangle, \tag{5}$$

Eqs.(4,5) are the Maxwell equations that we will be using in our analysis.

Next, we discuss the equations for the electron dynamics. We assume the presence of stong focusing field such that the electron motion is only one dimensional. We describe the longitudinal dynamics of the i^{th} electron in terms of phase ψ_i and energy (in units of rest mass energy) γ_i. It is straightforward to derive the following equation of motion for the electron in the presence of surface mode and the space-charge field:

$$\frac{\partial\gamma_i}{\partial t} + v\frac{\partial\gamma_i}{\partial z} = \frac{ev}{mc^2}(E + E_{sc})e^{i\psi_i} + c.c, \tag{6}$$

$$\frac{\partial\psi_i}{\partial t} + v\frac{\partial\psi_i}{\partial z} = \frac{\omega}{\beta^2\gamma^2}\frac{(\gamma_i - \gamma_p)}{\gamma_p}, \tag{7}$$

where e is the electronic charge and m is the electronic mass. Here, the electron velocity v is close to the phase velocity v_p of the zeroth order evanescent component of the surface mode . Eqations (4-7) are the full set of coupled Maxwell-Lorentz equations, which govern the behavior of the sheet beam SP-FEL with the given boundary conditions. These equations can be solved numerically. Before proceeding to the numerical solution, we define the following new dimensionless variables

$$\zeta = z/L, \tag{8}$$

$$\tau = (t - \frac{z}{v_p})\left(\frac{1}{v_p} + \frac{1}{v_g}\right)^{-1}\frac{1}{L}, \tag{9}$$

$$\eta_i = \frac{k_s L}{\beta^2\gamma^3}(\gamma_i - \gamma_p), \tag{10}$$

$$\mathcal{E} = \frac{4\pi}{I_A Z_0}\frac{k_s L^2}{\beta^2\gamma^3}E, \tag{11}$$

$$\mathcal{E}_{sc} = \frac{4\pi}{I_A Z_0}\frac{k_s L^2}{\beta^2\gamma^3}E_{sc}, \tag{12}$$

$$\mathcal{J} = 2\pi\frac{I}{I_A}\frac{\chi}{\Delta y}\frac{k_s L^3}{\beta^3\gamma^4}e^{-2\Gamma_0 b}, \tag{13}$$

where $I_A = 4\pi\epsilon_0 mc^3/e$ = 17 kA is the Alfven current. Here, ζ is the the dimensionless distance along the grating, which varies from 0 to 1. In terms of these dimensionless variables, the coupled Maxwell-Lorentz equations take the following form

$$\frac{\partial\mathcal{E}}{\partial\tau} - \frac{\partial\mathcal{E}}{\partial\zeta} = -\mathcal{J}\langle e^{-i\psi}\rangle, \tag{14}$$

$$\frac{\partial\eta_i}{\partial\zeta} = (\mathcal{E} + \mathcal{E}_{sc})e^{i\psi_i} + c.c., \tag{15}$$

$$\frac{\partial\psi_i}{\partial\zeta} = \eta_i, \tag{16}$$

$$\mathcal{E}_{sc} = i\frac{\mathcal{J}}{\chi L}(\chi_1 - e^{2\Gamma_0 b})\langle e^{-i\psi}\rangle. \tag{17}$$

Using the conservation of energy, we get the following expression for the power flowing backward in the surface mode

$$\frac{P}{\Delta y} = 2\frac{\beta\gamma}{Z_0\chi}\left(\frac{mc^2\beta^2\gamma^3}{ek_s L^2}\right)^2 e^{2\Gamma_0 b}|\mathcal{E}|^2. \tag{18}$$

Note that, since the power flows backward, the boundary condition for the field needs to be specified at the exit. For our case, since we want to study the growth of signal from the shot noise, we set $\mathcal{E}$ = 0 at ζ = 1.

Equations (14-17) can be linearized around the equilibrium solution and can be written conveniently in terms of collective variables as defined by Bonifacio et al. [9] for conventional FELs. We can assume a solution of the type $e^{\nu\tau}\Phi(\xi)$ and evaluate the growth rate ν in time. As shown in Ref. 7, we find that only if $I > I_s$, do we obtain Re(ν) > 0. The start surface current density $I_s/\Delta y$ is obtained to be

$$\frac{I_s}{\Delta y} = 7.68 I_A \frac{\beta^3\gamma^4}{2\pi\chi k_s L^3} e^{2\Gamma_0 b}. \tag{19}$$

Note that a similar equation was obtained by Swegel for BWO [10]. For our parameters, we obtain the start current density to be 36 A/m. For $I/\Delta y$ = 50 A/m, we find $\nu = 0.62 + i3.17$. The real part of ν gives us the growth rate in time. Converting to physical variables, this gives us the e-folding time of the power in the surface mode to be 0.17 ns. The imaginary part of ν tells us how much the actual wavelength is detuned from the resonant wavelength. Again converting to physical variables, this tells us that the power will actually build up at 694 microns, rather than at the resonant wavelength of 690 microns.

NUMERICAL SIMULATIONS

For numerically solving Eqs. (14-17), we use the approach used by Ginzburg et al. [11] and later also by Levush et al. [12] for BWO. The electron dynamics equations for a given field distribution along the interaction region are solved by the predictor-corrector method. Then, knowing the modified electron distribution in phase space, the field distribution at the next time step is obtained by solving the partial differential equation (Eq. 14) by the finite difference method. In our simulation, we have chosen the step size as $\Delta\tau$ = 0.01 and $\Delta\zeta$ = 0.02. As mentioned in Ref. 11, the accuracy of this method is $O(\Delta\tau^2 + \Delta\zeta^2)$, and the method is stable for $\Delta\tau < \Delta\zeta$.

For initializing the electron beam is phase space, we simulate the shot noise using the algorithm given by Penman and McNeil [13], which is commonly used in FEL codes.

We performed a couple of tests on the code we developed. We first checked for the convergence of the solution by increasing the number of particles and also by reducing the step size. Based on this convergence test, we chose the number of particles to be used in the simulation as 1024 and the step sizes as $\Delta\tau$ = 0.01 and $\Delta\zeta$ = 0.02. We also confirmed that the energy conservation is satisfied in the code at each integration step.

We now discuss the results obtained using this code. Parameters used in the simulation are same as discussed earlier in the paper. From the simulation, we find the start current density to be 37.5 A/m, which compares well with our analytic calculation. Figure 2(a) shows the growth of power at the grating entrance for an operating surface current density of 50 A/m. We find that the power builds up and saturates. The power per unit beam width $(P/\Delta y)$ at

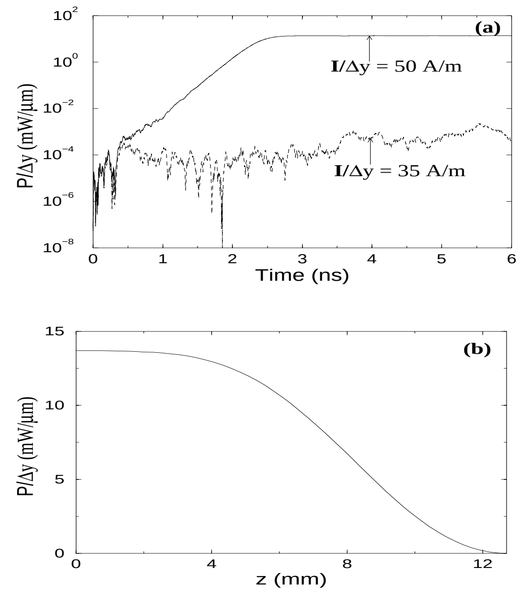

Figure 2: Plots of power per unit beam width in the surface mode (a) as a function of time at ζ = 0, and (b) as a function of z at saturation as obtained from the simulation.

saturation is 13.7 mW/μm. We also notice that the power builds up exponentially with time in the linear regime, and the power gets e-folded in 0.20 ns, which agrees quite well with the estimate of 0.17 ns based on analytic calculation in the previous section. We find that the power saturates in around 3 ns, which is around 15 times the e-folding time. As shown in the same figure, for a surface current density of 35 A/m, which is less than the start surface current density, we get random noise with a power level 4-5 orders of magnitude smaller. Figure 2(b) shows the growth of power along the interaction length after the system has reached saturation. We clearly see that power starts from zero at ζ = 1, i.e., at the end of the grating, and grows backward and reaches saturation at ζ = 0, i.e., at the beginning of the grating. We also looked at the power spectrum and found that it is a very narrow spectrum and perhaps the bandwidth is Fourier transform limited. We found that the power peaked at 694.5 μm, which means there is a detuning from the resonant wavelength as is common in FELs and BWOs. This detuning agrees quite well with the estimated detuning in the previous section based on analytic calculation.

Next, we looked at the evolution of longitudinal phase space. Figure 3(a) shows the phase space when the electrons enter the interaction region and also when the electrons exit the interaction region. The phase space is plotted here after the power has saturated. We clearly see that electrons get bunched due to the interaction with the surface mode. We plotted the amplitude of the bunching parameter $|\langle e^{-i\psi}\rangle|$ along the interaction length after power has saturated (Fig. 3(b)). We find that, when the electrons exit the interaction region, they are nicely bunched and the bunching parameter is around 0.7. We have noted that, if the beam current is increased further, although there is more power generated, the beam gets overbunched and the bunching factor starts decreasing as happens even in conventional FELs.

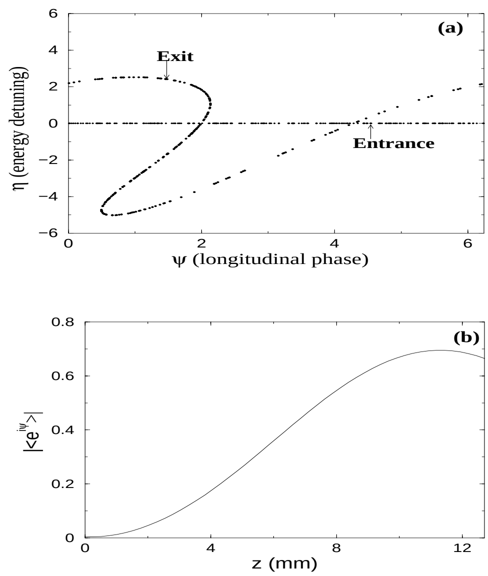

Figure 3: The phase space (a) of the electron beam at the exit and at the entrance. The bunching parameter along the interaction length is plotted in (b) after power saturation.

DISCUSSIONS

Let us first discuss the efficiency for power conversion in the SP-FEL system. For the 50 A/m case, the beam power per unit beam width is 1.75 W/μm and the output power per unit beam width is 13.7 mW/μm. Hence, the efficiency is $\sim$ 0.8%. We can get an analytic estimate for the upper bound of the efficiency by arguing that the maximum amount of energy that the electron can lose before saturation is such that it lags the co-propagating evanescent wave by half a wavelength during the transit through the grating. If the change in the velocity of the electron due to losing energy is Δv, then, as per this argument, $\Delta v L/v = \pi v/\omega$. This gives us the following expression for the efficiency η_{eff}:

$$\eta_{eff} = \frac{\pi c \beta^3 \gamma^3}{\omega L(\gamma - 1)}. \tag{20}$$

Using the above expression, we get an upper bound for the efficiency to be 2.0%. This compares well with 0.8% efficiency for 50 A/m case keeping in mind that this is only the estimate for the upper bound. We also simulated higher current and found that, for the 120 A/m case, the efficiency is 1.9%, but power is not stable and shows modulation.

We have so far assumed that the electron beam has infinite extent along the y-axis and so has the radiation beam. This means that the electron beam overlaps with the radiation beam completely. In practical cases, this is not going to be so, and, in general, the overlap may not be perfect. The radiation beam is guided by the grating in the x direction but not so in the y direction. Along the y direction, the radiation beam will be freely diffracting. The minimum average rms beam size over the length L due to diffraction effects is given by $\sqrt{\lambda_s L/4\pi}$ [14], which comes to around 500 μm for our parameters. Taking this as the effective electron beam radius in in the y direction, we obtain Δy = 1 mm. Using this value of Δy, we obtain the start current to be 37.5 mA. For I = 50 mA, the total power generated in the surface mode is then obtained to be 13.7 W.

There are several ways in which this power in the surface mode can be outcoupled to the freely propagating mode that can be used for experiments [7]. First, a significant amount of power will get outcoupled to the freely propagating mode at the grating entrance due to diffraction. Second, when the electron beam exits the grating, it will emit coherent diffraction radiation at the metallic edge since it is bunched. Third, we can have a second grating following the first grating with optimized parameters, where the bunched beam can radiate copiously due to coherent spontaneous emission.

To summarize, we have set up and solved the coupled Maxwell-Lorentz equations for an SP-FEL driven by a sheet beam. Our numerical calculation compares well with the analytic solution in the linear regime and shows saturation behavior. The analysis is useful for the design of future compact THz sources based on SP-FELs.

One of us (VK) would like to thank Y.-C. Chae for useful discussions on time-dependent FEL simulation.

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