ANALYTICAL SOLUTION OF PHASE SPACE EVOLUTION OF ELECTRONS IN A SASE FEL

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Abstract

I present an analytical solution for the phase space evolution of electrons in a self-amplified spontaneous emission Free Electron Laser (FEL) operating in the linear regime before saturation, by solving the one dimensional FEL equation together with the solution of the cubic equation, which represents the evolution of the FEL power. The analytical solutions for the phase space evolution are complementary to the solution for the optical evolution and hold until the optical amplitude grows greater than one-tenth of the amplitude in saturation. The amplitude in saturation obtained from a time dependent numerical calculation in which the analytical solutions are used as the initial values is shown to be equal to that obtained in the conventional theories.

INTRODUCTION

A self-amplified spontaneous emission (SASE) free electron laser (FEL) has been developed worldwide as an intense coherent x-ray radiation source [1, 2, 3]. The development has been supported by extensive theoretical studies [4, 5, 6], which can account for various types of experimental results such as the exponential increase of SASE power with the undulator length [1]. However, those studies have mainly focused on the property of the radiation field, and the phase space evolution of electrons of a SASE FEL has been studied only in numerical simulations so far [6].

In this paper, I present an analytical solution for the phase space evolution of electrons in a SASE FEL operating in the linear regime before saturation, by solving the one dimensional (1D) equations of electron motion together with the solution of the cubic equation, which represents the evolution of SASE power. The solutions for the energy and phase changes of electrons are respectively represented by sum of three independent terms similarly to the solution of the cubic equation; an exponentially growing term, an exponentially decaying term and an oscillating term. The 1D Maxwell equation results in the same field gain as the solution of the cubic equation, when the solution for the electron phase is substituted into the Maxwell equation. The solutions for the phase space evolution are thus complementary to the solution for the optical evolution. The present solutions hold until the optical amplitude grows greater than one-tenth of the amplitude in saturation. The field in non-linear regime near saturation is obtained from a numerical calculation which uses the analytical solutions as the initial values and solves the 1D FEL equations. The peak amplitude in saturation obtained in the calculation agrees well with those obtained in conventional theories [5, 6].

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1D FEL EQUATIONS

The present study starts with the Colson's dimensionless FEL equations under the slowly varying envelope approximation (SVEA) [7]. Some of the variables used in the present study are however defined differently from those of Colson's variables, as described later in this section. The simplest situation is considered in the present study where the electron pulse has a rectangular shape with density of n_e and an initial energy of $\gamma_0 mc^2$ with no energy spread. The electron pulse length is assumed to be longer than the slippage distance $N_w\lambda$. Here N_w is the number of undulator periods, $\lambda = \lambda_w(1 + a_w^2)/(2\gamma_0^2)$ is the resonant wavelength, $\lambda_w = 2\pi/k_w$ is the period of the undulator and a_w is the undulator parameter. The fundamental FEL parameter in MKSA units is given by

$$\rho = \frac{1}{\gamma_0}[ea_wF\sqrt{n_e/(\epsilon_0 m)}/(4ck_w)]^{2/3}, \qquad (1)$$

where F is unity for a helical undulator or Bessel function $[JJ]$ for a planar undulator [5]. The dimensionless time is defined by $\tau = ct/\lambda_w$, so that $\delta\tau = 1$ corresponds to the transit time of light through the undulator period. The longitudinal position of the ith electron is defined by $\zeta_i(\tau) = [z_i(t) - ct]/\lambda$, so that $\delta\zeta = 1$ corresponds to λ. The dimensionless field envelope is defined by

$$a(\zeta,\tau) = \frac{2\pi ea_w\lambda_wF}{{\gamma_0}^2mc^2}E(\zeta,\tau)\exp[i\phi(\zeta,\tau)], \qquad (2)$$

with phase $\phi(\zeta,\tau)$, which is equivalent to the Colson's dimensionless field envelope [8] divided by $2N_w^2$ and to the Bonifacio's envelope [5] multiplied by $(4\pi\rho)^2$. Here $E(\zeta,\tau)$ is the rms optical field strength. The dimensionless energy and phase of the ith electron are respectively defined by $\mu_i(\tau) = 4\pi[\gamma_i(t) - \gamma_0]/\gamma_0$ and $\psi_i(\tau) = (k_w + k)z_i(t) - \omega t$, where $k = 2\pi/\lambda$ is the wave number of the resonant wavelength λ.

In the present definition, evolutions of $a(\zeta,\tau)$, $\mu_i(\tau)$ and $\psi_i(\tau)$ are respectively given by [8]

$$\frac{d\mu_i(\tau)}{d\tau} = a[\zeta_i(\tau),\tau]\exp[i\psi_i(\tau)] + \text{c.c.}, \qquad (3)$$

$$\frac{d\psi_i(\tau)}{d\tau} = \mu_i(\tau), \qquad (4)$$

$$\frac{\partial a(\zeta,\tau)}{\partial\tau} = -(4\pi\rho)^3\langle\exp[-i\psi_i(\tau)]\rangle_{\zeta_i(\tau)=\zeta}. \qquad (5)$$

The angular bracket indicates the average of all the electrons in the volume V around ζ.

PHASE SPACE EVOLUTION

The lasing process in FELs starts with formation of a uniform field in time and space. This process is known as the spectrum narrowing in the frequency domain [4] or as the longitudinal phase mixing in the time domain [9]. The evolution of the uniform field before saturation is represented by three complex roots of the cubic equation [5, 8]. The field in τ for the steady-state region where $\zeta < -\tau$ is given by

$$a(\tau) = \frac{|a(0)|e^{i\phi(0)}}{3}[\exp(4\pi\rho\tau e^{i\pi/6}) + \exp(-4\pi\rho\tau e^{-i\pi/6}) + \exp(4\pi\rho\tau e^{-i\pi/2})] \quad (6)$$

The ith electron interacts with the field in the steady-state region due to the slippage, and the energy modulation at τ' during $\delta\tau'$ is given from Eq. (3) by $\delta\mu_i(\tau') = [a(\tau')e^{i\psi_i(\tau')} + \text{c.c.}]\delta\tau'$. Here the linear regime before saturation is defined as the regime where the electron phase remains almost unchanged due to the weak FEL field, i.e., $\psi_i(\tau') \approx \psi_i(0)$. The energy modulation in the linear regime is expressed by $\delta\mu_i(\tau') \approx [a(\tau')e^{i\psi_i(0)} + \text{c.c.}]\delta\tau'$. The energy change of the ith electron at time τ, $\mu_i(\tau)$, is given by the sum of those modulations during τ:

$$\mu_i(\tau) = \int_0^\tau \{a(\tau')\exp[i\psi_i(0)] + \text{c.c.}\}d\tau'. \quad (7)$$

The integration of Eq. (7) after substitution of Eq. (6) yields

$$\begin{aligned}\mu_i(\tau) = {} & \frac{2|a(0)|}{3(4\pi\rho)} \times \\ & \{e^{2\pi\sqrt{3}\rho\tau}\cos[\psi_i(0) + \phi(0) + 2\pi\rho\tau - \pi/6] \\ & -e^{-2\pi\sqrt{3}\rho\tau}\cos[\psi_i(0) + \phi(0) + 2\pi\rho\tau + \pi/6] \\ & +\cos[\psi_i(0) + \phi(0) - 4\pi\rho\tau + \pi/2]\}. \quad (8)\end{aligned}$$

The first term in the right hand side of Eq. (8) is the exponentially growing term, the second is the exponentially decaying term, and the third is the oscillating term.

The phase modulation at τ' during $\delta\tau'$ is given from Eq. (4) by $\delta\psi_i(\tau') = \mu_i(\tau')\delta\tau'$. The phase change of the ith electron at time τ is given by the sum of those modulations during τ:

$$\Delta\psi_i(\tau) = \psi_i(\tau) - \psi_i(0) = \int_0^\tau \mu_i(\tau')d\tau'. \quad (9)$$

The integration of Eq. (9) after substitution of Eq. (8) yields

$$\begin{aligned}\Delta\psi_i(\tau) = {} & \frac{2|a(0)|}{3(4\pi\rho)^2} \times \\ & \{e^{2\pi\sqrt{3}\rho\tau}\cos[\psi_i(0) + \phi(0) + 2\pi\rho\tau - \pi/3] \\ & +e^{-2\pi\sqrt{3}\rho\tau}\cos[\psi_i(0) + \phi(0) + 2\pi\rho\tau + \pi/3] \\ & +\cos[\psi_i(0) + \phi(0) - 4\pi\rho\tau + \pi]\}. \quad (10)\end{aligned}$$

Equation (10) has the exponentially growing term, the exponentially decaying term and the oscillating term, similarly to Eq. (8). Equations (8) and (10) are the analytical expressions for the phase space evolution in a SASE FEL operating in the linear regime; only numerical solutions for those have been obtained previously [6].

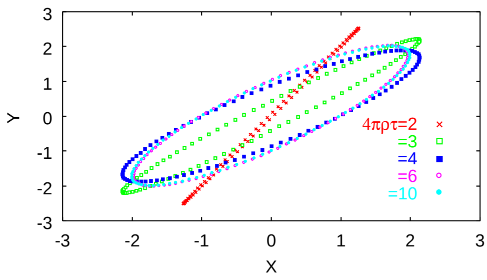

Figure 1: Electron distributions in a phase plane of $X = \Delta\psi_i(\tau)(4\pi\rho)^2/[|a(0)|\exp(2\pi\sqrt{3}\rho\tau)/3]$ and $Y = \mu_i(\tau)(4\pi\rho)/[|a(0)|\exp(2\pi\sqrt{3}\rho\tau)/3]$ derived from Eqs. (8) and (10) for $4\pi\rho\tau = 2, 3, 4, 6, 10$. The center of the electron microbunch is located at $(X, Y) = (0, -1)$ when $4\pi\rho\tau \geq 4$.

Using Eqs. (8) and (10), one can study the distribution of electrons in a longitudinal phase space. The electrons contained in the resonant wavelength at $\tau = 0$ are numbered from the front to back along the propagating direction in the present study. The relative position between two adjacent electrons is represented by $\zeta_i(0) - \zeta_{i+1}(0) > 0$, and the relative phase between two adjacent electrons is given by $\psi_i(0) > \psi_{i+1}(0)$, since $\psi_i(\tau) = 2\pi\zeta_i(\tau) + k_w z_i(t)$ by definition. One can calculate the values of Eqs. (8) and (10) for each i and plot the point in a phase space of $\Delta\psi_i(\tau)$ and $\mu_i(\tau)$. The point rotates counterclockwise in the phase space, as i increases. The distribution of the electrons at time instant τ in a longitudinal phase space of

$$X = \Delta\psi_i(\tau)\frac{3(4\pi\rho)^2}{|a(0)|e^{2\pi\sqrt{3}\rho\tau}} \quad (11)$$

and

$$Y = \mu_i(\tau)\frac{3(4\pi\rho)}{|a(0)|e^{2\pi\sqrt{3}\rho\tau}} \quad (12)$$

is elliptical as shown in Fig. 1. The shape of the distribution gradually changes when $4\pi\rho\tau < 4$ and remains almost constant when $4\pi\rho\tau \geq 4$.

The gain of the steady state field in the linear regime before saturation where $|\Delta\psi_i(\tau)| \ll 1$ is obtained by substitution of Eq. (10) into Eq. (5) as follows:

$$\begin{aligned}\frac{\partial a(\tau)}{\partial\tau} = {} & \frac{4\pi\rho|a(0)|e^{i\phi(0)}}{3}[\exp(4\pi\rho\tau e^{i\pi/6} + i\pi/6) \\ & -\exp(-4\pi\rho\tau e^{-i\pi/6} - i\pi/6) \\ & +\exp(4\pi\rho\tau e^{-i\pi/2} - i\pi/2)]. \quad (13)\end{aligned}$$

Equation (13) is the same as $\partial a(\tau)/\partial\tau$ obtained from differentiation of Eq. (6). This means that Eqs. (8) and (10) are complementary to Eq. (6). The magnitudes of the gain and phase shift are almost constant when $4\pi\rho\tau \geq 4$.

EVOLUTION IN HIGH GAIN REGIME

In this section, the field and electron phase space evolutions are studied in the high gain regime, which is defined by $4\pi\rho\tau \geq 4$ where the exponentially growing terms only survive in Eqs. (6), (8) and (10). In this case, the field is given by

$$a(\tau) \sim \frac{|a(0)|}{3} e^{2\pi\sqrt{3}\rho\tau} e^{i\phi(\tau)}, \qquad (14)$$

where $\phi(\tau) = 2\pi\rho\tau + \phi(0)$, and the energy and phase changes of the ith electron are respectively given by

$$\mu_i(\tau) \sim \frac{2|a(0)|e^{2\pi\sqrt{3}\rho\tau}}{3(4\pi\rho)} \cos\left[\psi_i(0) + \phi(\tau) - \frac{\pi}{6}\right], \quad (15)$$

$$\Delta\psi_i(\tau) \sim \frac{2|a(0)|e^{2\pi\sqrt{3}\rho\tau}}{3(4\pi\rho)^2} \cos\left[\psi_i(0) + \phi(\tau) - \frac{\pi}{3}\right]. \quad (16)$$

The shape of the distribution in a longitudinal phase space of X and Y is simply represented by the ellipse $X^2 + Y^2 - \sqrt{3}XY = 1$. The ellipse rotates clockwise as $\phi(\tau)$ increases linearly with τ. The electrons are lined along the ellipse counterclockwise as the identification number i increases. The electron at the microbunch center satisfies the condition of $\Delta\psi_i(\tau) = 0$. The electron just in front of the electron at the microbunch center satisfies the condition of $\Delta\psi_{i-1}(\tau) < 0$ for the bunch to be formed. Thus the microbunch center in the high gain regime is located at $(X, Y) = (0, -1)$ in Fig. 1 and the electrons inside the microbunch are concentrated around $\psi_i(0) + \phi(\tau) - \pi/3 = \pi/2$ when $4\pi\rho\tau \geq 4$.

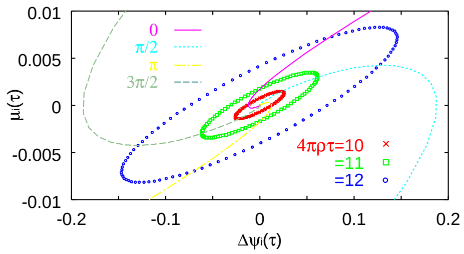

Figure 2: Electron distributions in a phase plane of $\Delta\psi_i(\tau)$ and $\mu_i(\tau)$ derived from Eqs. (8) and (10) when $4\pi\rho\tau$ = 10 (crosses), 11 (open squares), 12 (open circles). The time evolution of four different electrons, initial phases of which are $\psi_i(0)$ (solid line), $\psi_i(0) + \pi/2$ (dotted line), $\psi_i(0) + \pi$ (dash-dotted line) and $\psi_i(0) + 3\pi/2$ (dashed line), are also shown. The FEL parameter $\rho = 0.00447$ and $|a(0)| = 7.1 \times 10^{-5}\rho^{3/2}$ are used.

The evolution of the electron distribution in a longitudinal phase space of $\Delta\psi_i(\tau)$ and $\mu_i(\tau)$ is shown in Fig. 2, where $\rho = 0.00447$ and $|a(0)| = 7.1 \times 10^{-5}\rho^{3/2}$ are used. Those values are typical parameters in JAERI FEL [10]. The crosses are the distribution when $4\pi\rho\tau = 10$, the open squares when $4\pi\rho\tau = 11$ and the open circles when $4\pi\rho\tau = 12$. The figure also shows the time evolution of four different electron particles, initial phases of which are $\psi_i(0)$ (solid line), $\psi_i(0) + \pi/2$ (dotted line), $\psi_i(0) + \pi$ (dash-dotted line) and $\psi_i(0) + 3\pi/2$ (dashed line). One can find that the ellipse expands exponentially in size due to the exponential increase of $|a(\tau)|$ and rotates clockwise due to the linear increase of $\phi(\tau)$. The intersection of the ellipse and the line $\Delta\psi_i(\tau) = 0$ where $\mu_i(\tau) < 0$ is the location of the microbunch center. The exponential decrease of the energy of the microbunch center corresponds to the exponential decrease of the energy of the microbunch as a whole. The energy radiated by the microbunch is used for the field amplification.

The field gain in the linear regime where $|\Delta\psi_i(\tau)| \ll 1$ is given by Eq. (13). However, the gain deviates from Eq. (13) in the non-linear regime near saturation where the amplitude grows and $|\Delta\psi_i(\tau)| \ll 1$ does not hold any more for some electrons. The threshold amplitude for the non-linear regime can be roughly estimated from Eq. (5) in which Eqs. (14) and (16) are substituted as follows:

$$\frac{\partial \frac{|a(\tau)|}{(4\pi\rho)^2}}{\partial(4\pi\rho\tau)} = -\langle\cos\{\psi_i(0) + \phi(\tau) + 2\frac{|a(\tau)|}{(4\pi\rho)^2} \times \cos[\psi_i(0) + \phi(\tau) - \pi/3]\}\rangle_{\zeta_i(\tau)=\zeta}, \quad (17)$$

$$\frac{\partial\phi(\tau)}{\partial(4\pi\rho\tau)} = \frac{(4\pi\rho)^2}{|a(\tau)|}\langle\sin\{\psi_i(0) + \phi(\tau) + 2\frac{|a(\tau)|}{(4\pi\rho)^2} \times \cos[\psi_i(0) + \phi(\tau) - \pi/3]\}\rangle_{\zeta_i(\tau)=\zeta}. \quad (18)$$

One can calculate the values of the right hand sides of Eqs. (17) and (18) as a function of the value of $|a(\tau)|/(4\pi\rho)^2$ and find that Eq. (13) begins to deviate from Eqs. (17) and (18) around $|a(\tau)|/(4\pi\rho)^2 = 0.15$. In the calculation the value of $\psi_i(0) + \phi(\tau)$ is uniform over 2π.

SATURATION

One can calculate the efficiency and amplitude in the non-linear regime near saturation by solving the time dependent 1D FEL equations together with the initial values given by Eqs. (14), (15) and (16). In the high gain and linear regime, where $4\pi\rho\tau_x \geq 4$ and $|a(\tau_x)|/(4\pi\rho)^2 = x$ for $x \leq 0.15$, Eq. (14) is rewritten by

$$\frac{a(\tau_x)}{(4\pi\rho)^2} \sim \frac{|a(0)|}{3(4\pi\rho)^2} e^{2\pi\sqrt{3}\rho\tau_x} e^{i\phi(\tau_x)} = x e^{i\phi(\tau_x)}. \quad (19)$$

The energy and phase of the ith electron at τ_x are derived from substitution of Eq. (19) into Eqs. (15) and (16) and are respectively given by

$$\frac{\mu_i(\tau_x)}{4\pi\rho} \sim 2x\cos\{\psi_i(0) + \phi(\tau_x) - \frac{\pi}{6}\}, \quad (20)$$

$$\psi_i(\tau_x) \sim \psi_i(0) + 2x\cos\{\psi_i(0) + \phi(\tau_x) - \frac{\pi}{3}\}. \quad (21)$$

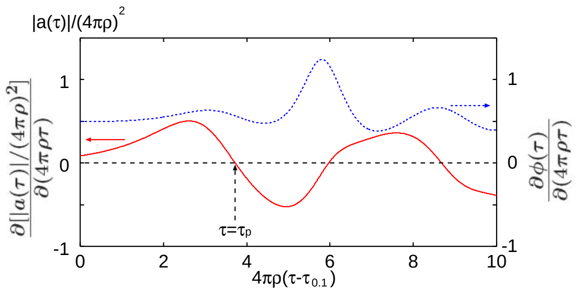

Figure 3: The amplitude gain $\partial[|a(\tau)|/(4\pi\rho)^2]/\partial(4\pi\rho\tau)$ (solid line) and phase shift $\partial\phi(\tau)/\partial(4\pi\rho\tau)$ (dotted line) calculated numerically from the 1D FEL equations given by Eqs. (22), (23), (24) and (25) as a function of $4\pi\rho(\tau - \tau_{0.1})$. The initial values for the calculation are derived from Eqs. (19), (20) and (21).

Solving the 1D FEL equations numerically with the initial values given by Eqs. (19), (20) and (21), one can calculate the phase space evolution of electrons and the field evolution. The equations for $\tau \geq \tau_x$ are written as follows:

$$\frac{d\frac{\mu_i(\tau)}{4\pi\rho}}{d(4\pi\rho\tau)} = 2\frac{|a(\tau)|}{(4\pi\rho)^2}\cos[\psi_i(\tau)+\phi(\tau)], \quad (22)$$

$$\frac{d\psi_i(\tau)}{d(4\pi\rho\tau)} = \frac{\mu_i(\tau)}{4\pi\rho}, \quad (23)$$

$$\frac{\partial\frac{|a(\tau)|}{(4\pi\rho)^2}}{\partial(4\pi\rho\tau)} = -\langle\cos[\psi_i(\tau)+\phi(\tau)]\rangle_{\zeta_i=\zeta}, \quad (24)$$

$$\frac{\partial\phi(\tau)}{\partial(4\pi\rho\tau)} = \frac{(4\pi\rho)^2}{|a(\tau)|}\langle\sin[\psi_i(\tau)+\phi(\tau)]\rangle_{\zeta_i=\zeta}. \quad (25)$$

Figure 3 shows $\partial[|a(\tau)|/(4\pi\rho)^2]/\partial(4\pi\rho\tau)$ and $\partial\phi(\tau)/\partial(4\pi\rho\tau)$ as a function of $4\pi\rho(\tau - \tau_x)$ when $x = 0.1$. It is found that $\partial[|a(\tau)|/(4\pi\rho)^2]/\partial(4\pi\rho\tau)$ decreases down to 0 when $4\pi\rho(\tau_p - \tau_{0.1}) = 3.7$. Here τ_p is the time when the efficiency and amplitude reach their peaks and the amplitude gain turns to negative.

Integration of $\partial|a(\tau)|/\partial(4\pi\rho\tau)$ from $4\pi\rho\tau_{0.1}$ to $4\pi\rho\tau_p$ yields

$$\int_{4\pi\rho\tau_{0.1}}^{4\pi\rho\tau_p}\frac{\partial|a(\tau)|}{\partial(4\pi\rho\tau)}d(4\pi\rho\tau) = 1.08(4\pi\rho)^2.$$

The peak amplitude is thus given by

$$|a(\tau_p)| = 1.18(4\pi\rho)^2, \quad (26)$$

which agrees well with the peak amplitude of the SASE in the steady-state regime obtained in a numerical calculation [5]. Equation (26) does not depend on x.

CONCLUSION

The phase space evolution of electrons in a SASE FEL operating in the linear regime before saturation where $|\Delta\psi_i(\tau)| \ll 1$ has been solved analytically from the 1D FEL equation. The evolutions of $\Delta\psi_i(\tau)$ and $\mu_i(\tau)$ are represented by sum of three independent analytical solutions similarly to the evolution of the SASE field; an exponentially growing term, an exponentially decaying term and an oscillating term. The distribution in a longitudinal phase space of $\Delta\psi_i(\tau)$ and $\mu_i(\tau)$ expands exponentially with time in size, rotating clockwise linearly in the high gain regime where $4\pi\rho\tau \geq 4$. These expansion in size and clockwise rotation corresponds to the exponential increase of the amplitude and linear increase of the phase of the radiation field, respectively. The microbunch center is located where $\Delta\psi_i(\tau) = 0$ and $\mu_i(\tau) < 0$, and the energy of the microbunch center decreases exponentially, which corresponds to the exponential increase of the SASE power. The analytical solutions hold until the optical amplitude grows greater than one-tenth of the amplitude in saturation. A numerical calculation which solves the 1D FEL equations together with initial values given by the present analytical solutions results in the peak amplitude in saturation $|a(\tau_p)| = 1.18(4\pi\rho)^2$, which agrees well with the conventional theories.

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