ANALYTIC ELECTRON TRAJECTORIES IN AN EXTREMELY RELATIVISTIC HELICAL WIGGLER: AN APPLICATION TO THE PROPOSED SLAC E166 EXPERIMENT

J. T. Donohue, Centre d'Etudes Nucléaires de Bordeaux-Gradignan, BP 120, 33175 Gradignan, France

Abstract

The proposed experiment SLAC E166 intends to generate circularly polarized gamma rays of energy 10 MeV by passing a 50 GeV electron beam through a meter long wiggler with approximately 400 periods. Using an analytic model formulated by Rullier and me, I present calculations of electron trajectories. At this extremely high energy the trajectories are described quite well by the model, and an extremely simple picture emerges, even for trajectories that that fail to encircle the axis of the wiggler. The calculations are successfully compared with standard numerical integration of the Lorentz force equations of motion. In addition, the calculation of the spectrum and angular distribution of the radiated photons is easily carried out.

PROPOSED EXPERIMENT AND ITS RELATION TO FELS

The experiment E-166 at SLAC proposes to produce circularly polarized photons of energy 10 MeV by sending a beam of electrons of energy 50 GeV through a helical undulator one meter long [1,2]. The wiggler period is 2.4 mm. The aim is to convert the polarized photons into longitudinally polarized positrons by pair production on a Ti target. Polarized leptons enable one to perform high precision tests of the Standard Model of weak and electromagnetic interactions, and the experiment is intended as a demonstration of principle. A key feature of the experiment is the helical wiggler Its properties, along with some beam properties are summarized in Table 1. Much information can be is found in the detailed report of Mikhailichenko [3], who designed the wiggler.

Table 1: Beam and Wiggler and Specifications

Energy	50 GeV
N_e/bunch	1×10^{10}
σ_x , σ_x	40 μm
$\gamma\varepsilon_x = \gamma\varepsilon_x$	3×10^{-5} m rad
Type	Helical
Period	0.24 cm
Length	1 m
Field on axis	0.76 T
Inner diameter	0.89 mm
Ω_w	0.1704
k_w	26.18 cm^{-1}

Although the wiggler is too short for any substantial bunching to occur, and consequently the system can't be classed as a Free Electron Laser (FEL), it does resemble many FELs that operated at much lower energy. . It also will be the highest energy photon source available, and may be seen as a test –bed for ultra high energy FELs. In particular, the FEL experiments performed at the CEA_CESTA facility [4] used helical wigglers and low energy (a few MeV) electron beams. Rullier and I developed a model to simulate the trajectories of the electrons in those experiments [5]. Our much earlier work on the analytic but approximate calculation of trajectories had been successful in describing trajectories in helical wigglers with an axial guide field [6], but was inapplicable in the absence of an axial field. A key advantage of our older approach is its ability to describe trajectories which don't encircle the axis of the wiggler. Our second approach is also capable of describing such trajectories. Now the radius of the SLAC E-166 beam (40 μm) is much greater than the radius of the ideal helical trajectory (0.665 nm), which implies that most electrons will be following trajectories that do not encircle the axis. While there is no major problem in calculating such trajectories by numerical integration of the Lorentz force equations of motion (I use both the NDSolve procedure in *Mathematica*, and the dsolve procedure in MAPLE), it is of interest to see what our model predicts. In fact, the conditions of the experiment are favorable, since the high energy limit of our model is quite simple. One can write simple closed form expressions for the position and velocity variables as functions of time.

An important experimental issue is the emission pattern of the radiation generated by the electrons during their passage through the wiggler. For the ideal helical trajectory, one may compute this most easily in the co-moving Lorentz frame, where the electron has only transverse motion. The resulting relative velocity is only 0.17, which means that the radiation is mainly at the fundamental frequency, with a small admixture of the second harmonic. Standard formula may be found, for example, in Jackson's book [7], where the problem 14.8 addresses the question (and provides the answer). A simple generalization furnishes the amplitudes for positive and negative helicity radiation. A straightforward Lorentz transformation then produces the angular distribution of the radiation in the laboratory frame. As might be expected the emission occurs mainly in a cone of half angle $1/\gamma_z$, where γ_z denotes the quantity

$\gamma_z = \left(1-\beta_z^{\,2}\right)^{-1/2}$ where β_z is the axial velocity of the electron in the laboratory. Both the photon energy and the polarization are highly correlated with the laboratory (lab) emission angle. The relations among the lab frequency ω, co-moving fundamental frequency $\tilde{\omega}_0$, on-axis wiggler field $\Omega_w = eB_w / mc^2 k_w$, lab emission angle θ, longitudinal rapidity y, and circular polarization P are:

$$\omega = \frac{\tilde{\omega}_0 e^y}{\cos^2\frac{\theta}{2} + e^{2y}\sin^2\frac{\theta}{2}}$$

$$\tilde{\omega}_0 = \frac{ck_w\gamma\beta_z}{\sqrt{1+\left(k_w\rho\gamma\beta_z\right)^2}} \cong \frac{ck_w\gamma\beta_z}{\sqrt{1+\Omega_w^{\,2}}}$$

$$\Omega_w = \frac{eB_w}{mc^2k_w} \cong 0.934 B_w\lambda_w$$

$$\beta_z = \tanh y$$

$$P \cong \frac{1-e^{2y}\tan^2\frac{\theta}{2}}{1+e^{2y}\tan^2\frac{\theta}{2}}$$

The relation between the radius of the ideal helix and the axial momentum is

$$\gamma\beta_z = 2\Omega_w I_1\left(k_w\rho\right)\left(1/\left(k_w\rho\right)^2+1\right) \cong \Omega_w / k_w\rho$$

In order to obtain both high photon polarization and energy, the emitted photons must make an angle of less than 5 μ radians with the axis. The conversion target is placed 10 m downstream from the wiggler, which means that its diameter must be about 100 μm .

THE MODEL

The model presented in ref. 5 is based on finding the fixed point of the Hamiltonian, making a series expansion keeping only quadratic terms, and then a transformation to two normal modes. The dynamics is simply that of two uncoupled harmonic oscillators. The squared frequencies of the oscillations were known from much earlier work; they may be found in the monograph of Freund and Antonen [8]. According to our model, the frequency of greater magnitude is positive, while, for zero axial field, that of lesser magnitude is negative. In fact, for small radius (In the remainder of this paper we shall use only dimensionless quantities with *mc* as the unit of momentum, $1/k_w$ the unit of length and ck_w the unit of frequency), one finds

$$\Omega_\pm \cong \pm\left(\beta_z\right)_f\left(1 \pm \tfrac{1}{\sqrt{2}}\rho_f - \tfrac{1}{2}\rho_f^{\,2}\right)$$

where ρ_f denotes the radius and $(\beta_z)_f$ the axial velocity at the fixed point. The dynamical variables are two conjugate coordinate and momentum pairs called (u,q) and (v,k). The key equation needed is the following, which is valid in the high energy limit:

$$x(t)+iy(t) = -\rho_f e^{iz(t)} + e^{i\left(z(t)-(\Omega_+-\Omega_-)t/2\right)} \times \left\{(u+iv)_0\cos\left(\frac{(\Omega_-+\Omega_+)t}{2}\right) + \frac{\sqrt{2}}{\Omega_w}(k+iq)_0\sin\left(\frac{(\Omega_-+\Omega_+)t}{2}\right)\right\}$$

where

$$(u+iv)_0 \equiv u(0)+iv(0)$$
$$= x(0)+iy(0)+\rho_f \cong x(0)+iy(0)$$

Similarly,

$$(k+iq)_0 \equiv k(0)+iq(0)$$
$$\cong \gamma\left(\beta_x(0)+i\beta_y(0)\right)+i\Omega_w$$

Since the quantity $(\Omega_-+\Omega_+)t/2 \simeq \rho_f z/\sqrt{2}$ is always small in a one-meter-long wiggler, one may replace the cosine by 1, and the sine by its argument. I find to a good approximation

$$x(t)+iy(t) \cong -\rho_f e^{iz(t)} + x(0)+iy(0)+z\left(\frac{\gamma\left(\beta_x(0)+i\beta_y(0)\right)+i\Omega_w}{\gamma\beta_z}\right)$$

This espression shows that the transverse position is a superposition of the FEL helix, with the period of the wiggler, an initial constant displacement, and a drift at constant velocity . The time derivative of this expression yields the transverse velocity,

$$\beta_x(t)+i\beta_y(t) \cong -i\beta_z\rho_f e^{iz(t)} + \beta_x(0)+i\beta_y(0)+i\Omega_w/\gamma$$

The resulting transverse motion is quite simple, consisting only of the ideal FEL helix and a constant drift. What turns out to be essential is matching the initial velocity as closely as possible to the ideal value,

$$\beta_x = 0, \quad \beta_y = -\Omega_w/\gamma$$

It will be essential to the success of the experiment that these conditions be realized as closely as possible.

COMPARISON WITH SIMULATION

In order to verify that the simple motion found above is indeed correct, we have calculated numerically some trajectories using the NDSolve package in *Mathematica*. The trajectories were calculated with high precision, and a typical 400 period trajectory took about 90 seconds on a PC.

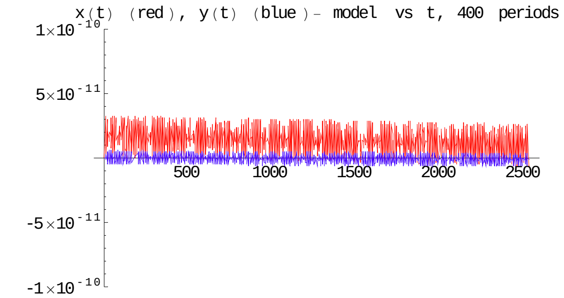

Figure 1. Difference between numerical solution and model keeping sine and cosine. Red is for $x(t)$, blue $y(t)$.

In figure 1 I show the differences between a numerically calculated trajectory and the model in which the sine and cosine are retained. The x-difference is shown in red, the y difference in blue. For this trajectory

$$x(0) = -1.7415\times10^{-6}, \qquad y(0) = 0.005,$$

$$\beta_x(0) = 1.7415\times10^{-8},\ \beta_z(0) = -1.7414\times10^{-6},$$

$$\beta_z(0) = 1 - 1.02\times10^{-10}, \qquad z = 0$$

For comparison the ideal injection would have $y(0) = 0$ and $\beta_x(0) = 0$. The initial y value is about 2 µm from the axis. In general the agreement between the model and the numerical calculation is excellent.

In Figure 2 I show the difference between the same numerical calculation and the simplified version of the model, in which the cosine is replaced by 1, and the sine by its argument.

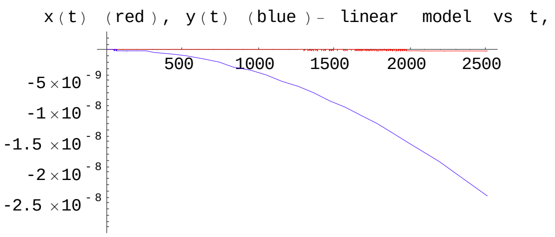

Figure 2. Difference between numerical solution and the simplified model with sine replaced by its argument and cosine by 1. Red is for $x(t)$, blue $y(t)$. Note change of scale compared to Figure 1.

Here the agreement remains surprisingly good for the x-coordinate, and slightly less so for the y-coordinate. However, the true value of the y-coordinate is approximately 0.005 throughout the trajectory, so the relative error remains small.

EFFECT ON RADIATION EMISSION

Without going into detail, one sees that for electrons that are injected with the ideal velocity, the resulting motion is basically the standard helix, except that it is not centered on the axis. For such trajectories, the radiation pattern is the same as the ideal one, except that the axis of the cone must be taken over the true axis of the helix. This necessarily leads to some “smearing” of the radiation pattern. A more serious problem occurs for those electrons whose injection velocity is not ideal. If the transverse drift is sufficiently large, the radiation pattern might be strongly affected. Further study is needed to settle this.

CONCLUSION

While we have not considered all the possible trajectories that are likely to occur in the proposed experiment, we are confident that the major part of them will be correctly described by our analytic approach. The even simpler approximation appears to be adequate for most purposes.

REFERENCES

1 http://www.slac.stanford.edu/exp/e166/

2 G. Moortgat-Pick, Eur. Phys. J. C 33 s1047 (2004).

3 A. A. Mikhailichenko, Report CBN 02-10, (2002). Available at site indicated in ref. [1].

4 T. Lefevre et al, Phys Rev. Lett. 1188 (2004).

5 J. T. Donohue, J. L. Rullier, Nucl. Instr. And Meth. A 507 (2003) 56.

6 J. T. Donohue, J. L. Rullier, Phys Rev. E 49 766 (1994).

7 J. D. Jackson Classical Electrodynamics, 2nd Edition, John Wiley & Sons, Inc. 1975.

8 H. P. Freund and T. M. Antonsen, Jr., Principles of Free-electron Lasers, 2nd Edition, Chapman and Hall, London .