OFF-AXIS ORBITS IN REALISTIC HELICAL WIGGLERS: FIXED POINTS AND TIME AVERAGED DYNAMICAL VARIABLES

J.T. Donohue, Centre d'Etudes Nucléaires de Bordeaux-Gradignan, BP 120, 33175 Gradignan, France
J. L. Rullier, CEA CESTA, BP 2, F-33114 Le Barp, France

Abstract

Many years ago Fajans, Kirkpatrick and Bekefi (FKB) studied off-axis orbits in a realistic helical wiggler, both experimentally and theoretically. They found that as the distance from the axis of symmetry to the guiding center increased, both the mean axial velocity and the precession frequency of the guiding center varied. They proposed a clever semi-empirical model which yielded an excellent description of both these variations. We point out that a approximate model proposed by us several years ago can be made to predict these delicate effects correctly, provided we extend our truncated quadratic Hamiltonian to include appropriate cubic and quartic terms. We develop an argument similar to the virial theorem to compare time averaged and fixed-point values of dynamical variables. Illustrative comparisons of our model with numerical calculation are presented.

INTRODUCTION

In 1985 Fajans, Kirkpatrick and Bekefi performed an experiment with a low-energy free electron laser (FEL) operating in the amplifier mode in the microwave region[1]. The wiggler was helical, and a uniform axial field was present. The electron beam was furnished by a pulse-line diode in single shot operation. Among other studies, they investigated what happened when the beam was injected off axis, simply by displacing their wiggler in the transverse direction. Both beam and FEL measurements were carried out. In general, they found that for small displacements, the FEL operation remained satisfactory. Two properties of the beam were measured quantitatively as a function of the off-axis injection distance. The mean axial velocity was observed to satisfy a simple quadratic law

$$\langle \beta_z(y) \rangle = \beta_z(0) + K_\beta y^2 + \cdots$$

where K_β is a number which depends on the FEL parameters, and y denotes the displacement of the beam centroid from the wiggler axis at injection. The symbol $\langle\ \rangle$ denotes the time average of the corresponding dynamical quantity Throughout this paper we shall use only dimensionless quantities with mc as the unit of momentum, $1/k_w$ the unit of length and ck_w the unit of frequency. A second important property was the precession of the displaced quasi-circular FEL orbits. Again, for small displacements, a simple quadratic behavior was found for the precession frequency ω_P.

$$\omega_P(y) = \omega_P(0) + K_P y^2 + \cdots$$

where K_P denotes another constant. Since the quantities $\beta_z(0)$ and $\omega_P(0)$ are just the values on the ideal orbit, they may be considered as known. The real task is to compute the quantities K_β and K_P .

The authors analyzed the wiggler magnetic field in detail, and proposed two formulas to describe the modification of the mean axial velocity and the precession frequency. The former is described by

$$\langle \beta_\perp \rangle = \frac{\langle \beta_z \rangle \Omega_w I_0(k_w y_g)\left(I_0(\lambda) - I_2(\lambda)\right)}{\gamma\langle \beta_z \rangle - \Omega_0 - 2\Omega_w I_0(k_w y_g) I_1(\lambda)}$$

$$\text{subject to } \gamma = \frac{1}{\sqrt{1+\langle \beta_z \rangle^2 + \langle \beta_\perp \rangle^2}}$$

where γ is the dimensionless energy. The quantities Ω_0 and Ω_w are eB_0/mc^2k_w and eB_w/mc^2k_w , respectively, y_g denotes the off-axis injection distance, and λ is $\pm k_w\rho$, where ρ is the radius of the FEL motion. A somewhat simpler expression had been proposed by Freund and Ganguly [2],

$$\langle \beta_\perp \rangle = \frac{\langle \beta_z \rangle \Omega_w I_0(k_w y_g)}{\gamma\langle \beta_z \rangle - \Omega_0}$$

$$\text{subject to} \qquad \gamma = \frac{1}{\sqrt{1+\langle \beta_z \rangle^2 + \langle \beta_\perp \rangle^2}}$$

For the precession, FKB proposed the formula

$$\omega_p = ck_w \frac{\langle \beta_\perp \rangle \Omega_w \left(I_0(k_w y_g) - I_2(k_w y_g)\right)}{\Omega_0 - 2\Omega_w I_0(k_w y_g) I_1(\lambda)}.$$

In the three cases investigated by FKB, their formula were remarkably successful. That of Freund and Ganguly for the diminution of the mean axial speed was somewhat less precise, but adequate.

Given the success of these formulas at describing the data, one might well consider the problem solved. However, having proposed an analytic (but approximate) method of calculating the trajectories in a helical wiggler with axial guide field [3], we felt challenged to show that our model could be used to generate equally successful expressions, perhaps in a more systematic way. We don't know the details of the adiabatic magnetic field used by FKB to inject into the wiggler, so we address a related but slightly different problem. Suppose that an electron is on the ideal axially centered helical trajectory, and then displace the electron by a small amount in a transverse direction, leaving its velocity vector unchanged. We remind the reader that the ideal orbit satisfies two conditions:

$$\psi = \phi - z = \begin{pmatrix} 0 \;\; \text{Group II} \\ \pi \;\; \text{Group I} \end{pmatrix},$$

$$\gamma\beta_z = \Omega_0 - 2\Omega_w I_1(\rho)\left(\frac{1}{\rho^2}+1\right)\cos\psi,$$

where ρ denotes the constant radius of the helix. The notation Group I corresponds to $\gamma\beta_z > \Omega_0$ and Group II to $\gamma\beta_z < \Omega_0$. There is also the reversed field configuration, studied by Conde and Bekefi [4], where the axial velocity is anti-parallel to the axial field, or $\gamma\beta_z\Omega_0 < 0$. If one chooses the z-direction such that $\Omega_0 > 0$, then the reversed field is a special case of Group II, typically with a very small radius. The effect of a small displacement is then calculated by linearizing the equations of motion around the ideal helix. This procedure is described in detail in the monograph of Freund and Antonsen [5]. The electron then has two independent normal modes of oscillation, whose frequencies are well-known. It turns out that one of these two frequencies is numerically close to the unperturbed constant axial velocity β_z.

In our approach to calculating the trajectories, the key role is played by the Helical Invariant, P_z, a conserved quantity which is a consequence of the screw symmetry of the wiggler field [6]. We find the fixed point of the Hamiltonian (where the first derivatives with respect to our chosen dynamical variables vanish), expand to second order in our variables, and then find the normal modes of oscillation of the resulting quadratic system. If we denote the complex normal mode amplitudes in our model by A_+ and A_-, with the Poisson brackets $\{A_\alpha, A_\beta^*\} = i\delta_{\alpha\beta}$, our Hamiltonian may be written as

$$H = H_{fp} + \Omega_+ |A_+|^2 + \Omega_- |A_-|^2 + O(A^3).$$

Neglecting the cubic and higher order terms, we find the simple dynamics,

$$A_\alpha(t) = A_\alpha(0)e^{i\Omega_\alpha t}.$$

With our rather complicated choice of dynamical variables the transformation to normal modes was straightforward, and we obtained a quantitatively accurate description of the transverse motion as a Ptolemaic superposition of three independent circular motions. One is the projection of the standard FEL helix, the second, driven mainly by the mismatch in transverse velocity, occurs at a frequency near the relativistic cyclotron field Ω_0/γ, while the third is a very slow motion, whose effective frequency is $\langle\beta_z\rangle - \Omega_\beta$, where Ω_β denotes that oscillation frequency which is close to β_z. It is this slow motion that is the precession seen by FKB.

In terms of our model, the calculation of the effects observed by FKB is straightforward. Displacing the electron from the ideal helical orbit produces a change in the Helical Invariant that is second order in the displacement. The resulting changes in the axial velocity and frequency are readily computed, and can be compared to experiment. Proceeding in this way, we find extremely **poor** agreement between our calculations and experiment.

TIME AVERAGES AND FIXED POINTS

The difficulty encountered in attempting to calculate the mean axial velocity was traced to the source, the fact that the time average of a dynamical variable in the neighborhood of a fixed point is not its value at the fixed point. The time averages of our normal modes of oscillation, which we had assumed to be zero, were in fact different from zero. If our quadratic approximation to the Hamiltonian were exact, this would not occur. However, the cubic terms in the Hamiltonian generate such non-zero time average values. While the inclusion of the cubic terms into our model makes it non-soluble, it is possible by using a perturbation approach to obtain the lowest order corrections by calculating third derivatives at the fixed point. The required labor is greatly facilitated by using symbolic manipulators such as MAPLE or *Mathematica*. If one is interested in calculating the dependence of the precession frequency on the displacement, some fourth derivatives at the fixed point are also needed. We present below a sketch of our method.

The essential tool in our approach is a proposition similar to the virial theorem in classical mechanics. Given a general Hamiltonian, and a complex dynamical variable $A(t)$ of the sort we use, we consider the time average of the following quantity

$$\left\langle \frac{d}{dt}\left(A(t)e^{i\omega t}\right)\right\rangle = \lim_{T\to\infty}\frac{A(T)e^{i\omega T} - Q(0)}{T} = 0$$

provided the variable is $A(t)$ bounded. But by Hamilton's equations

$$\left\langle \frac{d}{dt}\left(A(t)e^{i\omega t}\right)\right\rangle = i\left\langle\left(\omega A(t) + \frac{\partial H}{\partial A^*}\right)e^{i\omega t}\right\rangle.$$

For technical reasons, it is more convenient to compute the higher order terms using the squared Hamiltonian, and we arrive the following result

$$\left(\omega+\Omega_\alpha\right)\left\langle A_\alpha(t)e^{i\omega t}\right\rangle=-\frac{1}{2H}\left\langle\frac{\partial\left(H^2\right)_{cubic}}{\partial A_\alpha^*}e^{i\omega t}\right\rangle$$

where represents the cubic and higher order terms in the multi-variable Taylor series expansion of the squared Hamiltonian at the fixed point. For our purposes, we keep only the cubic terms, and find non-vanishing contributions only for the nine following values of the arbitrary frequency ω,

$$\omega_j=\left\{0,\pm2\Omega_+\pm2\Omega_-,\left(\Omega_+\pm\Omega_-\right),-\left(\Omega_+\pm\Omega_-\right)\right\}.$$

This means we may write, correct to second order in the displacement,

$$A_+(t)=A_+(0)e^{i\Omega_+t}+\sum_{j=1}^{9}a_{j+}e^{-i\omega_j t}$$

with a similar expression for $A_-(t)$. Here the quantities a_{j+} are second order in the displacement. We find

$$\left(\omega_j+\Omega_+\right)a_{j+}=\frac{-1}{2H}\left\langle\frac{\partial\left(H^2\right)_{cubic}}{\partial A_\alpha^*}e^{i\omega_j t}\right\rangle_{A(t)=A(0)e^{i\Omega t}}$$

where on the right-hand-side only the first approximation to the variables is to be used.

HIGHER ORDER TERMS

In order to compute the quantities a_{j+}, we need the cubic part of the squared Hamiltonian, which we write as

$$H^2_{cubic}=-iK_+\left|A_+^2\right|\left(A_+-cc\right)-iL_+\left|A_-^2\right|\left(A_+-cc\right)-iR_+\left(A_+^3-cc\right)$$
$$-iS_+\left(A_+^2A_--cc\right)-iT_+\left(A_+^2A_-^*-cc\right)+\left(+\leftrightarrow-\right)$$

where cc denotes complex conjugate. The ten quantities K_+ etc. may be computed most easily if we write the squared Hamiltonian in cylindrical coordinates, as in ref. [7],

$$H\left(\rho,\psi,z,p_\rho,p_\psi,P_z\right)=$$
$$\sqrt{1+\left(p_\rho+A_\rho(\rho,\psi)\right)^2+\left(\frac{p_\psi}{\rho}+A_\phi(\rho,\psi)\right)^2+\left(P_z-p_\psi\right)^2}$$

The complex dynamical variables A_+ and A_- are linearly related to the usual variables,

$$A_\pm=a_\pm p_\rho+ib_\pm\left(\rho-\rho_f\right)+ic_\pm\left(p_\psi-\left(p_\psi\right)_f\right)+d_\pm\left(\psi-\psi_f\right)$$
$$p_\psi=\left(p_\psi\right)_f+id_+\left(A_+-cc\right)+id_-\left(A_--cc\right)$$

where ρ_f, ψ_f and $\left(p_\psi\right)_f$ denote the values of the variables at the fixed point. The time averaged axial velocity is then

$$\left\langle\beta_z\right\rangle=\frac{P_z-\left\langle p_\psi\right\rangle}{H}$$
$$=\frac{P_z-\left(p_\psi\right)_f}{H}-\frac{2i\left(d_+\left\langle A_+\right\rangle+d_-\left\langle A_-\right\rangle\right)}{H}$$

Explicit calculation yields

$$\left\langle A_+\right\rangle=\frac{-i\left(2K_+\left|A_+^2\right|+L_+\left|A_-^2\right|\right)}{2\Omega_+H}$$

and similarly for $\left\langle A_-\right\rangle$. In these expressions the amplitudes are linear in the displacement y, and we thus can calculate the coefficient of y^2 in $\left\langle\beta_z\right\rangle$, K_β.

CHANGES IN TIME AVERAGED FREQUENCIES

In order to calculate the slope of the FKB precession frequency, we must calculate the change in the time average of the oscillation frequencies caused by the higher order terms in the Hamiltonian. The relevant equation is

$$\left\langle\Omega_+\right\rangle=\left\langle\frac{d\Im\ln A_+}{dt}\right\rangle=\frac{1}{4H}\left\langle\frac{\partial H^2}{A_+\partial A_+^*}+cc\right\rangle.$$

This receives contributions from both the cubic part and from three of the many quartic contributions. These may be written as

$$H^2_{quartic}=M_+\left|A_+^4\right|+M_-\left|A_-^4\right|+N\left|A_+^2\right|\left|A_-^2\right|+\cdots$$

The details of the explicit calculation are too long to be given in this paper, and we give only the final result:

$$\left\langle\Omega_+\right\rangle=\Omega_++C_{++}\left|A_+\right|^2+C_{+-}\left|A_-\right|^2,$$

$$C_{++}=\frac{M_+}{H}-\frac{1}{2H^2}\left(\frac{3\left(K_+^2+R_+^2\right)}{\Omega_+}+\frac{L_-^2}{\Omega_-}+\frac{S_+^2}{2\Omega_++\Omega_-}-\frac{T_+^2}{2\Omega_+-\Omega_-}\right),$$

$$C_{+-}=\frac{N}{2H}-\frac{1}{H^2}\left(\frac{K_+L_+}{\Omega_+}+\frac{K_-L_-}{\Omega_-}\right)-\frac{1}{H^2}\left(\frac{S_+^2}{2\Omega_++\Omega_-}+\frac{T_+^2}{2\Omega_+-\Omega_-}+\frac{S_-^2}{2\Omega_-+\Omega_+}+\frac{T_-^2}{2\Omega_--\Omega_+}\right),$$

For the quantity $\left\langle\Omega_-\right\rangle$ a similar expression holds provided one makes the substitution $+\leftrightarrow-$ throughout. Note that $C_{+-}=C_{-+}$.

COMPARISON WITH NUMERICAL SIMULATIONS

At this point we can calculate K_β and K_P. However, in order to verify this analysis, we carried out numerical calculations of the trajectories using solvers of differential equations available on MAPLE and Mathematica. Our approach is simple. We start with an electron on the ideal trajectory suited to its energy. We then displace the position of the electron tangentially through a distance y, keeping the velocity unchanged. We then calculate the trajectory of the electron for long times, typically 100 periods, in order to obtain reliable numerical estimations of the time averaged axial velocity and the precession frequency. We performed this calculation for the three cases studied by FKB. In each of these, the frequency concerned by the precession was Ω_+ so we added two more, for which the relevant frequency was Ω_-. These include a Group II configuration with a large axial field, which we label low-ρ, and a reversed field configuration, of the sort investigated by Conde and Belkefi. The results are summarized in the Table, which indicates various properties of the trajectory, the frequencies, the precession frequency intercept $\omega_P(0)$ (in the FKB units of $10^8 s^{-1}$). The last eight lines show comparisons of our theory (denoted by T) with the simulations (S). The agreement between these is quite good.

	FKB a group I	FKB b group II	FKB c group II	low ρ, Group II	reversed $\beta_z B_0 < 0$
B_0 (T)	0.16	1.312	0.4	1.8	1
B_w (T)	0.025	0.05833	0.025	0.063	0.147
V (MeV)	0.16863	1.2264	0.16863	0.750	0.750
λ_w (cm)	3.30	3	3.30	3.18	3.18
k_w (cm^{-1})	1.904	2.094	1.904	1.97584	1.97584
γ	1.33	3.4	1.33	2.46771	2.46771
ρ_f	0.22461	0.31543	0.21602	0.06073	0.08433
$(\beta_z)_f$	0.64327	0.911498	0.644436	0.912532	-0.9110
ψ_f	π	0	0	0	0
Ω_+	0.65416	0.90481	0.640016	1.25171	2.10919
Ω_-	-0.23792	0.19576	0.286011	0.91155	-0.90526
$\omega_P(0)$	-6.217	4.199	2.523	0.5812	-3.389
d_+	-0.00110	-0.00304	-0.00230	-0.0991	-0.1018
d_-	-0.11976	-0.38085	-0.16418	0.00047	-0.0008
$\frac{\langle \Im A_+ \rangle}{y^2}$ T	-1.580	-3.290	-2.793	-13.46	-7.133
$\frac{\langle \Im A_+ \rangle}{y^2}$ S	-1.575	-3.292	-2.795	-13.15	-7.128
$\frac{\langle \Im A_- \rangle}{y^2}$ T	-1.085	-1.246	-1.525	-20.68	-10.69
$\frac{\langle \Im A_- \rangle}{y^2}$ S	-1.085	-1.247	-1.524	-20.88	-10.76
K_β T	-0.00952	-0.0165	-0.0076	-0.00094	0.0017
K_β S	-0.0095	-0.0165	-0.0076	-0.00094	0.0017
K_P T	-2.4868	1.3272	0.89201	0.2112	0.07146
K_P S	-2.56	1.31	0.88	0.20	0.072

TABLE

Comparison of theoretical (T) and simulation (S) values for various quantities. Five different configuration were studied FKB a, b and c, low ρ Group II, and reversed field.

CONCLUSION

We may thus conclude that our approach of calculating the contributions of the higher perturbatively is successful. However, one word of caution is necessary. The generally small values we find for K_P, especially in columns 5 and 6, are the result of cancellations of much greater changes in the separate pieces. Indeed, the change in the fixed point frequency due to the displacement is almost exactly cancelled by the contribution of the higher order cubic and quartic terms. This suggests that in a yet more sophisticated approach such cancellations could be avoided.

REFERENCES

1 J. Fajans, D.A. Kirkpatrick, and G. Bekefi, Phys. Rev. A 32 (1985) 3488.
2 H. P. Freund, A. K. Ganguly, IEEE J. Quantum Electron., **QE-21** (1985) 1073
3 J. T. Donohue, J.L. Rullier, Phys. Rev. E 49 (1994) 766.
4 M. E. Conde, G. Bekefi, Phys. Rev. Lett. 67, (1991) 3082.
5 H. P. Freund and T. M. Antonsen, Jr., *Principles of Free-electron Lasers*, 2nd Edition, Chapman and Hall, London .
6 R. D. Jones, Phys. Fluids 24 (1981) 564.
7 J. T. Donohue, J. L. Rullier, Nucl. Instr. And Meth. A 507 (2003) 56.