BENCHMARK OF ASTRA WITH ANALYTICAL SOLUTION FOR THE LONGITUDINAL PLASMA OSCILLATION PROBLEM

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Abstract

During the design of X-FELs, space-charge codes are required to simulate the evolution of longitudinal plasma oscillation within an electron beam in connection with LSC microbunching instability [1] and certain pump-probe synchronization schemes [2]. In the paper [3] we presented an analytical solution to the initial value problem for longitudinal plasma oscillation in an electron beam. Such a result, besides its theoretical importance, allows one to benchmark space-charge simulation programs against a self-consistent solution of the evolution problem. In this paper we present a comparison between our results [3] and the outcomes of the simulation code *ASTRA*.

INTRODUCTION

Start-to-end simulations of X-FEL systems require the use of several codes, able to deal with different parts of the setup. In particular, at low energies, simulations must account for space-charge interactions. These codes are also used in connection with LSC microbunching instability problems [1] and certain pump-probe synchronization schemes [2]: benchmarking them correctly is of critical importance since it is the only way to build up confidence in their outcomes.

In this paper we present first results of an ongoing benchmark of the space-charge code *ASTRA*. Our work takes advantage of the first three-dimensional, analytical model for the description of plasma oscillations within a relativistic electron beam, which has been described in [3] and [4]. This is the first time that such an analytical, self-consistent model is used to benchmark space-charge codes, which are usually compared with solutions of the Poisson equation for specific charge distributions. We claim that our method should be considered as a standard for benchmarking these codes from now on. In the following we describe how the simulation is set up, how it is compared with analytical results and we report the outcomes of this comparison as regards the estimation of a typical evolution parameter.

ANALYTICAL STUDY

We consider a radially uniform electron beam with transverse dimension $r_0 = 1.0$ mm and length $l_b = 2.2$ cm propagating with kinetic energy $E_k = 6.0$ MeV and current $I = 44.97$ A. Longitudinal and transverse beam emittances are set to zero. A $\rho = 5\%$ density modulation is considered at a wavelength $\lambda_m = 1$ mm in the longitudinal direction. This modulation is uniform in the transverse direction.

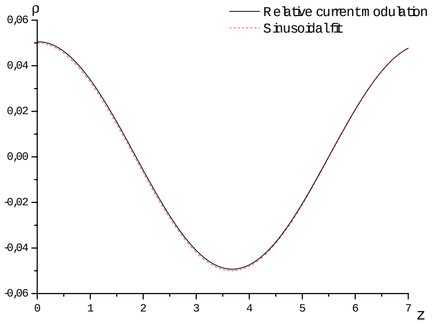

Figure 1: Relative current modulation level as a function of position inside the beam line (in m) and sinusoidal fit - analytical results. Parameters are given in the text.

Our analytical theory [4] deals with the longitudinal dynamics of the system, so it is applicable, roughly speaking, when the beam does not evolve, transversely, during the characteristic plasma oscillation length λ_p. This can be implemented in a simulation by focusing the beam using as we did, for instance, a longitudinal magnetic field B_z (compare with the next Section).

In our theory we describe the system evolution equivalently in terms of fields or currents. The former description is suitable, as has been remarked in [4] for numerical manipulations, while the latter gives us back a fully analytical model: the results out of these descriptions, of course, coincide. Our theory provides the longitudinal field $E_z(z,r)$ and current density $J_z(z,r)$ as a function of the longitudinal z and radial position r. As a first investigation, nevertheless, we were interested in how these quantities, averaged over the transverse beam distribution, compare with the analogous ones from *ASTRA*. Averaging J_z over the transverse distribution of the beam we get back the beam current $I(z) = \int_0^{r_0} J_z r dr$, while integrating E_z over z and averaging as for J_z we get back the momentum $p_z(z) = -2e/(cr_0^2) \int_0^r (\int_0^z E_z dz') r dr$.

In our theory the fields are presented as a superposition of propagating eigenmodes. The parameters described above correspond to a region where the one-dimensional asymptotic treatment of plasma oscillation is not applicable, and care was taken to choose them such that the

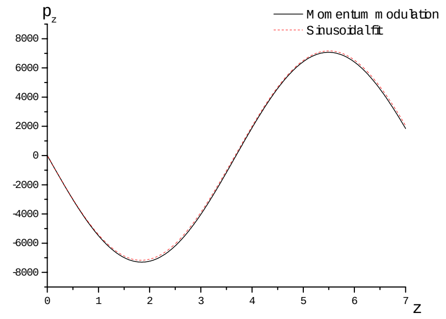

Figure 2: Momentum modulation (in eV/c) as a function of position inside the beam line (in m) and sinusoidal fit - analytical results. Parameters are given in the text.

first eigenmode is dominant with respect to the others. In this case the beam current and the longitudinal momentum, considered as functions of z, are nearly sinusoidal and therefore can be easily fitted [1] giving back, in both cases, the same estimation for the wavelength of plasma oscillations, $\lambda_p = 7.34$ m. In Fig. 1 we present the modulation level as a function of z, while in Fig. 2 we plot $p_z(z)$ as from our theory. Both figures also include sinusoidal fits. An alternative estimation of the plasma wavelength can be obtained, in the approximation when only the first eigenfunction is important, using the following equation (compare with [4]):

$$\lambda_p = \left[4I/(I_A r_0^2 \gamma \gamma_z^2)\right]^{1/2} / \Lambda_0 \simeq 7.31\mathrm{m}\ . \qquad (1)$$

Here $I_A = mc^3/e$ is the Alfven current, $\gamma = \gamma_z = 12.7417$ corresponds to a kinetic energy of 6 MeV and the first eigenvalue $\Lambda_0 = 0.3804$ is obtained from our analytical method: note that Λ_0 is sensibly different from unity, which is a clear signature of the fact that the one-dimensional theory cannot be applied.

Together with the benchmark for the full program, one can obtain, as a sort of by-product, a separate benchmark for the electromagnetic solver of *ASTRA* alone. In fact, at a short distance $z \ll \lambda_p$ the beam has not yet appreciably evolved so that E_z is the solution to the electrostatic problem of the initial charge distribution. Our analytical technique gives, at $z = 0.1$ m, an electric field on axis ($r = 0$) $E_z = 7235$ V/m. We checked that one can recover roughly the same value (with an accuracy of order 10^{-3}) from the well-known expression for the impedance per unit length of drift:

$$Z = \frac{4}{k_m r_0^2}\left[1 - \frac{k_m r_0}{\gamma} K_1\left(\frac{k_m r_0}{\gamma}\right)\right]\ , \qquad (2)$$

where $k_m = 2\pi/\lambda_m$ and K_1 is the modified Bessel function of the second kind of order 1. In fact $(I/I_A)Z m_e =$ 7242 V/m, m_e being the electron mass in eV.

[1] For this purpose we used the program *ORIGIN*.

The maximum field on axis, multiplied by $ez/c = 0.1$ e m/c is the maximum momentum modulation at $z = 10$ cm, which may be compared with *ASTRA* results. Indeed, we preferred to average $E_z(z, r)$ at $z = 10$ cm over the transverse beam distribution and compare with the analogous averaged value of p_z from *ASTRA*.

ASTRA SIMULATION

For our purposes we used a modified version of *ASTRA* [5] using dynamic memory allocation, which allowed us to set the number of macroparticles $N_{ptc} = 12 \cdot 10^6$ and we used the customary *Generator* program to produce a uniform, cold beam with no modulation. No particular option for noise reduction was used. A 5% modulation was then subsequently introduced by rearranging the z positions of the particles using a short code written $ad\ hoc$: this allowed to overcome numerical noise problems in the z direction, related with the *Generator* program, but not with *ASTRA* itself. A solenoidal field $B_z = 7$ T has been used to decouple the longitudinal from the transverse dynamics. The question obviously arises if the transverse particle motion is truly negligible. Because of B_z, particles undergo small circular trajectories in the transverse plane with Larmor frequency $\omega_L = qB_z/(\gamma m_e)$, q being the macroparticle charge. The radius of these circles are related to the transverse velocities of the particles which are altered though, during the beam evolution, by the transverse space charge force. A quick estimation shows that the particle-tracker maximal integration step (which was set to 10 ps, corresponding to about 3 mm) is too long to sample the Larmor motion correctly. Yet, if the effect of Larmor motion is negligible at any point in the z direction, this fact is not of concern. We checked that this is the case analyzing the transverse rms angular spread as a function of the z-position, and the maximal transverse momentum also as a function of the z-position. We set the beam line length $L = 7$ m, which is about one full plasma oscillation length, according to our analytical estimations. A simple and conservative estimate shows that, at $z = L = 7$ m, every particle moved, transversely, $6.5 \cdot 10^{-5}$ m $at\ most$. This distance should be compared with the smaller transverse dimension typical of our system. *ASTRA* divides the beam in several radial slices for the calculation of the space-charge interactions. In our simulations we used 10 radial rings, with a difference of a factor 2 in thickness between the outer and the inner ring. Given the beam dimension $r_0 = 1.0$ mm we have that throughout all the evolution a particle can move, at most, the dimension of the thinner ring. Moreover, one should account for the fact that the transverse motion of the particles introduce a small change in the longitudinal projected motion of order $(p_\perp/p_z)^2$; however this can be estimated to be less than 10^{-10} or, on a length of 7 m, less than 1 nm, which is completely negligible.

The previous discussion allows us to consider the transverse particle motion negligible and to compare *ASTRA*

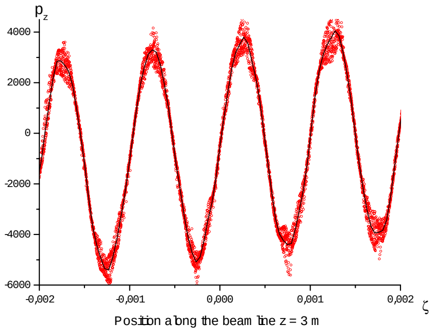

Figure 3: Particle longitudinal momenta p_z (in eV/c) as a function of the position inside the beam (in m) at 3 m inside the beam line. The solid line represents the momentum averaged over r. $N_{bin} = 400$.

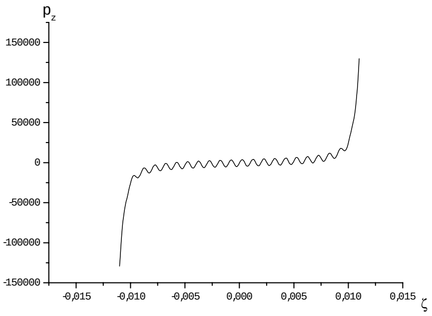

Figure 4: Average particle longitudinal momentum p_z (in eV/c) as a function of the position inside the beam (in m) at 3 m inside the beam line. $N_{bin} = 400$.

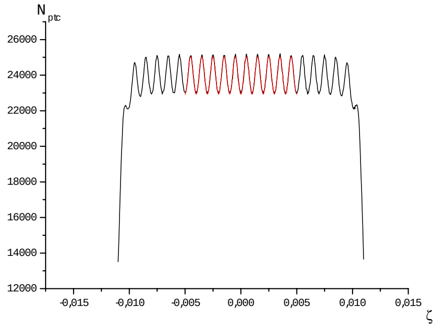

Figure 5: Particle density distribution normalized to N_{ptc} as a function of the position inside the beam (in m) at 3 m inside the beam line. $N_{bin} = 400$.

results with our analytical results.

In order to calculate the space-charge fields, *ASTRA* divides the beam also in a number N_{bin} of longitudinal slices, or bins: this is a critical parameter which should be studied to obtain correct results. A right estimation of the fields will occur only when a sufficiently large number of bins is chosen: considering a modulation wavelength λ_m, an intuitive guideline is that the number of bins per radiant should be larger than unity. At the same time, the number of particles in each bin should be large. We ran several *ASTRA* simulations with different longitudinal bins. The maximum number of bins was $N_{bin} = 800$, for a 2.2 cm-long bunch modulated at $\lambda_m = 1$ mm, which means that, when $N_{ptc} = 12 \cdot 10^6$, about 10^3 particles can be expected, roughly, to be present in each longitudinal slice for each radial ring.

The *ASTRA* output file consists of several information, among which the phase-space state of every particle. To give an example, in Fig. 3 we plot the particles longitudinal momentum p_z as a function of the longitudinal coordinate inside the bunch at 3 m down the beam line for $N_{bin} = 400$ [2]. We wrote a short code to get the average p_z and position (see the solid line in Fig. 3) of particles as a function of the longitudinal coordinate from *ASTRA* output: this program divides the bunch length (2.2 cm) in 500 parts, then the raw data from the simulation is read by the code and the average momentum and position is calculated for each slice. Results for the same example case in Fig. 3 are shown in Fig. 4 and Fig. 5.

The data in Fig. 4 and Fig. 5 and their analogous for every simulation run at any position down the beam line have been subsequently fit. The particle momenta have been fit with a tilted sinusoidal function $A + B\sin(2\pi z/\lambda_m) + Cz$, where the tilting is due to the fact that the tail of the bunch is decelerated, while the head is accelerated by the space-charge fields. The particle density distribution, instead, has been fitted using $A + B\sin(2\pi z/\lambda_m + \pi/2)$, B/A giving the relative modulation level. Only the inner 10 periods have been used in the fits (nearer the bunch center, in the longitudinal direction) since, as the bunch progresses through the beam line, spurious edge effects get more and more important.

For every *ASTRA* run (with a fixed number of longitudinal slices N_{bin}) we could plot the current modulation level and the absolute energy modulation and fit them, as before, using a sinusoidal function.

The results in the case of $N_{bin} = 400$ are shown in Fig. 6 and Fig. 7, from which we have two estimation for the plasma wavelength, $\lambda_p = 7.32$ m and $\lambda_p = 7.44$ m respectively, which differ of about 1.6%.

As said before several *ASTRA* runs were performed with different numbers of longitudinal slices N_{bin}. This allowed us to plot the plasma wavelength as a function of

[2] actually only one particle in 500 was selected for the plot, in order to make the figure file more easily manageable.

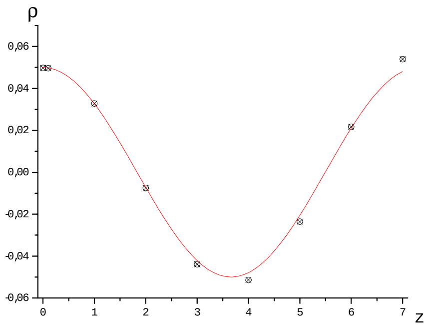

Figure 6: Averaged relative current modulation level as a function of the position in the beam line (in m) and sinusoidal fit. $N_{bin} = 400$.

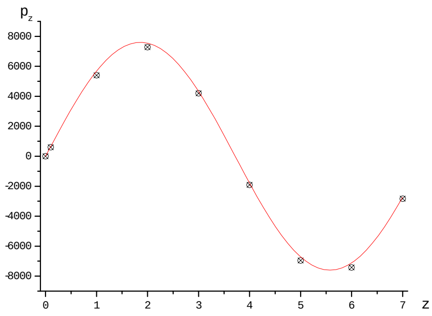

Figure 7: Average momentum modulation (in eV/c) as a function of the position in the beam line (in m) and sinusoidal fit. $N_{bin} = 400$.

the number of bins per modulation wavelength and to compare the results with analytical estimations. This result is presented in Fig. 8. As one can see at least 20 bins per wavelength are required in order to obtain acceptable results. Finally, to benchmark separately *ASTRA* electromagnetic solver, we plotted the electromagnetic fields estimated from the average momenta at $z = 10$ cm, as a function number of bins per wavelength and we compared these data with our analytical expectations. The result is shown in Fig. 9. It can be seen that results in Fig. 8 and Fig. 9 are in agreement: as the number of bins decreases, the fields are underestimated and the plasma wavelength is then badly overestimated.

CONCLUSIONS

We benchmarked *ASTRA* simulation by means of an analytical, self-consistent three-dimensional model developed by us in [4]. Results show a very good agreement between analysis and simulations, provided that critical parameters like the number of longitudinal bins for space-charge calculations are chosen correctly.

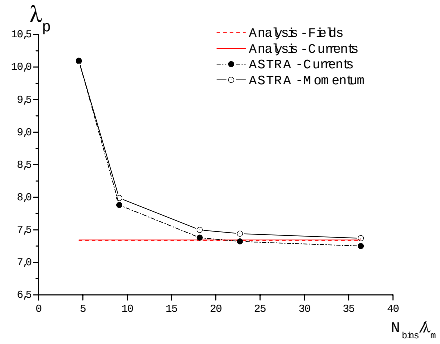

Figure 8: Benchmark of ASTRA - self-consistent problem: comparison between λ_p (in m) as a function of N_{bin}/λ_m calculated by ASTRA and with our analytical method.

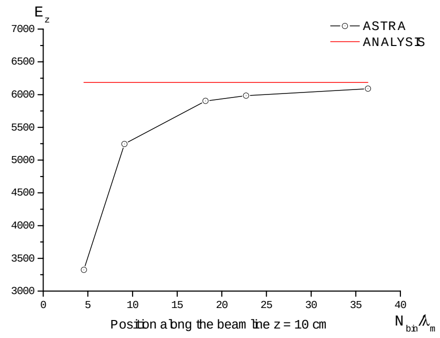

Figure 9: Benchmark of ASTRA - electromagnetic problem: comparison between the fields at 10 cm down the beam line (in V/m) as a function of N_{bin}/λ_m calculated by ASTRA and with our analytical method.

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