BACKWARD WAVE EXCITATION AND GENERATION OF OSCILLATIONS IN DISTRIBUTED GAIN MEDIA AND FREE-ELECTRON LASERS IN THE ABSENCE OF FEEDBACK

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Abstract

Quantum and free-electron lasers (FELs) are based on distributed interactions between electromagnetic radiation and gain media. In an amplifier configuration, a forward wave is amplified while propagating in a polarized medium. Formulating a coupled mode theory for excitation of both forward and backward waves, we identify conditions for phase matching, leading to efficient excitation of backward wave without any mechanism of feedback or resonator assembly. The excitations of incident and reflected waves are described by a set of coupled differential equations expressed in the frequency domain. The induced polarization is given in terms of an electronic susceptibility tensor. In quantum lasers the interaction is described by two first order differential equations, while in high-gain free-electron lasers, the differential equations are of the third order each. Analytical solutions of reflectance and transmittance for both quantum lasers and FELs are presented. It is found that when the solutions become infinite, the device operates as an oscillator, producing radiation at the output with no field at its input, entirely without any localized or distributed feedback.

INTRODUCTION

Conventional (quantum) lasers, microwave tubes and free-electron lasers (FELs) are based on distributed interactions between electromagnetic radiation and gain media. When such devices are operating in an amplifier configuration, a forward wave is amplified while propagating in a polarized medium, in a stimulated emission process [1]. In an oscillator configuration a resonator [2]-[4] or a distributed feedback [5] are employed to circulate the radiation, which is excited and amplified by the gain medium. If the single-pass gain is higher than the total losses, the radiation intensity inside the cavity increases and becomes more coherent. After several round trips, the radiation is built up until arriving at the nonlinear regime and saturation.

In this paper we suggest a mechanism of generation of laser oscillations, without any feedback means. It is shown that under conditions of phase-matching, both forward and backward waves can be excited in a distributed gain medium as illustrated schematically in Figure 1. The excitation of incident and reflected waves is described by a set of two differential equations coupled by the induced polarization of the gain media. The coupling coefficient is given in terms of the electronic susceptibility tensor.

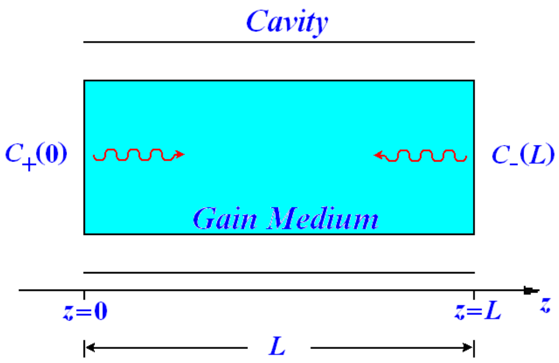

Figure 1: Schematic illustration of incident and reflected waves in a distributed gain medium.

Two cases are discussed: In quantum lasers, which are characterized by isotropic, homogeneous gain media, the interaction is described by two first order differential equations. In high-gain free-electron lasers, where the susceptibility is space dependent, the set includes two differential equations of the third order each. The coupled equations sets are solved analytically for both cases. Oscillation conditions are identified from the derived reflectance and transmittance coefficients.

EXCITATION OF FORWARD AND BACKWARD MODES

The total electromagnetic field is given by the time harmonic wave vector:

$$\mathbf{E}(\mathbf{r},t) = \Re\left\{\widetilde{\mathbf{E}}(\mathbf{r})e^{-j\omega t}\right\} \tag{1}$$

where $\widetilde{\mathbf{E}}(\mathbf{r})$ is the phasor of the wave oscillating at an angular frequency ω. The vector $\mathbf{r}$ stands for the (x,y,z) coordinates, where (x,y) are the transverse coordinates and

z is the axis of propagation. In the case of excitation of forward and backward modes, the phasor can be written as the sum [6]:

$$\widetilde{\mathbf{E}}(\mathbf{r}) = \left[C_+(z)\, e^{+jk_z z} \;+\; C_-(z)\, e^{-jk_z z}\right] \widetilde{\mathcal{E}}(x,y) \quad (2)$$

$C_+(z)$ and $C_-(z)$ are scalar amplitudes of forward and backward modes respectively, with profile $\widetilde{\mathcal{E}}(x,y)$ and axial wavenumber k_z. The evolution of the amplitudes of the excited modes is described by a set of two coupled differential equations:

$$\frac{d}{dz}C_\pm(z) = \mp\frac{1}{2\mathcal{N}}e^{\mp jk_z z}\int\!\!\int \widetilde{\mathbf{J}}(\mathbf{r})\cdot\widetilde{\mathcal{E}}^*(x,y)\,dx\,dy \quad (3)$$

The normalization of the mode amplitudes is made via the complex Poynting vector power:

$$\mathcal{N} = \int\!\!\int \left[\widetilde{\mathcal{E}}_\perp(x,y)\times\widetilde{\mathcal{H}}^*_\perp(x,y)\right]\cdot\hat{\mathbf{z}}\,dx\,dy \quad (4)$$

The total power carried by the forward and backward (propagating) modes is:

$$\begin{aligned} P(z) &= \frac{1}{2}\Re\int\!\!\int\left[\widetilde{\mathbf{E}}(\mathbf{r})\times\widetilde{\mathbf{H}}^*(\mathbf{r})\right]\cdot\hat{\mathbf{z}}\,dx\,dy \\ &= \frac{1}{2}\left[|C_+(z)|^2 - |C_-(z)|^2\right]\cdot\Re\{\mathcal{N}\} \quad (5)\end{aligned}$$

When the interaction takes place in a polarized gain medium, the driving current density $\widetilde{\mathbf{J}}(\mathbf{r})$ is given in terms of the induced polarization (dipole moment per unit volume) $\widetilde{\mathbf{P}}(\mathbf{r})$. In the time domain, the current density is the time derivative of the induced polarization. Thus, the phasor representation of the driving current density is given by:

$$\widetilde{\mathbf{J}}(\mathbf{r}) = -j\omega\widetilde{\mathbf{P}}(\mathbf{r}) = -j\omega\varepsilon_0\,\chi(\mathbf{r},\omega)\cdot\widetilde{\mathbf{E}}(\mathbf{r}) \quad (6)$$

where $\chi(\mathbf{r},\omega)$ is the electronic susceptibility tensor at the frequency ω (in a homogeneous isotropic medium it is a scalar). Using (6) in (3) results in:

$$\frac{d}{dz}C_\pm(z) = \pm j\frac{\omega\varepsilon_0}{2\mathcal{N}}e^{\mp jk_z z}\int\!\!\int\widetilde{\mathbf{E}}(\mathbf{r})\cdot\chi(\mathbf{r},\omega)\cdot\widetilde{\mathcal{E}}^*(x,y)\,dx\,dy \quad (7)$$

Substitution of the field expansion (2) in the excitation equations (7), the mode amplitudes $C_\pm(z)$ are described by a set of two coupled differential equations, that can be presented in a matrix form:

$$\frac{d}{dz}\begin{bmatrix} C_+(z) \\ C_-(z)\end{bmatrix} = \begin{bmatrix} +\kappa(z) & +\kappa(z)e^{-j2k_z z} \\ -\kappa(z)e^{+j2k_z z} & -\kappa(z)\end{bmatrix}\begin{bmatrix} C_+(z) \\ C_-(z)\end{bmatrix} \quad (8)$$

The coupling parameter:

$$\kappa(z,\omega) \equiv j\frac{\omega\varepsilon_0}{2\mathcal{N}}\int\!\!\int\widetilde{\mathcal{E}}(x,y)\cdot\chi(\mathbf{r},\omega)\cdot\widetilde{\mathcal{E}}^*(x,y)\,dx\,dy \quad (9)$$

is in general a complex, space-frequency dependent quantity.

QUANTUM LASER

We relate first to gain media, where the electronic susceptibility does not change along the axis of propagation z. This situation occurs in quantum lasers, where the atomic susceptibility of the gain medium is uniform [1]. In that case the coupling parameter is not yet space (z) dependent and can be presented in the form $\kappa(\omega) = \gamma(\omega) + j\beta(\omega)$, where $\gamma(\omega)$ is the field gain factor. Consequently, the set (8) can be written as two coupled first order linear differential equations:

$$\frac{d}{dz}\begin{bmatrix} C_+(z) \\ C_-(z)\end{bmatrix} = \begin{bmatrix} +\kappa & +\kappa e^{-j2k_z z} \\ -\kappa e^{+j2k_z z} & -\kappa\end{bmatrix}\begin{bmatrix} C_+(z) \\ C_-(z)\end{bmatrix} \quad (10)$$

Analytical solution of the coupled set (10) for a given forward mode amplitude $C_+(0)$ at the input $z = 0$, while the backward mode amplitude at the exit of the interaction region ($z = L$) is $C_-(L) = 0$, leads to the solution of incident and reflected wave amplitudes:

$$\begin{aligned}\frac{C_+(z)}{C_+(0)} &= \frac{(\kappa+jk_z)\sinh[S(L-z)] - S\cosh[S(L-z)]}{(\kappa+jk_z)\sinh(SL) - S\cosh(SL)}e^{-jk_z z} \\ \frac{C_-(z)}{C_+(0)} &= \frac{-\kappa\sinh[S(L-z)]}{(\kappa+jk_z)\sinh(SL) - S\cosh(SL)}e^{+jk_z z}\end{aligned} \quad (11)$$

where $S \equiv \sqrt{(\kappa+jk_z)^2-\kappa^2}$ is a complex parameter. The evolution of incident and reflected wave amplitudes along the gain medium are shown in Fig. 2. It is assumed that the interaction takes place in the vicinity of the resonance frequency, where $\kappa(\omega_0)$ is real.

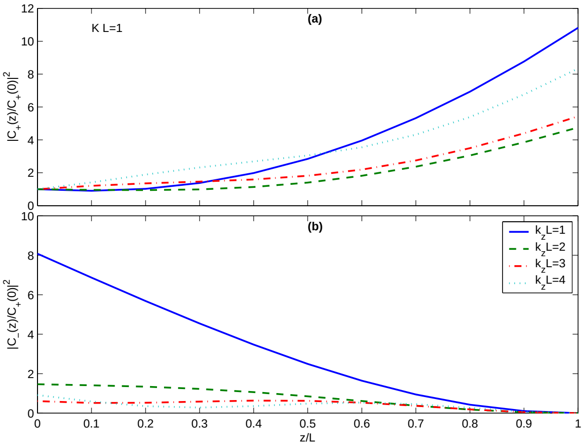

Figure 2: The evolution of (a) incident and (b) reflected wave amplitudes along the gain medium.

The transmission gain is defined by:

$$\frac{C_+(L)}{C_+(0)} = \frac{-SL}{(\kappa + jk_z)L\sinh(SL) - SL\cosh(SL)}\, e^{-jk_zL} \quad (12)$$

Respectively, the reflection gain is:

$$\frac{C_-(0)}{C_+(0)} = \frac{-\kappa L\sinh(SL)}{(\kappa + jk_z)L\sinh(SL) - SL\cosh(SL)} \quad (13)$$

Contour plots of the transmission and reflection power gain in the $(k_zL, \kappa L)$ plane are shown in Figure 3. An infinite gain singularities are inspected when the denominator of the gain dispersion relations given in (12) and (13) vanishes. This happens when:

$$\tanh(SL) = \frac{SL}{\kappa L + jk_zL} \quad (14)$$

In that case the forward and backward modes will be excited in the absence of an input signal, resulting in excitation and buildup of oscillations. Equation (14) expresses the oscillation condition, determining the threshold gain factor required for excitation of oscillations and their resultant frequencies at steady-state.

HIGH GAIN FREE-ELECTRON LASER

In free-electron lasers, the accelerated electrons serve as a gain medium and the interaction with the electromagnetic field takes place along the e-beam axis. Coupled mode theory for multi transverse mode excitation was developed previously, deriving an expression for the gain-dispersion relation in the linear regime of the FEL operation [7]. Set of equations (3) for the different modes were solved together with the small-signal moment equations describing the evolution in the driving current modulation. In the high gain limit the amplitudes of the forward and backward waves is described by two coupled third order linear differential equations:

$$\frac{d^3}{dz^3}\begin{bmatrix} C_+(z) \\ C_-(z) \end{bmatrix} = \begin{bmatrix} +\kappa & +\kappa e^{-j2k_zz} \\ -\kappa e^{+j2k_zz} & -\kappa \end{bmatrix} \begin{bmatrix} C_+(z) \\ C_-(z) \end{bmatrix} \quad (15)$$

where the coupling parameter:

$$\kappa = j\frac{\epsilon_0\zeta_q}{4\mathcal{N}}\frac{\omega_p^2}{v_{z0}^2}(k_z + k_w) \times \int\int f(x,y)\,\widetilde{\mathcal{E}}^{pm}(x,y)\,\widetilde{\mathcal{V}}_\perp^w \cdot \widetilde{\mathcal{E}}_\perp^*(x,y)\,dx\,dy \quad (16)$$

where $\widetilde{\mathcal{E}}_q^{pm}(x,y)$ is the pondermotive field, $f(x,y)$ is the transverse profile of the e-beam and ω_p is the plasma frequency of a relativistic beam with average axial electron velocity v_{z0}.

We solved the coupled set (15) analytically. The amplitude of the forward wave can be written as:

$$C_+(z) = e^{-jk_zz}\sum_{i=1}^{6} c_i e^{\lambda_i k_z z} \quad (17)$$

and the backward wave's amplitude is:

$$C_-(z) = e^{+jk_zz}\sum_{i=1}^{6}\frac{(j-\lambda_i)^3}{(j+\lambda_i)^3} c_i e^{\lambda_i k_z z} \quad (18)$$

where the six eigenvalues λ_i appear in the above solutions are found from a characteristic equation of the sixth order. Fortunately it can be written in a third order form:

$$\gamma^3 + 3\gamma^2 + 3\gamma(1 - jw) + 1 + jw = 0 \quad (19)$$

in which $\gamma = \lambda^2$ and $w = \frac{2\kappa}{k_z^3}$, enabling analytical solution of λ_i. The properties of the eigenvalues λ_i are discussed in the Appendix. The constants c_i are determined by the boundary conditions. Assuming that there is no prebunching in the electron beam, an initial amplitude $C_+(0)$ is assumed for the forward mode at $z = 0$ and all the other boundary conditions are $C'_+(0) = C''_+(0) = C_-(0) = C'_-(0) = C''_-(0) = 0$ (here $'$ denotes first order derivative $\frac{d}{dz}$). The analytical results were verified using a numerical algorithm solving the boundary condition problem. Figure 4 presents contour plots of the transmission and reflection power gain in the $(k_zL, \kappa L^3)$ plane. Oscillations are expected where an infinite gain is obtained.

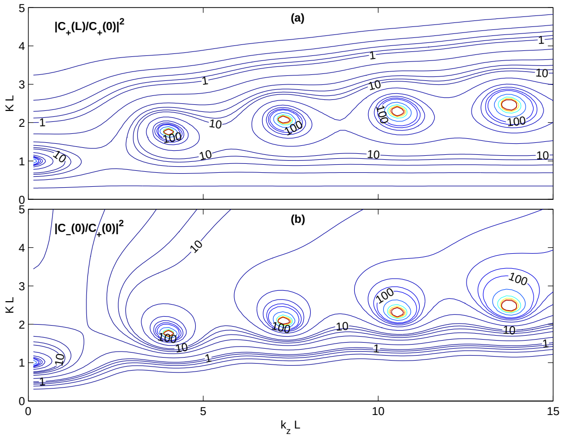

Figure 3: (a) Transmission and (b) reflection contours in the $(k_zL, \kappa L)$ plane for atomic laser.

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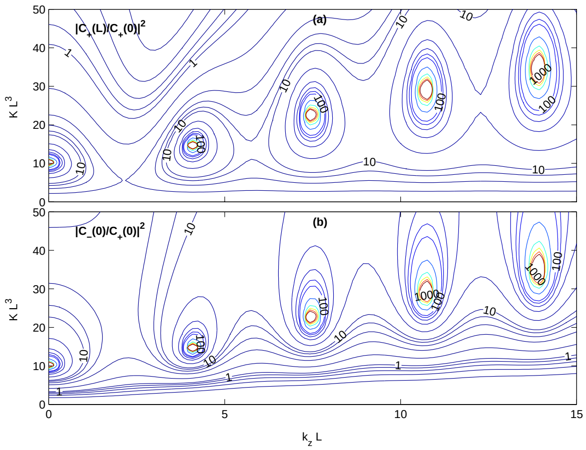

Figure 4: (a) Transmission and (b) reflection contours in the $(k_z L, \kappa L^3)$ plane for free-electron laser.

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APPENDIX

The six eigenvalues calculated from equation (19) come in positive-negative pairs $\lambda = \pm\sqrt{\gamma}$. Taking complex conjugate of equation (19) (assuming that w is real):

$$\gamma^{*3} + 3\gamma^{*2} + 3\gamma^{*}(1 + jw) + 1 - jw = 0 \qquad (20)$$

reveals that the solutions satisfy the relation:

$$\gamma^{*}_{m}(-w) = \gamma_n(w) \qquad (21)$$

in which m, n are the solution indices. However, since we have a total of three roots to equation (19) , at least one of the roots, say γ_1, satisfies the relation:

$$\gamma^{*}_{1}(-w) = \gamma_1(w) \qquad (22)$$

that is its real part is even with respect to w and its imaginary part is odd.

Equation (19) can be readily solved and the solutions are:

$$\begin{aligned}
\gamma_1 &= -1 + ba - \frac{a^2}{b} \\
\gamma_2 &= \frac{1}{2}[-\gamma_1(1+\sqrt{3}j) - 3 - \sqrt{3}j + 2\sqrt{3}jab] \\
\gamma_3 &= \frac{1}{2}[-\gamma_1(1-\sqrt{3}j) - 3 + \sqrt{3}j - 2\sqrt{3}jab] (23)
\end{aligned}$$

where:

$$a = (-jw)^{\frac{1}{3}} \quad b = (2 + \sqrt{4 - jw})^{\frac{1}{3}} \qquad (24)$$

We see that:

$$\gamma_3(w) = \gamma^{*}_{2}(-w) \qquad (25)$$

Hence one can deduce the behavior of the roots for negative w values from their behavior for positive values.

In the case $w \rightarrow 0$ we obtain $\lim_{w\rightarrow 0}\gamma_i = -1$ for $i \in [1, 2, 3]$. This is obviously a degenerate case in which the solution is not of an exponential type. Inserting the condition $w \rightarrow 0$ into equation (15) one is left with the trivial equation $\frac{d^3}{dz^3}\begin{bmatrix} C_+(z) \\ C_-(z) \end{bmatrix} = 0$ with the solution:

$$\begin{aligned}
C_+(z) &= \frac{1}{2}z^2 c_+^{(2)} + z c_+^{(1)} + c_+^{(0)} \\
C_-(z) &= \frac{1}{2}z^2 c_-^{(2)} + z c_-^{(1)} + c_-^{(0)} \qquad (26)
\end{aligned}$$

In the opposite limit in which $w \rightarrow \infty$ we obtain the following results:

$$\begin{aligned}
\lim_{w\rightarrow\infty} \gamma_1 &= \frac{1}{3} \\
\lim_{w\rightarrow\infty} \gamma_2 &= \sqrt{3}e^{\frac{5}{4}\pi j} w^{\frac{1}{2}} \\
\lim_{w\rightarrow\infty} \gamma_3 &= \sqrt{3}e^{\frac{1}{4}\pi j} w^{\frac{1}{2}} \qquad (27)
\end{aligned}$$

Since non of those are negative real numbers this means that we have three growing exponents and three decaying exponents in the case of large w those are:

$$\begin{aligned}
\lim_{w\rightarrow\infty} \lambda_1 &= \frac{1}{\sqrt{3}} \\
\lim_{w\rightarrow\infty} \lambda_2 &= -\frac{1}{\sqrt{3}} \\
\lim_{w\rightarrow\infty} \lambda_3 &\cong 3^{\frac{1}{4}}(-0.383 + 0.924j)w^{\frac{1}{4}} \\
\lim_{w\rightarrow\infty} \lambda_4 &\cong 3^{\frac{1}{4}}(0.383 - 0.924j)w^{\frac{1}{4}} \\
\lim_{w\rightarrow\infty} \lambda_5 &\cong 3^{\frac{1}{4}}(0.924 + 0.383j)w^{\frac{1}{4}} \\
\lim_{w\rightarrow\infty} \lambda_6 &\cong 3^{\frac{1}{4}}(-0.924 - 0.383j)w^{\frac{1}{4}} \qquad (28)
\end{aligned}$$

For large w, λ_5 is clearly the most dominant exponent. Although λ_1 & λ_2 asymptotically approach a finite number, the other eigenvalues continue to grow without limit.