MODELLING THE FRINGE FIELDS OF A MULTIPOLE DEVICE

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Abstract

The scalar and vector potentials of magnetic multipole devices are investigated. Special attention is paid to the fringe field region. Pseudo differential operators induced by Bessel functions are used to obtain the various multipole coefficients of the potential. The fringe fields are fitted using a finite element basis; the coefficients of the best fit are directly obtained from field measurements at the surface $r = R$.

1 INTRODUCTION

From Maxwell's equations for the static electromagnetic field, we find that, for a magnetic field $\vec{B}$ in a region without free charges or currents, there exist a scalar potential u, and a vector potential $\vec{A}$, satisfying:

$$\operatorname{grad} u = \vec{B} = \operatorname{curl} \vec{A}, \quad (1)$$
$$\Delta u = 0, \quad (2)$$
$$\operatorname{curl}\operatorname{curl} \vec{A} = \vec{0}. \quad (3)$$

The vector potential $\vec{A}$ will be chosen such that $\operatorname{div} \vec{A} = 0$. In this case, we have $\Delta \vec{A} = \vec{0}$.

We apply these equations to the magnetic field inside a magnetic multipole device which has the z-axis as its central axis. Our region of interest G is, in cilindrical coordinates, given by: $G : 0 \le r < R,\ -\pi \le \varphi \le \pi,\ -\infty < z < \infty$, with boundary $\Gamma : r = R$. Here, $R > 0$ is a convenient maximal radius of the multipole device, e.g. the aperture radius.

The potential problem is then given by:

$$\begin{cases} \Delta u = 0,\ x \in G, \\ \\ u(R, \varphi, z) = U_R(\varphi, z),\ x \in \Gamma. \end{cases} \quad (4)$$

In order to have a unique solution to this problem, we have to impose the additional conditions:

$$\begin{aligned} &|u(r,\varphi,z)| \text{ is bounded on } G, \\ &\lim_{|z|\to\infty} \nabla u(r,\varphi,z) = 0,\ r \le R, \\ &\int_{-\infty}^{\infty} |u(r,\varphi,z)| dz < \infty,\ r \le R. \end{aligned} \quad (5)$$

In practical cases, these conditions will always be satisfied, which guarantees that the solution to (4) is unique.

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2 HARMONIC POTENTIALS

In cylindrical coordinates (r, φ, z), (2) is given by:

$$\frac{\partial^2 u}{\partial r^2} + \frac{1}{r}\frac{\partial u}{\partial r} + \frac{1}{r^2}\frac{\partial^2 u}{\partial \varphi^2} + \frac{\partial^2 u}{\partial z^2} = 0. \quad (6)$$

Since we use cylindrical coordinates we can expand $u(r, \varphi, z)$ into a Fourier series:

$$u = \sum_{m=0}^{\infty} (a_m(r,z)\cos(m\varphi) + b_m(r,z)\sin(m\varphi)).$$

Having inserted this expansion into (6), we find that the coefficients of the φ-independent term and the terms with $\cos(m\varphi)$ and $\sin(m\varphi)$ must vanish independently. This yields:

$$\frac{\partial^2 a_0}{\partial r^2} + \frac{1}{r}\frac{\partial a_0}{\partial r} + \left(\frac{\partial}{\partial z}\right)^2 a_0 = 0,$$
$$\frac{\partial^2 a_m}{\partial r^2} + \frac{1}{r}\frac{\partial a_m}{\partial r} + \left(\left(\frac{\partial}{\partial z}\right)^2 - \frac{m^2}{r^2}\right) a_m = 0,$$
$$\frac{\partial^2 b_m}{\partial r^2} + \frac{1}{r}\frac{\partial b_m}{\partial r} + \left(\left(\frac{\partial}{\partial z}\right)^2 - \frac{m^2}{r^2}\right) b_m = 0.$$

In these equations, any coefficient a_m or b_m will be considered to be a function of r only, while z is merely a parameter. Then $\frac{\partial}{\partial z}$ can be seen as a linear operator, and the formal solutions to these equations are given by:

$$\begin{aligned} a_0(r,z) &= J_0(r\frac{\partial}{\partial z})A_0(z), \\ a_m(r,z) &= J_m(r\frac{\partial}{\partial z})A_m(z),\ m = 1, 2, \ldots \\ b_m(r,z) &= J_m(r\frac{\partial}{\partial z})B_m(z),\ m = 1, 2, \ldots \end{aligned}$$

where J_m denotes the Bessel function of the first kind of order m, and A_m and B_m are to be determined from the boundary conditions. The general solution for $u(r, \varphi, z)$ reads:

$$u = \sum_{m=0}^{\infty} J_m(r\frac{\partial}{\partial z})(A_m\cos(m\varphi) + B_m\sin(m\varphi)). \quad (7)$$

These expressions become meaningful if we take their Fourier transforms with respect to z:

$$\begin{aligned} \tilde{a}_m(r,\omega) &= i^m I_m(\omega r)\tilde{A}_m(\omega), \\ \tilde{b}_m(r,\omega) &= i^m I_m(\omega r)\tilde{B}_m(\omega). \end{aligned}$$

Using the standard recurrent relations for derivatives of Bessel functions, we can easily calculate the r- and φ-derivatives of terms like $a_m(r, z)\cos(m\varphi)$, etc. This will be used when determining a harmonic vector potential for $\vec{B} = \mathrm{grad}\, u$.

Using (1), we can determine a vector potential $\vec{A}$ for $\vec{B}$. We choose $\vec{A}$ such that $\mathrm{div}\, \vec{A} = 0$, which implies that $\vec{A}$ is harmonic. A possible solution is given by

$$A_r = \sum_{m=1}^{\infty} J_{m+1}(r\frac{\partial}{\partial z})(B_m \cos(m\varphi) - A_m \sin(m\varphi)),$$

$$A_\varphi = \sum_{m=0}^{\infty} J_{m+1}(r\frac{\partial}{\partial z})(A_m \cos(m\varphi) + B_m \sin(m\varphi)),$$

$$A_z = \sum_{m=1}^{\infty} J_m(r\frac{\partial}{\partial z})(-B_m \cos(m\varphi) + A_m \sin(m\varphi)).$$

This solution is not unique; the gradient of any harmonic scalar-valued function can be added in order to yield another valid solution.

3 INTRODUCING BOUNDARY CONDITIONS

Consider the boundary condition $u(R, \varphi, z) = U_R(\varphi, z)$. The function (or distribution) U_R can be expanded into a Fourier series:

$$U_R(\varphi, z) = \sum_{m=0}^{\infty} (U_{mR} \cos(m\varphi) + W_{mR} \sin(m\varphi)).$$

Inserting general solution (7) into the boundary condition yields the (weak) equation

$$a_m(R, z) = J_m(R\frac{\partial}{\partial z})A_m(z) = U_{mR}(z),$$

while a similar result holds for $b_m(R, z)$. Taking the Fourier transform of this equation yields

$$\tilde{A}_m(\omega) = \frac{\tilde{U}_{mR}(\omega)}{i^m I_m(\omega R)}. \tag{8}$$

Equation (8) determines the coefficients A_m and B_m, and therefore the potential u, uniquely. For example, the coefficients $a_m(r, z)$ are given by

$$a_m(r, z) = \left(\mathcal{F}^{-1} \frac{I_m(\omega r)}{I_m(\omega R)} \mathcal{F}\right) U_{mR}(z).$$

After expanding the Fourier integrals in this expression, one eventually finds:

$$a_m(r, z) = \int_{-\infty}^{\infty} g_m(r, z - \zeta) U_{mR}(\zeta) d\zeta, \tag{9}$$

where the basic function $g_m(r, z)$ is given by

$$g_m(r, z) = \frac{1}{\pi} \int_0^{\infty} \frac{I_m(\omega r)}{I_m(\omega R)} \cos(\omega z) d\omega.$$

In fact, $g_m(r, z)$ is the solution in the case that $U_{mR}(z) = \delta(z)$. This will be used in the next section.

By expanding the functions g_m into powers of r/R, one can expand the coefficients a_m and b_m into powers of r/R. For example:

$$\begin{aligned} a_m(r, z) &= \sum_{l=0}^{\infty} \alpha_{ml}(z) \left(\frac{r}{R}\right)^{m+2l}, \\ \alpha_{ml}(z) &= \int_{-\infty}^{z} g_{ml}(z - \zeta) U_{mR}(\zeta), \\ g_{ml}(z) &= \frac{1}{4^l l!(m + l)!\pi} \int_0^{\infty} \frac{(\omega R)^{m+2l}}{I_m(\omega R)} \cos(\omega z) d\omega. \end{aligned}$$

This will be useful for deriving particle trajectory equations for a beam guiding element that are accurate up to a certain order in r/R. The coefficient $\alpha_{m0}(z)$ can also be used to fit the multipole strength $(\frac{\partial^m}{\partial r^m} a_m)(0, z)$.

In general, we write for $k \geq m - 1$:

$$J_k(r\frac{\partial}{\partial z})A_m(z) = \int_{-\infty}^{\infty} g_m^k(r, z - \zeta) U_{mR}(\zeta) d\zeta,$$

where g_m^k is given by:

$$g_m^k(r, z) = \frac{1}{\pi} \int_0^{\infty} \frac{I_k(\omega r)}{I_m(\omega R)} i^{k-m} e^{i\omega z} d\omega.$$

This allows us to calculate the components of $\vec{A}$ and their derivatives from the boundary conditions at $r = R$. In fact, any quantity related to the magnetic field can be calculated, if the right basic function is used.

In the case that not the potential, but a component of $\vec{B}$, e.g. B_φ, is known at $r = R$, we find:

$$J_k(r\frac{\partial}{\partial z})A_m(z) = \int_{-\infty}^{\infty} \frac{R}{m} g_m^k(r, z - \zeta) B_{mR}(\zeta) d\zeta.$$

where $B_{mr}(\zeta) \cos(m\varphi)$ denotes the $2m$-pole contribution to B_φ at $r = R$. If B_r or B_z are known, appropriate basic functions can also be derived.

4 FITTING THE MULTIPOLE FIELD

In practice, we obtain approximations of U_{mR} or its derivatives by interpolating a discrete set of measurements. Since piecewise constant or piecewise linear interpolations are often employed, the special cases of U_{mR} being piecewise constant or linear will be considered here. First, assume U_{mR} is piecewise constant. Then there are pairs (λ_i, z_i) such that $U'_{mR} = \sum_i \lambda_i \delta(z - z_i)$. Then, after integration,

$$J_k(r\frac{\partial}{\partial z})A_m(z) = \sum_i \lambda_i G_m^k(r, z - z_i),$$

where $G_m^k(r, z) = \int_{-\infty}^{z} g_m^k(r, \zeta) d\zeta$. If U_{mR} is supposed to be piecewise linear, then $U''_{mR} = \sum_i \lambda_i \delta(z - z_i)$, in which case we have

$$J_k(r\frac{\partial}{\partial z})A_m(z) = \sum_i \lambda_i \tilde{G}_m^k(r, z - z_i),$$

where $\tilde{G}^k_m(r,z) = \int_{-\infty}^{z} G^k_m(r,\zeta)d\zeta$. As was shown above, related quantities can easily be derived by replacing the functions G_m and $\tilde{G}_m$ by functions related to these quantities, while retaining the pairs (λ_i, z_i).

At this point we shall show how to determine the various multipole contributions to a magnetic field from the values of B_z at $r = R$. At $r = R$, the B_z-component is given by:

$$B_z(R,\varphi,z) = \sum_{m=0}^{\infty} \left(U'_{mR}\cos(m\varphi) + W'_{mR}\sin(m\varphi)\right).$$

Assume B_z is known at the points (R, φ_i, z_j). Then U'_{mR} is the Fourier coefficient of $\cos(m\varphi)$, and $B_j = U'_{mR}(z_j)$ is obtained from:

$$B_j = \frac{1}{\pi}\int_{-\pi}^{\pi} B_z(R,\varphi,z_j)\cos(m\varphi)d\varphi$$

Choosing a piecewise constant approximation for $U'_{mR}(z)$, we find $U''_{mR}(z) = \sum_l (B_{l+1} - B_l)\delta(z - z_l)$, so $J_k(r\frac{\partial}{\partial z})A_m(z)$ is approximated by:

$$J_k(r\frac{\partial}{\partial z})A_m(z) = \sum_l (B_{l+1} - B_l)\tilde{G}_m(r, z - z_l).$$

Since $\tilde{G}_m(r,z)$ is a known function, $J_k(r\frac{\partial}{\partial z})A_m(z)$ can be approximated directly from the measurements. Elaborate calculations on the measurements are not necessary.

It should be noted that, in order to determine the $2m$-pole contribution, one needs measurements performed at $2n$ different angles φ_i.

From (9), we find that a_m depends linearly on U_{mR}. This allows us to calculate the effect of errors in U_{mR} on the calculation of a_m. It turns out, that

$$\sup_{z\in\mathbb{R}} |\delta a_m(r,z)| \leq \left(\frac{r}{R}\right)^m \sup_{z\in\mathbb{R}} |\delta U_{mR}(z)|,$$

where δa_m and δU_{mR} denote the errors in a_m and U_{mR} respectively. This justifies the use of the given approximations.

5 EXPERIMENTAL TEST OF THE PRESENTED THEORY

The theory developed in the previous sections has been verified using actual field measurements for a magnetic quadrupole, performed by G. Brooijmans [1]. Using a normal-orientated quadrupole, he measured the component B_y for $y = 0$ and numerous values of x and z. Using the measurements for the outermost value of x, we calculated B_y for the other values of x and compared the calculations to the measurements. Since $B_y = B_\varphi$ for $x > 0$ and $B_y = -B\varphi$ for $y < 0$, we used the basic function $\hat{G}_2(r,z) = \frac{R}{r}G_2(r,z)$ for the fitting procedure. The results of the calculations and the measurements are shown in figure 1.

The differences between the calculations and measurements result mainly from errors in the numerical calculation of the basic functions. This calculation is complicated because of the integration on $[0,\infty)$ that has to be carried out. Fortunately, the basic functions need to be calculated only once.

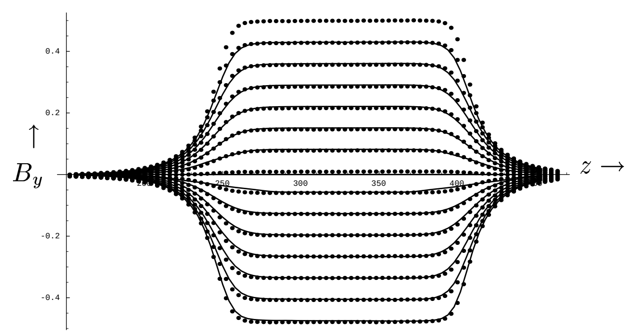

Figure 1: Comparison of the measured and calculated B_y as a function of z for various values of x. The dots represent the measurements, the curves the calculated field.

6 CONCLUSIONS

The magnetic field inside a magnetic multipole, and its harmonic scalar and vector potentials, have been explored in the area $0 \leq r < R$ and $-\infty < z < \infty$. The various multipole contributions to these quantities have been fitted using field measurements at the boundary $r = R$ and shifted basic functions. The same set of measurements and shiftings can be used to fit many field-related quantities.

The developed procedure is independent of the exact form of the boundary conditions and can be used to fit the field of one device or various consecutive devices.

The procedure works for any order multipole contribution, but will be the most useful for lower order multipole contributions, since higher order multipole contributions are more difficult to obtain from measurements their effect on particle trajectories will often be small.

Recently, M. Venturini and A. Dragt presented an article at CPO 5, in which they also derive multipole coefficients from boundary conditions [2]. This article presents a somewhat different view on the subject.

7 REFERENCES

[1] G.J.L.M. Brooijmans, Design of and measurements on the EUTERPE dipoles and quadrupoles, TUE internal report, VDF/NK 88-15 (1988).

[2] M. Venturini and A. Dragt, Computation of exact transfer maps from magnetic field data, presented at the 5th Charged Particle Conference (1998), to be published in Nucl. Instr. and Meth.