A MOMENT EQUATION APPROACH TO A MUON COLLIDER COOLING LATTICE

C.M. Celata and A. M. Sessler, Ernest Orlando Lawrence Berkeley National Laboratory, Berkeley, CA 94720 USA
P. B. Lee, B. A. Shadwick, and J. S. Wurtele, Univ. of CA at Berkeley, Berkeley, CA 94720 USA

Abstract

Equations are derived which describe the evolution of the second order moments of the beam distribution function in the ionization cooling section of a muon collider. Ionization energy loss, multiple scattering, and magnetic fields have been included, but forces are linearized. A computer code using the equations agrees well with tracking calculations. The code is extremely fast, and can be used for preliminary design, where such issues as beam halo, which must be explored using a tracking code, are not the focus.

1 INTRODUCTION

One of the most fundamental problems in designing a muon collider is the initial cooling of the muons. Not only must the muons be cooled in order to accelerate efficiently, but the success of the cooling effectively determines the acceptance of the machine, thus putting an upper limit on the luminosity. Decreases in transverse emittance of almost three orders of magnitude, and in longitudinal by more than one, are necessary.

Because of the short lifetime of the muons, the optimal cooling method appears to be ionization cooling. This method produces transverse cooling by passing the beam through material, thus decreasing the momentum of each particle in the direction of its motion. Longitudinal momentum is then restored by means of RF cavities. Longitudinal cooling can be achieved by using a dispersive element to correlate longitudinal momentum and transverse position, then passing the beam through a material whose thickness is dependent on transverse position. Multiple scattering and energy straggling are competing processes which determine the lower limit on the beam emittance. Theoretical estimates of the cooling, as well as a discussion of the effects of multiple scattering and straggling can be found in papers by Skrinsky [1], Vsevolozhskaya [2], and Neuffer [3,4]. The ionization cooling concept has yet to be experimentally proven, but a prototype experiment at FNAL has been proposed.

Extensive tracking studies have been conducted by other authors [4]-[7], resulting in an initial design for the cooling section. Tools include the tracking code ICOOL [7], the results of which have been used as a standard of comparison for the code discussed in this paper. In this report we describe a moment equation approach to cooling section design, which is accurate for arbitrary distribution function, and is much faster than tracking. The main approximation of the theory is linearization of the forces in the problem.

2. A CLOSED SET OF SECOND ORDER MOMENT EQUATIONS

We consider first the case of a beam propagating through a straight lattice containing cooling absorbers and acceleration. The transverse equations of motion for a single particle in the cooling section can be written as:

$$\frac{dp_X}{dt} = q\left(\vec{\mathrm{v}} \times \vec{B}\right)_X + \frac{\mathrm{v}_X}{\mathrm{v}}\left(\frac{dE}{ds}\right)_{Total} + F_X \qquad (1)$$

and similarly for Y, where X and Y are the transverse coordinates and $\vec{B}$ is the focusing field. We neglect transverse components of the accelerating electric field, since the rf cavity for the cooling section will be a closed pillbox, with no holes for the beam passage. F_X is the X component of the force associated with multiple scattering, and $(dE/ds)_{Total}$ is the energy change per unit length in the direction of the particle's motion, due to passage through matter (note: $dE/ds<0$). $(dE/ds)_{Total}$ can be written as $(dE/ds)_{avg} + \Delta$, where $(dE/ds)_{avg}$ is the mean energy loss per unit length for a distribution of particles with a given β, and Δ is the energy straggling due to statistics of the ionization process. Below we drop the subscript "avg", so that "dE/ds" is the mean loss. We will assume that $\mathrm{v}_X^2+\mathrm{v}_Y^2<<\mathrm{v}_Z^2$. We wish to take moments of the single particle equations with the transverse coordinates and momenta, and to arrive at a closed set of equations. In order to eliminate higher order moments which would couple the 2nd order equations to those of higher order, we linearize the forces on the right side of the equations by expanding in the transverse coordinates. Taylor expanding $\vec{B}$ about X=Y=0, so that $B_Z \approx B_{Z0}$, $B_X \approx B_{X0} + C_{11}X + C_{12}Y$, and $B_Y \approx B_{Y0} + C_{21}X + C_{22}Y$, we obtain:

$$X'' + \frac{(\beta_Z\gamma)'}{\beta_Z\gamma}X' = \frac{1}{m_0c^2\beta_z{}^2\gamma}\Big[q\beta_z(Y'B_{z0} - C_{21}X - C_{22}Y) + X'\frac{dE}{ds} + F_X\Big] \qquad (2)$$

$$Y'' + \frac{(\beta_Z\gamma)'}{\beta_Z\gamma}Y' = \frac{1}{m_0c^2\beta_z{}^2\gamma}\Big[q\beta_z(-X'B_{z0} + C_{11}X + C_{12}Y) + Y'\frac{dE}{ds} + F_Y\Big] \qquad (3)$$

The prime signifies d/dZ, β=v/c, and m_0 is the rest mass. Next we assume that $\beta_Z=\beta_{Z0}+\chi$, where β_{Z0} is β_Z for a reference particle, and to maintain consistent ordering in equations (2) and (3) we set $\beta_Z=\beta_{Z0}$ and $\gamma=\gamma_0$. For the same reason (consistent ordering), we assume that dE/ds is evaluated at the β of the reference particle in Eqs. (2)-(12). The quantities we are interested in– e.g., for the calculation of emittance– are coordinates and momenta measured with respect to the beam centroid. Therefore we average equations (2) and (3) over the particle distribution of all the particles in a beam slice located within Z increment ΔZ, obtaining for the centroid motion:

$$\overline{X}'' + \frac{p_{z0}'}{p_{z0}}\overline{X}' = \frac{q\beta_{Z0}}{M}\left[\overline{Y}' B_{z0} - C_{21}\overline{X} - C_{22}\overline{Y}\right] + \frac{\overline{X}'}{M}\frac{dE}{ds} \quad (4)$$

$$\overline{Y}'' + \frac{p_z'}{p_z}\overline{Y}' = \frac{q\beta_{z0}}{M}\left[-\overline{X}' B_{z0} + C_{11}\overline{X} + C_{12}\overline{Y}\right] + \frac{\overline{Y}'}{M}\frac{dE}{ds}, \quad (5)$$

where $M=m_0c^2\beta_{Z0}^2\gamma_0$. These equations can now be subtracted from equations (2) and (3), to give equations for x and y, where $x\equiv X-\overline{X}$, and $y\equiv Y-\overline{Y}$. Because the equations are linear, the equations of motion for x and y will be identical to equations (2) and (3), with x and y everywhere replacing X and Y. B_{X0} and B_{Y0} = 0, since we have assumed a straight lattice. We now take moments of the equations of motion with respect to x, y, x′, and y′, to get:

$$M\left(\tfrac{1}{2}\overline{x^2}'' - \overline{x'^2} + \frac{p_{z0}'}{p_{z0}}\overline{xx'}\right) = q\beta_{z0}c\left(\overline{xy'}B_{z0} - C_{21}\overline{x^2} - C_{22}\overline{xy}\right) + \overline{xx'}\frac{dE}{ds}, \quad (6)$$

$$M\left(\tfrac{1}{2}\overline{y^2}'' - \overline{y'^2} + \frac{p_{z0}'}{p_{z0}}\overline{yy'}\right) = q\beta_{z0}c\left(-\overline{x'y}B_{z0} + C_{12}\overline{y^2} + C_{11}\overline{xy}\right) + \overline{yy'}\frac{dE}{ds}, \quad (7)$$

$$M\left[(\overline{xy})'' + \frac{p_{z0}'}{p_{z0}}(\overline{xy})' - 2\overline{x'y'}\right] = q\beta_{z0}c\left[(\overline{yy'} - \overline{xx'})B_{z0} + (C_{12}-C_{21})\overline{xy} + C_{11}\overline{x^2} - C_{22}\overline{y^2}\right] + (\overline{xy})'\frac{dE}{ds}, \quad (8)$$

$$M\left[\frac{p_{z0}'}{p_{z0}}(\overline{x'y} - \overline{xy'}) + (\overline{x'y} - \overline{xy'})'\right] = q\beta_{z0}c\left[(\overline{xx'} + \overline{yy'})B_{z0} - C_{11}\overline{x^2} - C_{22}\overline{y^2} - (C_{12}+C_{21})\overline{xy}\right] + (\overline{x'y} - \overline{xy'})\frac{dE}{ds}, \quad (9)$$

$$M\left[(\overline{x'y'})' + 2\frac{p_{z0}'}{p_{z0}}\overline{x'y'}\right] = q\beta_{z0}c\left[(\overline{y'^2} - \overline{x'^2})B_{z0} + C_{11}\overline{xx'} - C_{22}\overline{yy'} + C_{12}\overline{x'y} - C_{21}\overline{xy'}\right] + 2\overline{x'y'}\frac{dE}{ds}, \quad (10)$$

$$M\left[\frac{1}{2}(\overline{x'^2})' + \frac{p_{z0}'}{p_{z0}}\overline{x'^2}\right] = q\beta_{z0}c\left(\overline{x'y'}B_{z0} - C_{21}\overline{xx'} - C_{22}\overline{x'y}\right) + \overline{x'^2}\frac{dE}{ds} + \frac{1}{2}m_0c^2\beta_{z0}^2\gamma_0\frac{d\theta_0^2}{dz}, \quad (11)$$

$$M\left[\frac{1}{2}(\overline{y'^2})' + \frac{p_{z0}'}{p_{z0}}\overline{y'^2}\right] = q\beta_{z0}c\left(-\overline{x'y'}B_{z0} + C_{11}\overline{xy'} + C_{12}\overline{yy'}\right) + \overline{y'^2}\frac{dE}{ds} + \frac{1}{2}m_0c^2\beta_{z0}^2\gamma_0\frac{d\theta_0^2}{dz}. \quad (12)$$

Here θ_0 is the change in rms v_x/v_z due to multiple scattering. These equations, with equations (4) and (5) and an equation for p_{Z0}' (see below), give a complete closed set of transverse moment equations for the case of a lattice with no bends.

A computer code has been written to solve Eqs. (4)-(12). The Bethe-Bloch model is used for dE/ds, and the Lynch-Dahl model [8] for multiple scattering. We assume all particles have the same γ. $d\gamma/dt$ is determined by dE/ds and the accelerating fields. In the future, energy spread will be added, using Eqs. (14) and (15) below. This type of code has the advantage over particle tracking that it is much faster (~2 orders of magnitude), and therefore useful for preliminary lattice design. It is trivial to include space charge in the model, should it become important, in the approximation of linear space charge forces. Results have been found to agree well in the appropriate regime with results from the ICOOL tracking code of R. C. Fernow.

We can obtain a linear single particle equation for the Z direction in a similar fashion to what was done above for the transverse plane by using the linearized magnetic field, and linearizing the ionization energy loss and rf terms about the position and momentum of the reference particle. This gives:

$$m_0c^2\beta_0^2\gamma_0^3\left[\delta'' + \left(2\beta_{z0}\beta_{z0}' + \beta_{z0}\beta_0' + 3\beta_{z0}^3\beta_0'\gamma_0^2 + \frac{\beta_{z0}'}{\beta_{z0}}\right)\delta'\right] = q\left(E_z'\right)_{Z=Z_0}\delta + \beta_{z0}\left(\frac{\partial}{\partial\beta}\frac{dE}{ds}\right)_{Z=Z_0}\delta'. \quad (13)$$

Here $\delta= Z-Z_0$, where Z_0 is the longitudinal position of the reference particle, and E_z is the longitudinal component of the accelerating field. Note that to this order, multiple scattering does not appear in the longitudinal equation, and while p_{Z0} appears as a parameter in the transverse equations, the longitudinal dynamics are not otherwise coupled to the transverse. We can thus average Eq. (13) over all particles in the bunch, getting an equation for the Z component of the bunch centroid. Subtracting the centroid equation from Eq. (13) and taking moments with $\xi \equiv \delta - \bar{\delta}$, and ξ', we get equations for the z envelope and velocity spread:

$$\overline{\xi^2}'' + \left[\kappa - \frac{1}{m_0\beta_{z0}c^2\gamma_0^3}\left[\frac{\partial}{\partial\beta}\left(\frac{dE}{ds}\right)_{Total}\right]_{z=z_0}\right]\overline{\xi^2}' - \frac{2q}{m_0\beta_{z0}^2c^2\gamma_0^3}\left(E_z'\right)_{z=z_0}\overline{\xi^2} = 2\overline{\xi'^2} \qquad (14)$$

$$\overline{\xi'^2}' + 2\left(\kappa - \frac{\beta_{z0}}{M\gamma_0}\left[\frac{\partial}{\partial\beta}\left(\frac{dE}{ds}\right)_{Total}\right]_{z=z_0}\right)\overline{\xi'^2} = \frac{q}{m_0\beta_{z0}^2c^2\gamma_0^3}\left(E_z'\right)_{z=z_0}\overline{\xi^2}, \qquad (15)$$

where $\kappa \equiv 2\beta_{z0}\beta_{z0}' + \beta_{z0}\beta_0' + 3\beta_{z0}\beta_0'\gamma_0^2 + \beta_{z0}'/\beta_{z0}$. The averages of Eqs. (14) and (15) can be understood to be averages over the whole bunch, or over a transverse slice if the dynamics of a slice are to be followed.

3. MOMENT EQUATIONS WITH BENDS

The single particle equations of motion for sections of the lattice with dipole fields can easily be written down in the usual fashion. We neglect here B_z due to fringe fields, and skew quadrupole components, and assume that γ is constant within the dipole. If we consider a bend in the X-Z plane and linearize the magnetic field about the reference orbit, we have the analog to Eq. (2):

$$X'' = -\left(\frac{1}{\rho_0^2} + \frac{q}{p_0}C_{21}\right)X + \frac{1}{\rho_0}\delta', \qquad (16)$$

where $\rho_0 = p_0/(qB_{y0})$ is the reference orbit radius of curvature, and the primes now signify derivatives with respect to s, the distance along the reference orbit. The equation has been linearized, so corrections coming from the difference between the particle's radius of curvature and momentum and that of the reference particle have been dropped. The Y equation is unchanged from the case without bends, except that skew quadrupole and B_{Z0} terms are neglected here, and we assume no material inside the dipoles. To this order, the s equation of motion is $\xi''=0$, i.e., there is no effect of the fields on the s motion.

It is now straightforward to derive moment equations, as was done above for the case without bends. First the average of Eq. (16) is taken, to give the equation for the centroid motion. Then this is subtracted from Eq. (16) to give an equation for x. This equation will be of the same form as Eq. (16), with $X \rightarrow x$ and $\delta \rightarrow \xi$. Moments of the three equations of motion with the coordinates and momenta may then be taken, averaging over a slice of beam within a length Δs.

4. SUMMARY AND CONCLUSIONS

We have derived second order moment equations describing the beam in the cooling section of a muon collider, including the effect of ionization energy loss in materials and multiple scattering. A code to compute the evolution of the transverse moments along the accelerator has been written and tested, and is now ready for use. Results agree well with ICOOL runs in the appropriate regime, and the code is extremely fast-- essentially instantaneous. Apertures in X, Y, and Z will be added, to simulate approximately the effects of dynamic apertures and rf buckets of finite size. The formalism is very general, and space charge can be easily included, in the approximation of linear space charge forces. The main restriction of this theory is the approximation of linearity of forces, both in transverse coordinates and in distance and momentum difference from the reference particle.

5. ACKNOWLEDGMENTS

This work was supported by the U.S. Department of Energy, Office of Energy Research, Office of High Energy and Nuclear Physics, under Contract No. DEA-AC03-76SF0098 and Grant No. PDDEFG-03-95ER-40936. The authors are grateful to Edward P. Lee for many useful discussions, and to Richard C. Fernow for the use of the code ICOOL, and much discussion concerning its use.

6. REFERENCES

[1] A.N. Skrinsky, Beam Dynamics and Technology Issues for $\mu^+\mu^-$ Colliders, Proc. of the 9th Advanced ICFA Beam Dynamics Workshop, Montauk, NY, p. 133 (1996).
[2] Tatiana A. Vsevolozhskaya, *ibid*., p. 159.
[3] David Neuffer, Particle Accelerators 14, 75 (1983).
[4] David Neuffer, A. Van Ginneken, NIM A 403, 1 (1998).
[5] "Ionization cooling in the muon collider", Proc. of the 1997 DPF/DPB Summer Study "New Directions for HEP", Snowmass, CO, June 25-July 12, 1996, Chapter 6 (BNL Report # 52503).
[6] "Transverse Emittance Cooling in the Muon Collider", R.C. Fernow et. al., Submitted to the 1998 Advanced Accelerator Conference, July 5-11, 1998, Baltimore, MD.
[7] "Ionization Cooling Research and Development Program for a High Luminosity Muon Collider", FNAL Proposal P904, April 1998.
[8] G.R. Lynch and O.I. Dahl, Nucl. Inst. and Meth. B58, 6 (1991).