TUNE VALUE EVALUATION FOR COMBINED FUNCTION LATTICE

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Abstract

On a compact synchrotron with combined function lattice, a tune value evaluated by the conventional method may not be accurate because of fringing and connecting region, which are difficult to treat accurately with the conventional method. We evaluate a tune value by reconstructing transfer matrix from particle orbits which are tracked in given magnetic field.

1 INTRODUCTION

For cancer therapy, compactness and easy handling in daily operation are required for a synchrotron system. We can simplify the power supply system and many tuning components by introducing combined function magnets. The simplification of the tuning components, however, reduces the tunability of operating point after machining. Thus it is necessary to evaluate the accurate tune value in designing stage. Because the dipole and the quadrupole components decays differently in the fringing area and the transition region of the combined function magnet, the direct derivation of the transfer matrix by the conventional method may not be accurate. Then we took an indirect method, which reconstructs a transfer matrix from betatron oscillation orbits. In the following paper, we report our calculation method and its results.

2 BASIC THEORY

We can obtain a orbit of particle in the given magnetic field by integrating the equation of motion:

$$\gamma m \frac{d\boldsymbol{v}}{dt} = q\boldsymbol{v} \times \boldsymbol{B}, \qquad (1)$$

where $d\gamma/dt$ is assumed to be zero. Then, we reconstruct transfer matrix from betatron oscillating orbits and extract both beta-function and tune-value from reconstructed transfer matrix.

2.1 Central Orbit and Coordinate

To find the central orbit, we determine a central orbit condition from closed orbit conditions using symmetry of orbit. Integrating equation (1) with such condition, we obtain the central orbit $\boldsymbol{r}_0(s)$ $(s(t) = \int_0^t |\frac{d\boldsymbol{v}}{dt}|dt)$. Thus, we introduce a curvilinear coordinate(see Fig.1) by Servet-Frenet

formula:

$$\boldsymbol{r}(x, s, z) = \boldsymbol{r}_0(s) + x \cdot \boldsymbol{n}(s) + z \cdot \boldsymbol{b}(s), \qquad (2)$$

$$\boldsymbol{t}(s) = \frac{d\boldsymbol{r}_0(s)}{ds}, \qquad (3)$$

$$\boldsymbol{b}(s) = \boldsymbol{n}(s) \times \boldsymbol{t}(s), \quad \text{and} \qquad (4)$$

$$\frac{d\boldsymbol{t}(s)}{ds} = -G(s) \cdot \boldsymbol{n}(s), \qquad (5)$$

where $G(s)$ is curvature. In the curvilinear coordinate, a particle position in phase space is denoted by 6-parameters $(s, x, x', z, z', \triangle p/p_0)$.
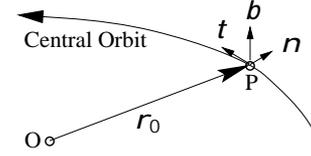
Figure 1: Curvilinear Coordinate

2.2 Transfer Matrix Reconstruction

The particle motion is considered as the mapping of $(x, x', z, z', \triangle p/p_0)$ in s space. The transfer matrix is defined as a first order term of this mapping from $s = s_0$ plane to $s = s_1$ plane:

$$\begin{pmatrix} x_1 \\ x'_1 \\ z_1 \\ z'_1 \\ (\frac{\triangle p}{p_0})_1 \end{pmatrix} = \boldsymbol{M}(s_1|s_0) \begin{pmatrix} x_0 \\ x'_0 \\ z_0 \\ z'_0 \\ (\frac{\triangle p}{p_0})_0 \end{pmatrix}$$
$$+ \text{ nonlinear term.} \qquad (6)$$

The coordinate at $s = s_1$ plane is derived from the coordinate at $s = s_0$ plane by integrating equation of motion. Thus the transfer matrix is reconstructed by Least Square Method using betatron orbits which have an amplitude small enough to neglect nonlinear term of x, x', z, z', and $\triangle p/p_0$. Considering that $x - z$ coupling is negligible and $\triangle p$ is fixed to zero, equation (6) is rewritten by the least square method as:

$$\sum_n \begin{pmatrix} \xi_1^{(n)} \\ \xi_1'^{(n)} \end{pmatrix} (\xi_0^{(n)} \xi_0'^{(n)})$$
$$= \boldsymbol{M}(s_1|s_0) \sum_n \begin{pmatrix} \xi_0^{(n)} \\ \xi_0'^{(n)} \end{pmatrix} (\xi_0^{(n)} \xi_0'^{(n)}), \quad (7)$$

where ξ represents either x or z and n is the orbit index. For calculation simplicity, we choose $(\xi_0^{(n)}, \xi_0'^{(n)})$ so that $\sum_n \xi_0^{(n)} \xi_0'^{(n)} = 0$ and rewrite equation (7) as

$$
\boldsymbol{M}(s_1|s_0) = \sum_n \begin{pmatrix} \xi_1^{(n)} \xi_0^{(n)} & \xi_1^{(n)} \xi_0'^{(n)} \\ \xi_1'^{(n)} \xi_0^{(n)} & \xi_1'^{(n)} \xi_0'^{(n)} \end{pmatrix}
$$
$$
\times \begin{pmatrix} \sum_n |\xi_0^{(n)}|^2 & 0 \\ 0 & \sum_n |\xi_0'^{(n)}|^2 \end{pmatrix}^{-1}. \quad (8)
$$

Using product rule, a relation between one revolution transfer matrix $\boldsymbol{M}_{rev}(s)$ and $\boldsymbol{M}_{rev}(s_0)$ is obtained as:

$$
\begin{aligned}
\boldsymbol{M}_{rev}(s) &= \boldsymbol{M}(s+C|s) \\
&= \boldsymbol{M}(s+C|s_0+C)\boldsymbol{M}(s_0+C|s_0)\boldsymbol{M}(s_0|s) \\
&= \boldsymbol{M}(s|s_0)\boldsymbol{M}_{rev}(s_0)\boldsymbol{M}(s|s_0)^{-1}, \quad (9)
\end{aligned}
$$

where C is one revolution length of the ring. We obtain one revolution transfer matrix which have any periodic boundary from transfer matrix $\boldsymbol{M}(s|s_0)$ ($s_0 < s \leq s_0 + C$) reconstructed by one path tracking of betatron orbits.

2.3 Tune Value

In Hill equation formalism, one revolution transfer matrix $\boldsymbol{M}_{rev}$ is related to phase advance and twiss parameters by following equation:

$$
\boldsymbol{M}_{rev} = \begin{pmatrix} \cos\mu + \alpha\sin\mu & \beta\sin\mu \\ -\gamma\sin\mu & \cos\mu - \alpha\sin\mu \end{pmatrix}, \quad (10)
$$

where μ is phase advance. α, β, and γ are twiss parameters at the revolution boundary. Applying equation (10) to reconstructed one revolution transfer matrix $\boldsymbol{M}_{rev}(s)$ and considering relationship between twiss parameters $1 + \alpha^2 = \beta\gamma$, we obtain phase advance μ and twiss parameters:

$$
\mu = \cos^{-1}\frac{m_{11} + m_{22}}{2}, \quad (11)
$$

$$
\beta = \frac{1}{\sqrt{-\frac{m_{21}}{m_{12}} - \left(\frac{m_{11}-m_{22}}{2m_{12}}\right)^2}}, \quad (12)
$$

$$
\alpha = \frac{m_{11} - m_{22}}{2m_{12}}\beta, \text{ and } \gamma = -\frac{m_{21}}{m_{12}}\beta, \quad (13)
$$

where m_{ij} is the component of $\boldsymbol{M}_{rev}(s)$ defined by

$$
\boldsymbol{M}_{rev}(s) = \begin{pmatrix} m_{11} & m_{12} \\ m_{21} & m_{22} \end{pmatrix}. \quad (14)
$$

Thus s dependence of twiss parameters are obtained from one revolution transfer matrix $\boldsymbol{M}_{rev}(s)$. And the tune value ν is given by the definition $\nu = \mu/2\pi$.

3 TRACKING METHOD

The transfer matrix is reconstructed by particle tracking in the magnetic field which is given by TOSCA [1].

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[1] 3D magnetic field calculation code: Vector Fields Limited(24 Bankside, Kidlington, Oxford OX5 1JE, England)

3.1 Ring with Combined Function Magnets

The proton synchrotron ring[1] which we consider has median plane symmetry and 60 degree rotational symmetry (see Fig.2). Each bending magnet is an FDF combined
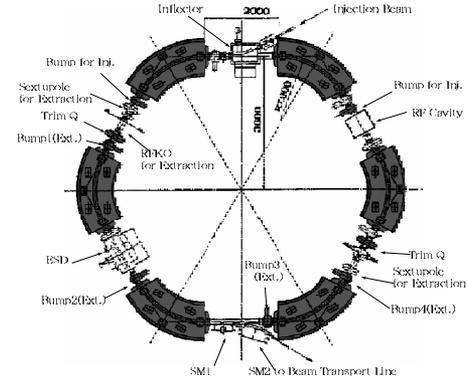
Figure 2: Proton Synchrotron Ring which we consider

function magnet and bending angle of F sector and D sector are 15.25deg and 29.5deg, respectively. A simple central orbit has 60 degree rotational symmetry and stays on median plane. Considering the symmetry of the central orbit, equation (10) can be applied to one cell transfer matrix $\boldsymbol{M}_{cell}(s) = \boldsymbol{M}(s + C/6|s)$ to calculate the phase shift in a cell and twiss parameters. Thus the product rule gives relations $\boldsymbol{M}_{rev} = (\boldsymbol{M}_{cell})^6$ and $\nu = \frac{3}{\pi}\mu_{cell}$, where μ_{cell} is the phase shift in a cell.

3.2 Magnetic Field

We must interpolate the magnetic field because TOSCA gives magnetic field values at discrete points. Considering the sector shape of the magnet, it is convenient to use cylindrical coordinate(ρ, θ, Z) for magnetic field calculation. Because of median plane symmetry, magnetic field component B_ρ and B_θ vanish on central orbit plane(median plane). Considering that $\text{rot}\boldsymbol{B} = 0$ and $\text{div}\boldsymbol{B} = 0$, magnetic field close to median plane can be expanded as follows:

$$
B_Z(\rho, \theta, Z) = B_Z(\rho, \theta, 0) + O(Z^2), \quad (15)
$$

$$
B_\rho(\rho, \theta, Z) = Z\frac{\partial B_Z}{\partial \rho}(\rho, \theta, 0) + O(Z^2), \text{ and } (16)
$$

$$
B_\theta(\rho, \theta, Z) = \frac{Z}{\rho}\frac{\partial B_Z}{\partial \theta}(\rho, \theta, 0) + O(Z^2). \quad (17)
$$

In order to avoid the direct numerical differentiation, firstly we expanded the field maps into two dimension Fourier series and then the derivatives of the Fourier components are summed up. Fourier expansion on θ axis gives quick convergence because of the cell periodicity. The magnetic field distribution on ρ axis, however, is not periodic and we added mirror image(see Fig.3). We modify a few points nearby both the mirror boundaries, which are far away from the orbit, for C^2 class continuity by following conditions:
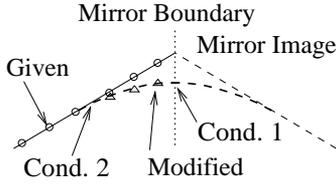
Figure 3: Mirror Image Connection

1. first and second order derivative vanish at the mirror boundary,

2. field component and first order derivative are connected continuously and second order derivative vanish at the boundary between modified region and given data region.

3.3 Tracking

The equation of motion is integrated by Runge Kutta Gill method and the betatron oscillation amplitude for initial condition is determined by following method:

1. calculate tune value ν_n from given amplitude series $a_n = a_0 r^n$ $(0 < r < 1$, we use $r \sim 1/3)$,

2. select the amplitude a_n to satisfy $|\nu_{n+1} - \nu_n|$ less than required tolerance.

4   PERTURBATION METHOD

We can obtain the excitation dependence of the tune value by the perturbation method for comparison. From the given central orbit and beta function, the tune shift is obtained as:

$$\triangle\nu = \frac{1}{4\pi} \oint_{Ring} \beta(s)\triangle k(s)ds, \qquad (18)$$

where $\beta(s)$ is beta function and $\triangle k(s)$ is an error term of Hill equation, which is given by the lowest order expansion of equation of motion:

$$\triangle k_x = \frac{1-n}{\rho^2}, \qquad \triangle k_z = \frac{n}{\rho^2} \qquad (19)$$

$$n = -\frac{\rho}{B_Z}\frac{\partial B_Z}{\partial x}, \qquad \rho = -\frac{p}{qB_Z}, \qquad (20)$$

where q and p are particle charge and momentum, respectively. In this calculation, the central orbit and its beta function at the energy $E = 26$MeV are used.

5   RESULTS

Figure 4 and 5 show a beta function and an excitation dependence of the tune value, respectively. In Figure 5, the circle points show the tune values calculated by the tracking method and the square points show the tune values calculated by the perturbation method. There is good agreement between two methods. Figure 6 shows the integral term of the equation (18). Because the beta function in the

D sector is larger than that in the F sectors, the n value variation in the D sector makes the bend on the tune shift around the energy $E = 150 \sim 180$MeV.
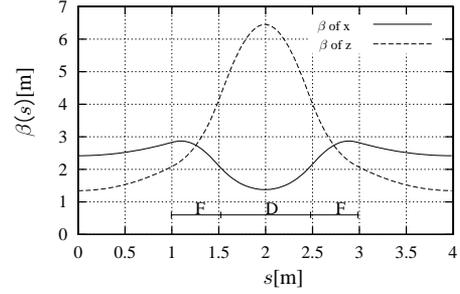
Figure 4: Beta function at the energy $E = 26$MeV
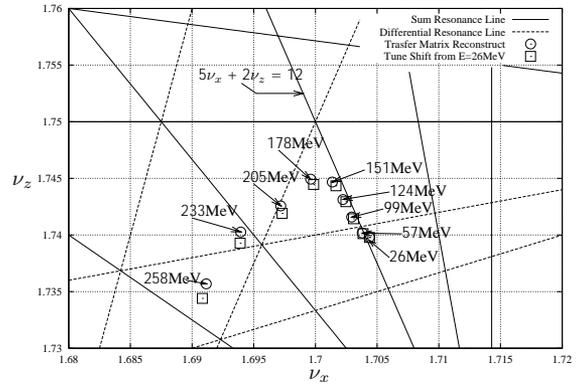
Figure 5: Excitation dependence of the tune value with resonance lines up to 8th order
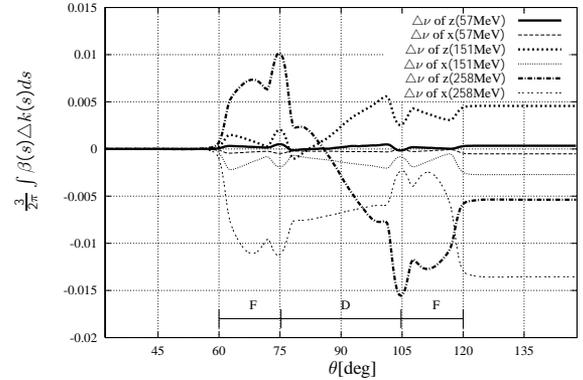
Figure 6: θ dependence of integral term $\int_{s_O}^{s} \beta(s)\triangle k(s)ds$

6   REFERENCES

[1] Akira NODA, et al., "Development of Compact Proton Synchrotron with Combined Function Dedicated for Cancer Therapy", Proceedings of the 11th Symposium on Accelerator Science and Technology, October 21-23, 1997, SPring-8, Harima Science Garden City, Hyogo, Japan