HEAD-TAIL MODE INSTABILITY CAUSED BY FEEDBACK

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Abstract

In [1], Kikutani uses a two-particle model to demonstrate the existence of a "head-tail" instability caused by the presence of a feedback system. This paper demonstrates that this instability is a general feature of storage rings with a transverse low-frequency feedback system which damps the rigid (m=0) modes but attempts to leave the head-tail (m=1) modes unaffected. The growth rate is an effect of transverse mode coupling, but doesn't have a threshold. The growth rate increases quadratically (or even cubically under some assumptions) with current for small currents for a given feedback gain. For low gains, the effect is linear in the feedback gain. The formulation given is based on a Vlasov equation analysis, incorporating an impedance-like representation for the feedback system [2]. The associated growth rates for the $m = 1$ modes can be computed in the presence of an arbitrary impedance and feedback transfer function. In general, one needs to consider impedance together with the feedback system to get the correct effect; ignoring the impedance will give the incorrect result. The effects can be computed just as easily for a symmetric multibunch system as for a single-bunch system.

1 INTRODUCTION

This paper approaches the analysis of the stability of the beam in a circular storage ring under the influence of collective effects and feedback using the Vlasov equation. An equilibrium distribution of particles is assumed, and perturbations to that distribution are Fourier analyzed in turn number. The resulting equations will only have self-consistent solutions for certain eigenfrequencies. If these eigenfrequencies have positive imaginary parts, the system is unstable.

The question to be addressed is what effect a transverse feedback system has on these eigenvalues in addition to the desired effect, namely to provide a *negative* imaginary part to one of the eigenvalues (the "rigid bunch" or "$m = 0$" mode). It turns out that except in exceptional circumstances, the feedback system will introduce a positive imaginary part into at least one of the "head-tail" or "$m = 1$" modes, creating a potential instability.

The paper will first introduce the equations describing the system in a fairly general form. Then the necessary perturbation theory will be briefly reviewed. Finally, some standard simplifications of the eigenvalue system are given, which will allow one to easily apply the perturbation theory results.

1.1 Eigenvalue System

Begin with the Fourier analyzed Vlasov equation. In obtaining this, one has assumed that one can normalize the Hamiltonian consisting of the lattice without collective effects plus potential well distortion. The action-angle variables of this system are $\boldsymbol{J}$ and $\boldsymbol{\theta}$. The Vlasov equation is then Fourier analyzed in turn number (frequency Ω) and $\boldsymbol{\theta}$ (index $\boldsymbol{m}$). Multiple bunches are allowed, but for simplicity the equations are given assuming that all bunches have the same charge and are symmetrically placed. This is not essential to the method.

In addition, any tune shift with amplitude in the normalized Hamiltonian is ignored. Adding tune shift with amplitude changes the character of the eigenfrequency spectrum, and a different analysis is required. The analysis here holds to the extent that the frequency shifts from their zero-current values are large compared to the nonlinear frequency spread.

With these assumptions, the eigenvalue equation can be written

$$\left[\Omega - \boldsymbol{m}\cdot\boldsymbol{\omega} + i\beta c\frac{\partial}{\partial s}\right]\Psi_{\boldsymbol{m}p}(\boldsymbol{J},s) = i(2\pi)^3\frac{q^2NM}{\gamma m_q L}\boldsymbol{m}\cdot\frac{\partial\Psi_0}{\partial\boldsymbol{J}}\sum_{\substack{\bar{\boldsymbol{m}}\\ \bar{p}=p+p'M}}\Bigg[\int Z^{\mathrm{FB}}_{\boldsymbol{m}\bar{\boldsymbol{m}}}(\boldsymbol{J},\bar{\boldsymbol{J}},s,\bar{s},\omega_{\bar{p}})e^{-i\bar{p}\omega_0(s-\bar{s})/\beta c}\Psi_{\bar{\boldsymbol{m}}p}(\bar{\boldsymbol{J}},\bar{s})\,d\bar{\boldsymbol{J}}\,d\bar{s} + \int Z_{\boldsymbol{m}\bar{\boldsymbol{m}}}(\boldsymbol{J},\bar{\boldsymbol{J}},s,\omega_{\bar{p}})\Psi_{\bar{\boldsymbol{m}}p}(\bar{\boldsymbol{J}},s)\,d\bar{\boldsymbol{J}}\Bigg]. \quad (1)$$

Here βc is the velocity of a particle with charge q and mass m_q following the ideal orbit of length L. ω_0 is the angular circulation frequency for that ideal particle. γ is just $(1-\beta^2)^{-1/2}$. There are M bunches with N charges in each bunch. The frequencies of oscillation for a single particle in the field of the lattice plus potential well distortion (due to an equilibrium distribution $\Psi_0(\boldsymbol{J})$) are $\boldsymbol{\omega}$. Ω is the eigenvalue which we're trying to find, with eigenfunctions $\Psi_{\boldsymbol{m}p}(\boldsymbol{J},s)$ which are periodic in s. ω_p is a shorthand for $p\omega_0+\Omega$.

The function $Z_{\boldsymbol{m}\bar{\boldsymbol{m}}}(\boldsymbol{J},\bar{\boldsymbol{J}},s,\omega)$ represents the interaction due to passive objects described by an impedance; it is defined to be

$$Z_{\boldsymbol{m}\bar{\boldsymbol{m}}}(\boldsymbol{J},\bar{\boldsymbol{J}},s,\omega) = \sum_\alpha z_\alpha(\omega,s) Y_{\alpha\boldsymbol{m}}(\boldsymbol{J},\omega,s)\tilde{Y}^*_{\alpha\bar{\boldsymbol{m}}}(\bar{\boldsymbol{J}},\omega,s) \quad (2)$$

Here $z_\alpha(\omega,s)$ is the impedance of type α per unit length at the point s. For example, it might be the transverse impedance per unit length, or the longitudinal impedance per unit length multiplied by $\beta c/\omega$. $Y_{\alpha\boldsymbol{m}}$ is defined to be

$$Y_{\alpha\bar{\boldsymbol{m}}}(\boldsymbol{J},\omega,s) = \frac{1}{(2\pi)^3}\int f_\alpha(\boldsymbol{J},\boldsymbol{\theta},s)e^{i\omega z(\boldsymbol{J},\boldsymbol{\theta},s)/\beta c}e^{-i\boldsymbol{m}\cdot\boldsymbol{\theta}}d^3\boldsymbol{\theta} \quad (3)$$

and $\tilde{Y}_{\alpha\bar{\boldsymbol{m}}}$ is the same except with f_α replaced by g_α. f_α and g_α are part of the definition of the impedance: f_α multiplied by z_α (or really its Fourier transform, the wake function) multiplied by the integral of g_α times the distribution gives a function whose gradients are the collective force on a test charge. For example, for a transverse wake, both f_α and g_α would be the coordinate y written in terms of action-angle variables. For a longitudinal wake, both f_α and g_α would be 1. The function z in the exponent is just the arrival time relative to the arrival time a particle following the ideal orbit, multiplied by $-\beta c$.

Transverse wakes are of particular interest for this paper, and both $Y_{\alpha\boldsymbol{m}}(\boldsymbol{J},\omega,s)$ and $\tilde{Y}_{\alpha\boldsymbol{m}}(\boldsymbol{J},\omega,s)$ for that case are

$$\frac{1}{2\pi}\sqrt{2J_y\beta_y(s)}e^{i[\pm\Delta\psi_y(s)+m\Delta\psi_z(s)]}(i)^m J_m\left(\frac{(\omega\mp\omega_\xi)\sigma_z(s)}{\beta c}\sqrt{\frac{2J_z\beta\gamma m_q c}{\epsilon_\ell}}\right). \quad (4)$$

$\boldsymbol{m} = (0,\pm1,m)$ here. $\Delta\psi_{y,z}(s)$ are the phase advance at a point s relative to a phase linearly advancing with s, ω_ξ is $\xi_y\omega_y/\eta$, ξ_y is the chromaticity, η is the frequency slip factor (positive above transition), σ_z is the r.m.s. bunch length (in dimensions of length), and ϵ_ℓ is the r.m.s. longitudinal emittance in energy-time units.

The function $Z^{\rm FB}_{\boldsymbol{m}\bar{\boldsymbol{m}}}(\boldsymbol{J},\bar{\boldsymbol{J}},s,\bar{s},\omega_p)$ represents the interaction due to the feedback system. It can be written as

$$Z^{\rm FB}_{\boldsymbol{m}\bar{\boldsymbol{m}}}(\boldsymbol{J},\bar{\boldsymbol{J}},s,\bar{s},\omega) = \sum_\alpha z^{\rm FB}_\alpha(\omega,s,\bar{s})Y^*_{\alpha\boldsymbol{m}}(\boldsymbol{J},\omega,s)\tilde{Y}_{\alpha\bar{\boldsymbol{m}}}(\bar{\boldsymbol{J}},\omega,\bar{s}) \quad (5)$$

Here $z_\alpha(\omega,s,\bar{s})$ is the Fourier transform of the feedback response per unit length at s per until length of pickup at $\bar{s}$. The equations are somewhat simplified if one assumes that $Z^{\rm FB}_{\boldsymbol{m}\bar{\boldsymbol{m}}}$ can be written in the form

$$Z^{\rm FB}_{\boldsymbol{m}\bar{\boldsymbol{m}}}(\boldsymbol{J},\bar{\boldsymbol{J}},s,\bar{s},\omega) = \bar{Z}^{\rm FB}_{\boldsymbol{m}\bar{\boldsymbol{m}}}(\boldsymbol{J},\bar{\boldsymbol{J}},s,\omega)\delta\big(s-\bar{s}-\Delta s(s)\big)e^{i\omega\Delta s(s)/\beta c} \quad (6)$$

2 PERTURBATION THEORY

Now one must break the eigenvalue equation (1) into a "unperturbed" part plus a perturbation part. The perturbation will consist of the feedback term plus the impedance terms for which $\boldsymbol{m} \neq \bar{\boldsymbol{m}}$. The reason for not using the zero-current system as the unperturbed system is that it is desirable to be able to do nondegenerate perturbation theory: for zero current, all the radial modes corresponding to a single azimuthal mode index have degenerate eigenfrequencies.

2.1 *Nondegenerate Perturbation Theory*

Begin with a matrix A with eigenvalues ξ_k and corresponding eigenvectors w_k, such that $Aw_k = \xi_k w_k$. Assume that the ξ_k are all different. There are also vectors $\tilde{w}_k$ such that $\tilde{w}_k^\dagger w_\ell = \delta_{k\ell}$. Say we want to find eigenvectors and eigenvalues for $A+\epsilon B$, where ϵ is small. Denote the new eigenvectors and eigenvalues through $(A+\epsilon B)v_k = \lambda_k v_k$, and write λ_k and v_k as power series in ϵ:

$$\lambda_k = \sum_\ell \lambda_{k\ell}\epsilon^\ell \quad v_k = \sum_\ell v_{k\ell}\epsilon^\ell = \sum_{\ell m} c_{k\ell m}\epsilon^\ell w_m \quad (7)$$

Then equate equal powers of ϵ and multiply by $\tilde{w}_m^\dagger$ in the original eigenvalue equation, and choose $\lambda_k = \xi_k$ and $v_k = w_k$. Assuming nondegeneracy, the first few coefficients are:

$$\lambda_{k1} = \tilde{w}_k^\dagger B w_k \qquad c_{k1m} = \frac{\tilde{w}_m^\dagger B w_k}{\xi_k-\xi_m}(1-\delta_{km}) \quad (8)$$

$$\lambda_{k2} = \sum_{j\neq k}\frac{\tilde{w}_k B w_j \tilde{w}_j B w_k}{\xi_k-\xi_j} \quad (9)$$

$$c_{k2m} = (1-\delta_{km})\left[\sum_{j\neq k}\frac{\tilde{w}_m^\dagger B w_j \tilde{w}_j^\dagger B w_k}{(\xi_k-\xi_j)(\xi_k-\xi_m)} - \frac{\tilde{w}_m^\dagger B w_k \tilde{w}_k^\dagger B w_k}{(\xi_k-\xi_m)^2}\right] \quad (10)$$

$$\lambda_{k3} = \sum_{\substack{j\neq k\\ n\neq k}}\frac{\tilde{w}_k^\dagger B w_j \tilde{w}_j^\dagger B w_n \tilde{w}_n^\dagger B w_k}{(\xi_k-\xi_n)(\xi_k-\xi_j)} - \sum_{j\neq k}\frac{\tilde{w}_k^\dagger B w_k \tilde{w}_k^\dagger B w_j \tilde{w}_j^\dagger B w_k}{(\xi_k-\xi_j)^2} \quad (11)$$

2.2 *Feedback*

One can now use this perturbation expansion to find the eigenvalues in the presence of azimuthal mode coupling and feedback. Assume that the mode to be damped by the feedback system is $k=0$.

Examine each order in perturbation theory. To first order, λ_0 shifts by $\epsilon\tilde{w}_0^\dagger B w_0$. This is the intended effect of the feedback system, and therefore B is generally chosen so that this is a pure negative imaginary number. The second order terms give

$$\lambda_{02} = \sum_{k\neq 0} \frac{\tilde{w}_0^\dagger B w_k \tilde{w}_k^\dagger B w_0}{\xi_0 - \xi_k} \quad \lambda_{k2} = \frac{\tilde{w}_k^\dagger B w_0 \tilde{w}_0^\dagger B w_k}{\xi_k - \xi_0}$$

If there were no feedback, these would just give the standard mode coupling effects; for a single bunch with wakefields lasting less than one turn, these would be purely real since $\tilde{w}_k^\dagger B w_0 = -[\tilde{w}_0^\dagger B w_k]^*$.

With a feedback system, one can write $B = B_Z + B_F$, B_Z being the term from the impedance, B_F coming from the feedback. For a feedback system where the feedback response is for a short time, and when the feedback is designed to only give damping to the $m = 0$ mode, one finds that $\tilde{w}_k^\dagger B_F w_0 = [\tilde{w}_0^\dagger B_F w_k]^*$. Thus, there are terms in λ_{k2} which for a single bunch would be purely imaginary due to the opposite symmetries of B_F and B_Z:

$$\frac{\tilde{w}_k^\dagger B_F w_0 \tilde{w}_0^\dagger B_Z w_k + \tilde{w}_k^\dagger B_Z w_0 \tilde{w}_0^\dagger B_F w_k}{\xi_k - \xi_0} \qquad (12)$$

Finally, the dominant contribution of the feedback to the third order term is

$$\lambda_{k3} = \frac{\tilde{w}_0^\dagger B w_0 \tilde{w}_k^\dagger B w_0 \tilde{w}_0^\dagger B w_k}{(\xi_k - \xi_0)^2} = \frac{\lambda_{01}\lambda_{k2}}{\xi_k - \xi_0} \qquad (13)$$

There is also an additional term which doesn't involve the feedback, as well as an additional effect on λ_0 which should be small compared to the first order effect.

To find the growth or damping rate in the higher order mode due to the feedback system from these third order terms, multiply the growth rate for the $m = 0$ mode by the shift in the higher order mode due to azimuthal mode coupling divided by the separation between the two modes. With mode coupling, the mode frequencies generally approach each other more rapidly with increasing current than they would if there were no mode coupling. Thus λ_{k2} and $\xi_k - \xi_0$ will have opposite signs (they are complex numbers in the multibunch case, so this is really "their complex phases differ by around π"). Therefore a feedback system which damps the $m = 0$ modes will cause the higher order modes that it couples to to grow! In the single bunch case where $m = 0$ and $m = 1$ are coupled, this is exact (one must be careful with synchro-betatron modes [3]: the $+\omega_y$ $m = 0$ mode does not give instability when approaching the $-\omega_y$ $m = 1, 3, \ldots$ mode, and a feedback system will damp these modes as well: if the modes give instability when coupling, then the feedback system will cause instability).

Whether the second or the third order term will dominate will depend on an interplay between the mode coupling terms and the two dominant terms that the feedback will induce (the direct term and the coupling terms). If the coupling term $\tilde{w}_k^\dagger B w_0$ induced by the feedback is negligible (this will be true if the phase of the feedback transfer function is kept constant, and the delay is exactly correct), then the third order term will dominate. While this is often assumed, in practice, even with a very narrow band feedback system such as that used in the LHC, this is not the case, and the second order term dominates.

Finally, this discussion assumes that the feedback has no direct effect on the modes other than $k = 0$ (neglecting $\tilde{w}_k^\dagger B w_k$ for $k \neq 0$). Such a feedback system would have to have a rather large bandwidth: it must have gain at frequencies higher than those corresponding to the bunch length.

3 EXAMPLE: GAUSSIAN BUNCHES

The eigenvalue problem for transverse impedance and feedback can be simplified by ignoring a) terms due to the s variation of the impedance and feedback; b) coupling between modes near $\Omega = +\omega_y$ and modes near $\Omega = -\omega_y$; and c) terms which are at least first order in ϵ_y/ϵ_z, the ratio of the transverse to longitudinal emittance. Under these assumptions, the transverse eigenvalue problem becomes

$$\begin{aligned}(\Omega - \omega_y - m\omega_s)\Psi_{m\ell p} = -i\frac{q^2 NM}{2\gamma m_q L^2} \sum_{\substack{\bar{\ell}\bar{m} \\ \bar{p}=p+p'M}} \\ [\langle \beta_y Z_\perp(\omega_{\bar{p}})\rangle + \beta_y \langle Z_\perp^{\mathrm{FB}}(\omega_{\bar{p}}) e^{-i\bar{p}\Delta s/\beta c}\rangle] \\ h_{m\ell}(\omega_{\bar{p}}) h^*_{\bar{m}\bar{\ell}}(\omega_{\bar{p}}) \Psi_{\bar{m}\bar{\ell}p} \qquad (14)\end{aligned}$$

For Gaussian bunches, the $h_{m\ell}$ are [4]

$$h_{m\ell}(\omega + \omega_\xi) = \frac{1}{\sqrt{\ell!(m+\ell)!}} \left(\frac{\sigma_\ell \omega}{\sqrt{2}\beta c}\right)^{m+2\ell} e^{-\sigma^2\omega^2/2\beta^2 c^2}$$

The eigenfunctions of the "unperturbed" problem will of course vary depending on what the impedance is. However, because the impedance is generally dominated by frequencies which are small or comparable to the frequencies corresponding to the bunch length, one can safely consider only the $\ell = 0$ terms in (14), and the results will be correct to within about 20%. Since there is only one term for a given m, one can immediately obtain estimates of this effect.

In the LHC, for example, the second order terms dominate, giving growth rates which are in the range of 10000-20000 turns (see Figs. 5 and 7 of [5], comparing to Fig. 3). This is roughly a factor of 4 slower than what one expects to be able to tolerate because of Landau damping.

Finally, note that the perturbation theory arguments are very general, and apply to longitudinal as well as transverse impedances and feedback.

4 REFERENCES

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