STATIONARY SELF-CONSISTENT DISTRIBUTION OF BUNCHED BEAM IN RF FIELD

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Abstract

Emittance conservation of a high brightness particle beam in RF accelerator is an issue for existing and future high intensity accelerator projects. If the beam is matched with external focusing field, its distribution function as well as beam emittance are conserved. Finding matched conditions for the beam requires solutions of the self-consistent problem for beam distribution function in phase space. In this paper an analytical approximate solution of Vlasov-Poisson equations for self-consistent particle equilibrium in RF field is found. Solution is attained in approximation of high brightness beam. Distribution function in phase space is determined as a stationary function of the energy integral. Equipartitioning for beam distribution between degrees of freedom follows directly from the choice of stationary distribution function. Analytical expression for r-z equilibrium beam profile in RF field is obtained.

1 INTRODUCTION

The problem of stationary self-consistent particle distribution in RF field was considered in several books and papers. In Ref. [1] solution of one-dimensional problem for longitudinal phase space was found. Space charge density of cylindrical bunch was found to be constant in every cross section of the bunch, but dependent on longitudinal coordinate. In Ref. [2] spatial particle distribution in 3-dimensional configuration space was calculated numerically. In this paper an analytical approximate solution for 3-D self-consistent particle equilibrium is attained [3].

2 BEAM EQUILIBRIUM IN RF FIELD

Consider an intense bunched beam of particles with charge q and mass m, propagating in an uniform focusing channel with an applied accelerating RF field. Single-particle Hamiltonian of particle motion is given by [1]:

$$H = \frac{p_x^2 + p_y^2}{2\, m\, \gamma} + \frac{p_z^2}{2\, m\, \gamma^3} + q\, U_{ext} + q\, \frac{U_b}{\gamma^2}\ , \qquad (1)$$

$$U_{ext} = \frac{v_s E}{\omega}\ [\ I_o(\frac{\omega r}{\gamma v_s}) \sin(\varphi_s - \frac{\omega}{v_s}\zeta) - \sin\varphi_s + \frac{\omega}{v_s}\zeta \cos\varphi_s\ ] + \frac{m\gamma}{2}\ \Omega_r^2 r^2\ , \qquad (2)$$

where U_{ext} is a potential of external field, U_b is a potential of space charge field of the beam, γ is a particle energy, E is an amplitude of accelerating field, v_s is a velocity of synchronous particle, φ_s is a synchronous phase, ω is an RF frequency, Ω_r is a transverse frequency of particle oscillation, r is a particle radius, and $\zeta = z - z_s$ is a longitudinal deviation from synchronous particle.

General approach to find a self-consistent beam distribution function is to represent it as a function of Hamiltonian f = f(H). Convenient way is to use an exponential function $f = f_o \exp(- H / H_o)$:

$$f = f_o \exp(- \frac{p_x^2 + p_y^2}{2\, m\, \gamma H_o} - \frac{p_z^2}{2\, m\, \gamma^3 H_o} - q\, \frac{U_{ext} + U_b \gamma^{-2}}{H_o}). \qquad (3)$$

Let us rewrite distribution function (3) as follow

$$f = f_o \exp(- 2\, \frac{p_x^2 + p_y^2}{p_t^2} - 2\, \frac{p_z^2}{p_l^2} - q\, \frac{U_{ext} + U_b \gamma^{-2}}{H_o}), \qquad (4)$$

where $p_t = 2\sqrt{<p_x^2>} = 2\sqrt{<p_y^2>}$ and $p_l = 2\sqrt{<p_z^2>}$ are double root-mean-square (rms) beam sizes in phase space. Transverse, ε_t, and longitudinal, ε_l, rms beam emittances are:

$$\varepsilon_t = 2\, \frac{p_t}{mc} \sqrt{<x^2>} = 2\, \frac{p_t}{mc} \sqrt{<y^2>}\ , \qquad (5)$$

$$\varepsilon_l = 2\, \frac{p_l}{mc} \sqrt{<\zeta^2>}\ . \qquad (6)$$

Taking together Eqs. (3) - (6), the value of H_o can be expressed as a function of beam parameters:

$$16 \cdot H_o = \frac{m\, c^2}{\gamma} \frac{\varepsilon_t^2}{<x^2>} = \frac{m\, c^2}{\gamma} \frac{\varepsilon_t^2}{<y^2>} = \frac{mc^2}{\gamma^3} \frac{\varepsilon_l^2}{<\zeta^2>}\ , \text{ or} \qquad (7)$$

$$\frac{\varepsilon_t}{R} = \frac{\varepsilon_l}{\gamma\, l}\ , \qquad (8)$$

where $R = 2\sqrt{<x^2>}$ is a beam radius and $l = 2\sqrt{<\zeta^2>}$ is a half-size of the bunch length.

Equations (7) and (8) express the equipartitioning condition for the beam in RF field [4, 5]. From the above derivations it is clear, that equipartitioning is a consequence of a stationarity of the beam distribution function, Eq. (3).

3 SPACE CHARGE FIELD OF THE BUNCH

To find a self-consistent particle distribution, one has to solve a nonlinear Poisson's equation for unknown space charge potential of the beam. Space charge density of the beam is :

$$\rho\,(x,y,\zeta) = q\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} f\, dp_x\, dp_y\, dp_z$$

$$= \rho_o \exp\left(-q\frac{U_{ext} + U_b\gamma^{-2}}{H_o}\right) , \qquad (9)$$

where ρ_o is the space charge density in the center of the bunch. The value of ρ_o is unknown at this point due to the unknown space charge potential of the beam U_b. Let us introduce an average value of space charge density, which is equal to the density of an equivalent cylindrical bunch with the same beam radius and the same half-bunch length:

$$\bar{\rho} = \frac{I\,\lambda}{2\pi R^2\, l\, c} , \qquad (10)$$

where I is an average beam current and $\lambda = 2\pi c/\omega$ is a RF wavelength. The value of ρ_o differs from the average value of space charge density $\bar{\rho}$ as a factor of k:

$$\rho_o = k\,\bar{\rho} . \qquad (11)$$

Introducing dimensionless variables:

$$V_{ext} = \frac{q\,U_{ext}}{H_o} ,\quad V_b = \frac{q\,U_b}{H_o} ,\quad \xi = \frac{r}{a} ,\quad \eta = \frac{\zeta}{a} , \qquad (12)$$

where a is a channel radius, Poisson's equation in cylindrical polar coordinates can be expressed as

$$\frac{1}{\xi}\frac{\partial V_b}{\xi} + \frac{\partial^2 V_b}{\partial \xi^2} + \frac{\partial^2 V_b}{\partial \eta^2 \gamma^2}$$

$$= -\frac{8\,k\,b}{\delta\varphi}\left(\gamma\,\frac{a}{R}\right)^2 \exp -\left(V_{ext} + \frac{V_b}{\gamma^2}\right) . \qquad (13)$$

Here, $\delta\varphi \int \Delta\varphi/(2\pi)$, $\Delta\varphi$ is a phase bunch length, $b \int 2IR^2/(\beta\gamma I_c \varepsilon_t^2)$ is a dimensionless value of beam brightness, and $I_c \int 4\pi\varepsilon_o mc^3/q$ is a characteristic value of beam current. To solve the nonlinear equation (13), let us follow the method suggested in Ref. [6].

Represent unknown space charge potential of the beam by Fourier-Bessel series:

$$V_b = V_o + \sum_{n=0}^{\infty}\sum_{m=1}^{\infty} J_o(\upsilon_{om}\,\xi)\,[\,A_{nm}\cos(k_z n\eta)$$

$$+\; B_{nm}\sin(k_z n\eta)] , \qquad (14)$$

where $J_o(\zeta)$ is a Bessel function, υ_{om} is a m-th root of the equation $J_o(\zeta) = 0$, and $k_z \int \omega a/v_s$ is a wave number. Expansion (14) obeys Dirichlet boundary condition $V_b(a) = V_o$ at the perfect conductive surface of the channel. Constant V_o is defined in such a way that the total potential of the structure vanishes at the bunch center: $V_{ext}(0, 0) + V_b(0, 0)\,\gamma^{-2} = 0$.

To find an approximate solution of Poisson's equation, let us take only the first term in the near-center expansion of exponential function as $\exp(-V_{ext} - V_b\gamma^{-2}) \approx 1 - V_{ext} - V_b\gamma^{-2}$. Poisson's equation then becomes:

$$\sum_{n=0}^{\infty}\sum_{m=1}^{\infty}\left[1 + \frac{\upsilon_{om}^2 + (k_z n)^2\gamma^{-2}}{8kb}(\delta\varphi)\left(\frac{R}{a}\right)^2\right]$$

$$J_o(\upsilon_{om}\xi)\,[A_{nm}\cos(k_z n\eta) + B_{nm}\sin(k_z n\eta)]$$

$$= (1 - V_{ext})\,\gamma^2 - V_o . \qquad (15)$$

Space charge dominated beam transport is achieved, if b >> 1. It gives a possibility to simplify the Poisson's equation (15). Expression in square brackets in Eq. (15) is

$$1 + \frac{\upsilon_{om}^2 + (k_z n)^2\,\gamma^{-2}}{8\,k\,b}(\delta\varphi)\left(\frac{R}{a}\right)^2 = 1 + \delta , \qquad (16)$$

$$\delta = \frac{\upsilon_{om}^2 + (k_z n)^2\,\gamma^{-2}}{8\,k\,b}(\delta\varphi)\left(\frac{R}{a}\right)^2 . \qquad (17)$$

Roots of the Bessel function are $\upsilon_{o1} = 2.408$, $\upsilon_{o2} = 5.52$. Parameter k_z is close to unity:

$$k_z = 2\pi\left(\frac{a}{\lambda\,\beta}\right) \approx 1 . \qquad (18)$$

Taking into account, that $\delta\varphi \le 1$, $R/a \approx 0.5$, the value of δ is much smaller than unity for a high brightness beam:

$$\delta \approx \frac{1}{bk} << 1 . \qquad (19)$$

Therefore, expression, Eq. (16), can be taken out of the sum in Eq. (15). With this approximation, self-consistent space charge dominated beam potential is:

$$V_b = -\frac{\gamma^2}{1+\delta}\,V_{ext} . \qquad (20)$$

Second approximation to the self-consistent potential is given by holding one more term in the expansion of exponential function as $\exp(-V_{ext} - V_b\gamma^{-2}) \approx 1 - V_{ext} - V_b\gamma^{-2} + 0.5(V_{ext} + V_b\gamma^{-2})^2$, so that we have

$$V_b = \gamma^2\left[1 + \delta - V_{ext} - \sqrt{(1+\delta - V_{ext})^2 - V_{ext}(V_{ext} - 2)}\,\right] . \qquad (21)$$

With increasing of beam brightness ($\delta \to 0$), solution of Eq. (20) becomes close to that of Eq.(21).

Taking the first approximation to the space charge potential of the beam (20), the Hamiltonian corresponding to the self-consistent bunch distribution is as follows:

$$H = \frac{p_x^2 + p_y^2}{2\, m\, \gamma} + \frac{p_z^2}{2\, m\, \gamma^3} + q\left(\frac{\delta}{1+\delta}\right) U_{ext} \quad . \qquad (22)$$

Equation (22) indicates that in the presence of intense, bright bunched beam ($\delta << 1$) the stationary longitudinal phase space of the beam becomes narrow in momentum spread, remaining, in the first approximation, the same in coordinate. This is in a qualitative agreement with the study of Ref. [1].

4 STATIONARY BEAM PROFILE

Self consistent space charge distribution of matched beam in the channel is given from the Poisson's equation by

$$\rho_b = - \varepsilon_o \Delta U_b \quad . \qquad (23)$$

Substitution of Eq. (21) into Eq. (23) gives the stationary particle density distribution inside the bunch:

$$\rho(r,\zeta) = \rho_o \left\{ 1 - \frac{\delta}{\sqrt{(1+\delta)^2 - 2\delta V_{ext}}} - \frac{\delta^2}{32\gamma^2} \frac{\varepsilon_t^2}{<x^2>} \left(\frac{c}{a\Omega_r}\right)^2 \frac{\left(\frac{\partial V_{ext}}{\xi}\right)^2 + \left(\frac{\partial V_{ext}}{\gamma\,\eta}\right)^2}{[(1+\delta)^2 - 2\delta V_{ext}]^{3/2}} \right\} . \qquad (24)$$

For high brightness beam, parameter $\delta << 1$, therefore, space charge density is close to constant within the bunch. From Eq. (20) it follows, that, in the first approximation, space charge potential of the beam is the same function of coordinates, as the external potential, with opposite sign. Therefore, equation U_{ext} (r, ζ)= const gives the family of equipotential lines of space charge field of the beam:

$$I_o\left(\frac{\omega r}{\gamma v_s}\right)\sin\left(\varphi_s - \frac{\omega \zeta}{v_s}\right) - \sin\varphi_s + \frac{\omega\zeta}{v_s}\cos\varphi_s + C\, r^2 = \text{const}, \qquad (25)$$

$$C = \frac{m\,\gamma\,\Omega_r^{\,2}\,\omega}{2\, v_s\, E} \quad . \qquad (26)$$

Consider bunch with boundary R(ζ), defined by nonlinear equation :

$$I_o\left(\frac{\omega R}{\gamma v_s}\right)\sin\left(\varphi_s - \frac{\omega\zeta}{v_s}\right) - \sin\varphi_s + \frac{\omega\zeta}{v_s}\cos\varphi_s + C_1 R^2 = \text{const} \quad .(27)$$

In general case, bunch boundary R(ζ) does not create an equipotential surface. Nevertheless, space charge potential of uniformly populated bunch with boundary, Eq. (27), is close to that, given by Eq. (2). The value of constant in Eq. (27) can be defined from the condition, that longitudinal bunch size is, in the first approximation, the same as for zero - current mode. Therefore, at R(ζ)=0, the left bunch boundary is $\zeta = - 2\varphi_s$ and the value of constant is

$$\text{const} = 2\varphi_s \cos\varphi_s - 2 \sin\varphi_s \quad . \qquad (28)$$

Finally, the first approximation to the beam profile is given by the following equation

$$F(R, \zeta) = I_o\left(\frac{\omega R}{\gamma v_s}\right) \sin\left(\varphi_s - \frac{\omega\zeta}{v_s}\right) + \sin\varphi_s - \left(2\varphi_s - \frac{\omega\zeta}{v_s}\right)\cos\varphi_s + C_1 R^2 = 0 \quad . \qquad (29)$$

In Fig. 1 the uniformly populated bunch with boundary, Eq. (29), is presented. As seen, bunch boundary in configuration space is similar to separatrix shape.

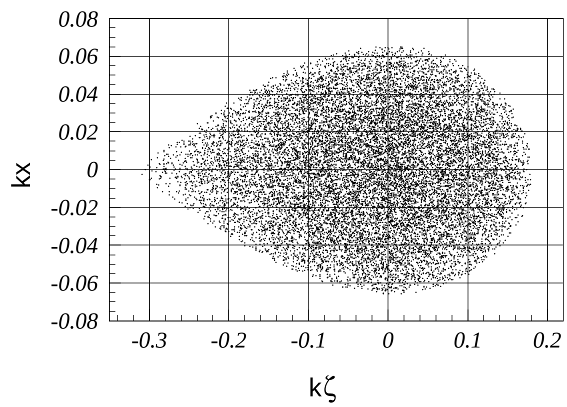

Fig. 1. Approximate stationary particle distribution in RF filed for $\varphi_s = -1$, $C_1 = 3.8$, $k = \omega/v_s$.

5 CONCLUSIONS

An approximate self-consistent solution for a bunched beam in an uniform focusing channel with applied RF acceleration field has been obtained. Analytical derivations were performed in the limit of a high brightness beam, when space charge forces are dominated. Nonlinear equation for stationary beam profile as well as expression for space charge density of the beam inside the bunch are derived.

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