Decoherence of Displaced Beam With Binomial Amplitude Distribution

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Abstract

Decoherence is the dephasing of individual particle motions due to a spread of oscillation frequencies. This has the effect of "washing out" any undriven collective motion, such as might occur by displacing the beam. When the frequency spread arises from amplitude dependence of tune, one must damp the beam (by feedback) well before decoherence is completed, so as to preserve emittance. By means of the Vlasov equation, in action–angle (J, θ) coordinates, we calculate the evolution of the transverse centroid, of a particle distribution which, before displacement, has the binomial form $\psi_0(J) = (1 - J/J_0)^\alpha$. The spread of betatron angular frequency, ω, is modelled by a linear dependence on the action as bJ, where b is the strength of an octupole like force arising from, say, image charge and the chromaticity is assumed to be zero. For free oscillations, the amplitude remains constant and the evolution of the distribution is completely determined by the retarded angle, $\theta - \omega t - bJt$. Only the dipole moment of the off-set distribution is required in performing the assemble averaging for the beam centroid. The amplitude of the centroid oscillation is found to decay inversely with time t to leading order, as $(bt)^{-1}$ for a distribution with $\alpha = 1$, and as $(bt)^{-2}$ for $\alpha = 2,\ 3$.

1 INTRODUCTION

When a previously stationary charged particle beam is displaced transversely from the closed orbit, it oscillates from side to side. Experimentally, the oscillation can be observed with a beam position monitor, which gives the centroid signal. In the absence of collective effects, the oscillation decoheres and the centroid signal decays. Decoherence is caused by a spread in the betatron tunes, and arises from nonlinearity, due to the dependence of tune on amplitude, and/or from chromaticity due to the dependence of tune on longitudinal momentum. We limit ourselves to coasting beams only, and assume the tune spread comes entirely from nonlinearity. Nonlinear forces are always present due to image charges and high multipole fields in the magnets.

It is of interest to calculate the motion of the beam centroid under decoherence. This information is relevant when interpreting beam centroid measurements and useful in the design of a feedback damping system. The decoherence of a Gaussian amplitude distribution, as appropriate for electron beams, due to nonlinearity and chromaticity had been considered in [1, 2]. Here we consider the binomial amplitude distribution which is more suitable for proton beams which typically lack radiation damping. Unlike the Gaussian distribution which extends indefinitely, the binomial distribution has well defined limits. Hence, in comparison, a binomial distribution has a limited tune spread due to nonlinearity, while a Gaussian distribution has an infinite spread and may lead to over estimating of the rate of decoherence.

2 DECOHERENCE DUE TO NONLINEARITY

The decoherence of a displaced particle distribution function, ψ, satisfies the Vlasov equation. First, we outline the formal solution of the Vlasov equation for any general distribution in coordinate and momentum $(x, \dot{x})$ using action and angle coordinates (J, θ). We then apply this solution to the case of a displaced binomial distribution and calculate the beam centroid by forming the ensemble average of the displacement.

The action and angle coordinates are defined in terms of the phase space coordinates $(x, \dot{x})$ as

$$\sqrt{2J}\ \cos\theta = x/\sqrt{\beta}, \qquad (1)$$

$$\sqrt{2J}\ \sin\theta = -\sqrt{\beta}\,\dot{x} + \dot{\beta}x/2\sqrt{\beta}, \qquad (2)$$

where $\beta(s)$ is the betatron function and the derivative is with respect to the path length s. The averaged Hamiltonian for a particle in the presence of a octupole-like force can be written

$$H = \omega J + \frac{1}{2}\, b\, J^2 + \frac{1}{3}(\cos 4\theta + 4\cos 2\theta)\, b J^2, \qquad (3)$$

where ω is the unperturbed angular frequency and b is the averaged octupole strength over one period. The derivative of the last two terms with respect to θ gives the rate of change of J and leads to distortion of the closed phase space paths. However, the distortion is periodic in θ and averages to zero. With the approximate Hamiltonian, the Vlasov equation can be written

$$\frac{\partial\psi}{\partial t} + (\omega + b\, J)\frac{\partial\psi}{\partial\theta} = 0, \qquad (4)$$

where the time parameter t is equal to s divided by the beam velocity. The evolution of an initial distribution $\psi(J, \theta)$ that satisfies the equation is easily found to be

$$\psi(J, \theta, t) = \psi_0[J,\ \theta - (\omega + b\ J)t]. \qquad (5)$$

Note that the solution has the same functional dependence on J as the initial distribution, which is consistent with the approximation that J is a constant. As time increases, the distribution $\psi(J,\theta,t)$ disperses from a localized distribution and eventually occupies the annulus defined by the smallest and largest values of J in the distribution.

The stationary binomial amplitude distribution can be written in terms of the action J as

$$\begin{aligned} \psi_0(J) &= k\left[1-\frac{J}{J_0}\right]^{\alpha}, \\ &= 0 \quad \text{for} \quad J > J_0, \end{aligned} \tag{6}$$

where J_0 is the maximum action, k is the normalization factor, and α is a positive number which characterizes the sharpness of the distribution. See Figure 1. Without loss of generality, assume at time $t = 0$ the distribution is displaced by a distance D but with no change in the divergence. The displaced distribution ψ now becomes a function of both J and θ. The centroid is calculated by taking the ensemble average of the displacement $\sqrt{2J}\cos\theta$,

$$\overline{x}(t) = \int\!\!\int \sqrt{2J}\cos\theta\,\psi[J,\,\theta-(\omega+b\,J)t]\;dJ\,d\theta, \tag{7}$$

where the integration limits are the increasingly complicated boundary of the displaced distribution. Note, the difficulty of prescribing the boundary does not arise for distributions which extend to infinity, such as the Gaussian, because the shifted distribution still extends to infinity and over all angles. However, the integration limits can be simplified, if we Taylor expand the displaced distribution in terms of the stationary distribution whose boundary is simple and well defined. We first Taylor expand $\psi(x-D,\,\dot{x})$ in the $(x,\dot{x})$ coordinates because the displacement is defined in this coordinate system and then make a change of variables to (J,θ) as required by Equation 7. The expansion up to third order in D is

$$\begin{aligned} \psi(J,\theta) = \psi_0(J) + D\,\psi_0'(J)\sqrt{2J}\cos(\theta) \\ +D^2\{\psi_0''(J)(\sqrt{2J}\cos(\theta))^2 + 2\psi_0(J)\} \\ +D^3\{\psi_0'''(J)(\sqrt{2J}\cos(\theta))^3 + 6\psi_0''(J)\sqrt{2J}\cos(\theta)\}\;, \end{aligned} \tag{8}$$

where the primes denote derivatives w.r.t. J. Substituting the expansion (8) into Equation 7, we can consider the contribution of each order to the integration. It is clear that all the even order terms do not contribute because of the odd powers of $\cos\theta$. The next order that contributes is the third, which is two orders of magnitude smaller if $D \ll \sqrt{2J_0}$.

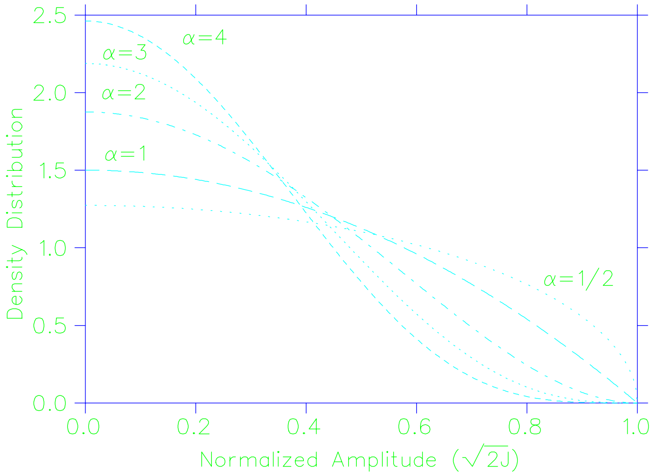

Figure 1: The binomial distribution for α = 1,2,3,4.

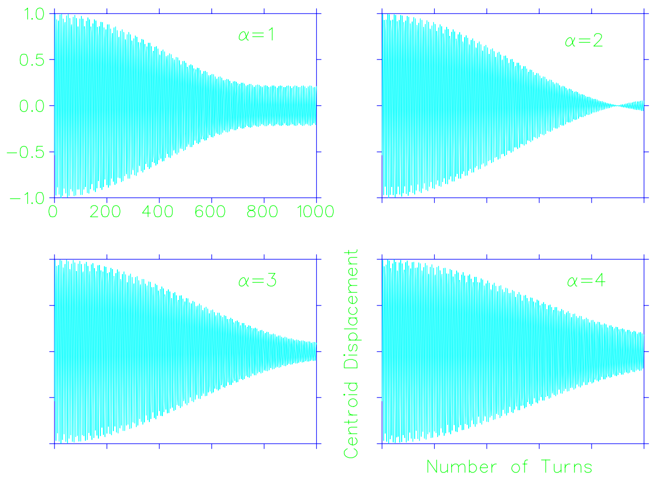

Figure 2: Beam centroid from turns 0 to 1000.

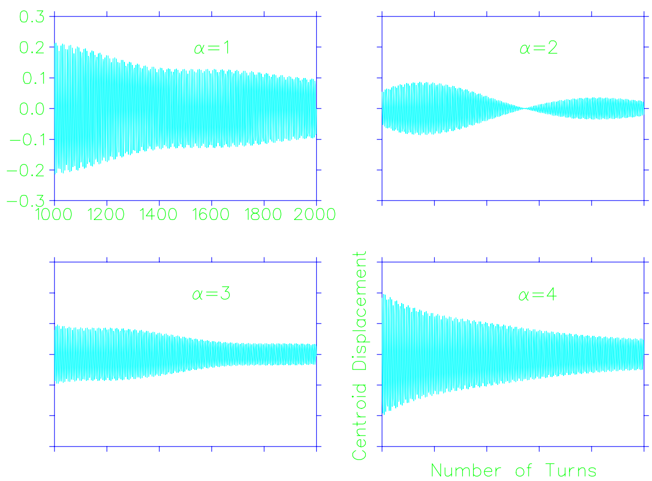

Figure 3: Beam centroid from turns 1000 to 2000.

For displacements which are small compared to the half width of the distribution, the contributions of the third and higher orders are negligible and can be neglected. To first order, the beam centroid can be evaluated with

$$\overline{x}(t) = D\int_0^{J_0}\int_0^{2\pi} 2\psi_0'(J)J\cos\theta\cos[\theta-(\omega+bJ)t]dJd\theta. \quad (9)$$

Generally, the integration can be evaluated analytically for positive integer values of α since the integrand is just a J polynomial times $\cos\theta$. For fractional α, there may be only a few cases where the integral can be evaluated analytically. Specifically, we have obtained expressions for the beam centroid $\overline{x}(t)$ for binomial distributions with $\alpha = 1/2$, 1, 2, 3, and 4. For $\alpha = 1$, we have

$$\overline{x}(t) = \frac{2D}{(\Delta\omega\, t)^2}\left\{\begin{array}{r}\cos(\omega t) - \cos(\widehat{\omega}t)\\ -\Delta\omega\, t\sin(\widehat{\omega}t)\end{array}\right\}, \quad (10)$$

where $\Delta\omega = bJ_0$, which is the difference between the unperturbed angular frequency and the maximum frequency $\widehat{\omega}$ at the edge of the distribution. For $\alpha = 2$, 3, and 4, we have respectively

$$\overline{x}(t) = \frac{6D}{(\Delta\omega\, t)^3}\left\{\begin{array}{r}\Delta\omega\, t\cos(\omega t) + \Delta\omega\, t\cos(\widehat{\omega}t)\\ +2\sin(\omega t) - 2\sin(\widehat{\omega}t)\end{array}\right\} \quad (11)$$

$$\overline{x}(t) = \frac{12D}{(\Delta\omega\, t)^4}\left\{\begin{array}{r}4\Delta\omega\, t\sin(\omega t) + 2\Delta\omega\, t\sin(\widehat{\omega}t)\\ +6\cos(\widehat{\omega}t) - 6\cos(\omega t)\\ +(\Delta\omega\, t)^2\cos(\omega t)\end{array}\right\} \quad (12)$$

$$\overline{x}(t) = \frac{20D}{(\Delta\omega\, t)^5}\left\{\begin{array}{r}(\Delta\omega\, t)^3\cos(\omega t) + 6(\Delta\omega\, t)^2\sin[(\omega t)]\\ -18\Delta\omega\, t\cos(\omega t) - 6\Delta\omega\, t\cos(\widehat{\omega}t)\\ 24\sin(\widehat{\omega}t) - 24\sin(\omega t)\end{array}\right\} \quad (13)$$

These expressions show that the oscillation frequency of the centroid contains components of ω and $\widehat{\omega}$ and is time dependent. For $\alpha = 1/2$, the expression involves hypergeometric series and is too lengthy to report here; but a plot for this case is shown in Figure 4.

3 DISCUSSION

Plots of the beam centroid for all the five values of α are shown for two time 'windows' in Figures 2, 3, and 4. The initial displacements for all cases are normalized for comparison. To exaggerate the effect of decoherence, the octupole strength is set such that $\Delta\omega = 0.01\omega$. For $\alpha = 1$, the amplitude decreases initially as $1/t^2$; and then, at later times, as $1/t$. This is evident in Equation 10. There is some beating of the envelope but never completely destructive. For $\alpha = 2$, the amplitude decreases as $1/t^2$ at all times, and the envelope beating is periodic; this can be seen from Equation 11, where the components have equal amplitudes. For $\alpha = 3$, the amplitude decreases as $1/t^3$ initially and as $1/t^2$ at later times; there is some beating but it is irregular. For $\alpha = 4$, the amplitude starts to decrease at a slower rate than for $\alpha = 2, 3$; and at longer times it falls as $1/t^2$. There is no prominent beating.

To understand this qualitative behaviour, one must realize that the centroid is proportional to the ensemble average of $\psi_0' \times J$, and that there are competing effects. For small values of α the distributions are 'flat' and the derivative of ψ is small, but there are proportionately more particles at large amplitudes. For large values of α, the magnitude of the derivative is large, but (for distributions normalized to a fixed J_0) there are rather fewer large amplitude particles. One can imagine two extreme cases that show no decoherence: (i) when $\alpha \rightarrow \infty$ (and we have a δ-function distribution), and (ii) when $\alpha = 0$ (and we have a δ-function derivative). The effect of distribution gradient on the rate of decoherence was previously noted by Hereward[3]; and is the reason why the distribution with $\alpha = 1$ decoheres more quickly than for $\alpha = 1/2$, despite the latter case having more large amplitude particles. Further, according to Equation 9, a uniform distribution ($\alpha \rightarrow 0$) does not decohere ($\overline{x} = D\cos\widehat{\omega}t$) because there is no gradient.

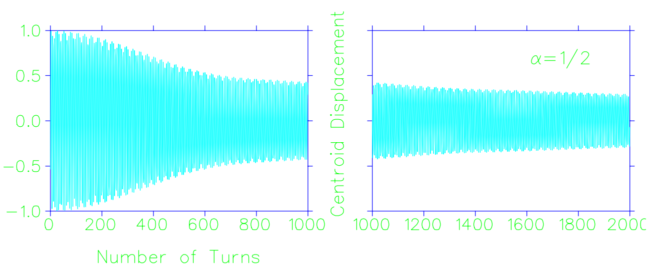

Figure 4: Beam centroid for $\alpha = 1/2$

4 CONCLUSION

We have calculated the motion of the beam centroid as a displaced binomial amplitude distribution dephases. In performing the ensemble average, the displaced distribution was Taylor expanded in terms of the stationary distribution to make use of its simple boundary; with the result that, to first order, the rate of decoherence depends on the average gradient of the stationary distribution.

5 REFERENCES

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