BEAM WITH UNEQUAL BUNCHES VERSUS A WIDE–BAND IMPEDANCE IN A SYNCHROTRON: STABILITY CRITERION

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Abstract

The paper expounds a technique to find a sufficient condition of (longitudinal, transverse) stability of a beam with unequal bunches, partial orbit filling or bunch trains included. It proceeds from a computer solution of eigenvalue problems of moderate dimensions for an observable within-bunch motion (multipole and head-tail modes, their higher-order radial modes) at a given normal coupled-bunch mode of a conventional basic beam — a closed train of identical and equispaced bunches. Then, its complex eigenvalues and non-diagonal Gram matrices of eigenvectors are used to find boundary of a convex field of complex Rayleigh-Ritz ratios which yields an 'upper' estimate of eigenvalue locus and, thus, stability safety margin for any arbitrary beam which is a subset of the basic one. As an example of application, the technique is applied to transverse resistive-wall head-tail instability of bunches in the UNK.

1 INTRODUCTION

Consider two beam configurations. The first, *the basic beam*, is a conventional closed pattern of M identical equispaced bunches filling the orbit entirely. The second, *a subset beam*, is an arbitrary beam which is derived from the former one by imposing unequal population to bunches and(or) arranging bunch trains and beam gaps. There is a long-standing issue on how to relate asymptotic (at $t \to \infty$) stability safety margins of basic and subset beams.

An algebraic approach to this problem (a subset beam with empty bunches) is pioneered by [1] where instability growth rates and coherent tune shifts are found as complex eigenvalues of a bunch-to-bunch interaction matrix. Ref.[2] treats a wider subset beam with unequally populated bunches (an empty bunch = a bunch of zero population). It also shrinks a rectangular estimate for subset-beam eigenvalue locus of [1] to that of a convex hull[1] around the basic-beam eigenvalues. Still, the qualitative outcome of [1, 2] is that "stability of the basic beam always ensures that of a subset one".

Unfortunately, the present paper would show that this optimistic statement holds true only in frames of a simplified dynamical model in which bunch-to-bunch interaction is treatable via a 'planar' $M \times M$ matrix. Accounting for impedance bandwidth extension and, say, transverse chromaticity requires a 'non-planar' bunch-to-bunch interaction array of higher dimension $M \times M \times N$ with $N \neq 1$. This tends to expel subset-beam eigenvalue locus beyond that of the basic beam. It is a clear symptom of deterioration, though potentially so, of the situation with coherent stability of a subset w.r.t. the basic beams.

2 MAJOR SET OF EQUATIONS

Let $\vartheta = \Theta - \omega_0 t$ be azimuth in a co-rotating frame, where Θ is azimuth around the ring in the laboratory frame, ω_0 is angular velocity of a reference particle, t is time. Let us numerate bunches of the basic beam with $j = 0, 1, \ldots, M-1$, and denote as $\vartheta_j = -2\pi j/M$ coordinates of their centers in co-frame. Let $J_{0b}^{(j)}$, J_{0b} be bunch currents (averaged over orbit) of a subset and the basic beams, respectively. Introduce bunch weights $\nu_j = J_{0b}^{(j)}/J_{0b} \le 1$ with $\nu_j = 0$ standing for an empty bunch in the j-th orbital position.

Let $x^{(j)}(\vartheta, t) = x^{(j)}(\vartheta + 2\pi, t)$ be a variable to describe coherent motion of the j-th bunch (a perturbed longitudinal current, a transverse dipole moment). It can be decomposed into plane waves

$$x^{(j)}(\vartheta, t) = \frac{1}{2\pi} \sum\nolimits_k \int d\Omega x_k^{(j)}(\Omega)\, \mathrm{e}^{ik(\vartheta - \vartheta_j) - i\Omega t} \tag{1}$$

with Ω being a frequency of Fourier Transform in t w.r.t. co-frame. In lab-frame, Ω is seen as a side-band $\omega = k\omega_0 + \Omega$. Harmonics $x_k^{(j)}(\Omega)$ are calculated in the coordinate system $\vartheta - \vartheta_j$ attached to the j-th bunch center. A set of x_k can be arranged into a $N \times 1$ column-vector

$$\vec{x} = (\ldots, x_{k-1}, x_k, x_{k+1}, \ldots)^T \in \mathrm{C}_N$$

from a linear complex vector space C_N. In practice, N is treated as its finite truncated dimension. Still, one can allow $N \to \infty$ if all the relevant limits do exist.

Stability problem of a subset beam can be stated as an eigenvalue problem for a linear operator acting in the extended space $\mathrm{C}_{N\cdot M} = \oplus \sum_j \mathrm{C}_N^{(j)}$ where $\mathrm{C}_N^{(j)} \ni \vec{x}^{(j)}$,

$$\lambda\, x_k^{(j)} = \frac{1}{M} \sum\nolimits_{k',j'} \nu_j P_{kk'}(\Omega)\, \mathrm{e}^{ik'(\vartheta_j - \vartheta_{j'})} x_{k'}^{(j')}. \tag{2}$$

Matrix $P_{kk'}(\Omega)$ operates in C_N. Its form varies slightly between longitudinal (L) and transverse (T) cases,

$$P_{kk'}(\Omega) = \begin{cases} Y_{kk'}^{(L)}(\Omega)\, Z_{k'}^{(L)}(k'\omega_0 + \Omega)/k'; \\ Y_{kk'}^{(T)}(\Omega)\, Z_{k'}^{(T)}(k'\omega_0 + \Omega). \end{cases} \tag{3}$$

[1] A convex hull $\mathrm{Co}(a_i)$ of array a_i is a set of linear combinations $\sum_i c_i a_i$ with $c_i = c_i^*$, $c_i \ge 0$, and $\sum_i c_i = 1$.

Coupling impedances $Z_k^{(L)}(\omega)/k$ and $Z_k^{(T)}(\omega)$ exhibit similar reflection properties w.r.t. $\omega, k = 0$. Bunch transfer functions $Y_{kk'}^{(L,T)} \propto J_{0b}$ can be found elsewhere and include effects of Landau damping, decomposition into multipole (L) or head-tail dipole (T) modes, etc.

Functions of integer variable j are conveniently decomposed with Discrete Fourier Transform (DFT):

$$\vec{x}^{(n)} = M^{-1}\sum\nolimits_j \vec{x}^{(j)} \exp\left(2\pi i n j/M\right), \qquad (4)$$

$$\vec{x}^{(j)} = \sum\nolimits_n \vec{x}^{(n)} \exp\left(-2\pi i n j/M\right) \qquad (5)$$

where $n = 0, 1, \ldots, M-1$ is a wave number of DFT.

To simplify notations, let $\widehat{I}_n$ be a projection operator from C_N to $\mathrm{C}_{N/M}$. Its matrix elements are $\delta_{kk'}\sum_l \delta_{k,n+Ml}$ with δ_{ij} the Kronecker delta-symbol, $\sum_n \widehat{I}_n = \mathrm{diag}(\ldots,1,1,\ldots)$, the matrix unit.

Transformation of Eq.2 to DFT-images yields

$$\lambda\, \vec{x}^{(n)} = \sum\nolimits_{n'} \Delta_{nn'}\, \widehat{P}(\Omega)\widehat{I}_{n'}\, \vec{x}^{(n')} \qquad (6)$$

where $\Delta_{nn'}$ denotes an interference factor

$$\Delta_{nn'} = M^{-1}\sum\nolimits_j \nu_j \exp\left(2\pi i (n-n')j/M\right). \qquad (7)$$

Eq.6 shows that instability in a subset beam with $\nu_j \neq \mathrm{const}$ mixes all DFT-harmonics of perturbation. Characteristic equation of instability and sufficient condition of beam stability are, respectively,

$$\lambda_p(\Omega) = 1, \qquad \max\nolimits_{p,\Omega_1} |\lambda_p(\Omega = \Omega_1 + i0)| \leq 1. \qquad (8)$$

Here p is a generalized index of the beam oscillation eigenmode, its eigenfrequency Ω_p being a root of the first of Eqs.8. Instability occurs when $\mathrm{Im}\Omega_p > 0$.

3 BASIC BEAM

Let us label its eigenvalues and eigenvectors with a subscript "$\bullet$". Now that $\nu_j = 1$ and $\Delta_{nn'} = \delta_{nn'}$, Eq.6 splits into M independent problems

$$\lambda_\bullet\, \vec{x}_\bullet^{(n)} = \widehat{P}(\Omega)\widehat{I}_n\, \vec{x}_\bullet^{(n)}, \quad n = 0, 1, \ldots, M-1. \qquad (9)$$

The perfect beam periodicity decouples all M DFT-harmonics. Each of them describes a possible coupled-bunch (CB) mode. Thus, the first sub-index of the basic-beam eigenmode index $\ell = (n, m)$ is naturally found, the latter m being an index of inside-bunch mode whose components are identified in solving specific problems. Spacial structure of mode $\ell = (n, m)$ is merely

$$\vec{x}_{\bullet\ell}^{(n')} = \vec{x}_{\bullet\ell}^{(n)}\delta_{nn'}, \;\; \vec{x}_{\bullet\ell}^{(j)} = \vec{x}_{\bullet\ell}^{(n)} \exp\left(-2\pi i n j/M\right) \qquad (10)$$

which is an usual CB oscillation with a phase shift of $2\pi n/M$ between adjacent bunches. These modes are mutually orthogonal being treated as hyper-vectors from extended space $\mathrm{C}_{N\cdot M}$ with a scalar product

$$\langle \breve{a}, \breve{b}\rangle = M^{-1}\sum\nolimits_{k,j} w_k\, a_k^{(j)}\, b_k^{(j)*}; \;\; \breve{a}, \breve{b} \in \mathrm{C}_{N\cdot M} \qquad (11)$$

where w_k is a real positive weight. Vector $\vec{x} \in \mathrm{C}_N$ is a projection of $\breve{x} \in \mathrm{C}_{N\cdot M}$.

Eq.9 has only each M-th its eigenvalue $\lambda_{\bullet\ell} \not\equiv 0$. These can be obtained on projecting Eq.9 into a subspace $\vec{x}\,' = \widehat{I}_n\vec{x} \in \mathrm{C}_{N/M}$ of a smaller dimension,

$$\lambda_\bullet\, \vec{x}_\bullet^{\,\prime(n)} = \widehat{I}_n\widehat{P}(\Omega)\, \vec{x}_\bullet^{\,\prime(n)}. \qquad (12)$$

Eq.12 is amenable to a straightforward search for $\lambda_{\bullet\ell} \not\equiv 0$ and $\vec{x}_{\bullet\ell}^{\,\prime(n)}$ with computer codes available. Then, Eq.9 can be used to recover full-component vectors $\vec{x}_{\bullet\ell}^{(n)} \in \mathrm{C}_N$ which describe observable motion of individual bunches. All these formally belong to a range of operator $\widehat{P}\widehat{I}_n$.

On the contrary, vectors from a null space of $\widehat{P}\widehat{I}_n$ with $\lambda_{\bullet\ell} \equiv 0$ describe a hidden, unobservable motion. Eq.9 shows that the null space of $\widehat{P}\widehat{I}_n$ is a set of $\vec{x}$: $\widehat{I}_n\vec{x} = \vec{0}$. Hence, eigenvectors of the hidden motion may be composed as, say, a natural orthogonal set of $N\times 1$ column-vectors $\vec{x}_{\bullet\ell}^{(n)} = (\ldots,0,1,0,\ldots)^T$ with a single non-trivial component '1' put consequently into every line save for the $(n+Ml)$-th ones, l is an integer.

Thus, in practice, a complete set of eigenvalues $\lambda_{\bullet\ell}$ and eigenvectors $\vec{x}_{\bullet\ell}^{(n)}$ of Eq.9 can be found. For a given n, the span over $\vec{x}_{\bullet\ell}^{(n)}$ originates the entire space C_N. Hence, $\vec{x}_{\bullet\ell}^{(n)}$ can be used as a coordinate basis in C_N. Its (Hermitian positive defined) Gram matrix is

$$G_{mm'}^{(n)} = \left(\vec{x}_{\bullet(n,m)}^{(n)}, \vec{x}_{\bullet(n,m')}^{(n)}\right) \qquad (13)$$

where $(\ldots)$ is a scalar product consistent with Eq.11,

$$\left(\vec{a}, \vec{b}\right) = \sum\nolimits_k w_k\, a_k b_k^*; \;\; \vec{a}, \vec{b} \in \mathrm{C}_N. \qquad (14)$$

Operator $\widehat{P}\widehat{I}_n$ is not a normal one, hence basis of $\vec{x}_{\bullet\ell}^{(n)}$ is non-orthogonal, and $G_{mm'}^{(n)}$ is a non-diagonal matrix.

Eq.12 is commonly treated at length in an instability theory. Naturally, the desire arises to use its supposedly known spectrum $\lambda_{\bullet\ell}$ (and $G_{mm'}^{(n)}$) to localize spectrum λ_p of a subset beam and study its stability with the second of Eqs.8.

4 SUBSET BEAM

Multiply both sides of Eq.2 by $M^{-1}w_k x_k^{(j)*}/\nu_j$ and sum over k, j. Rewrite the result so as to arrange formally a Rayleigh-Ritz ratio $\mathcal{R}$ for a linear operator $Q_{kk'}^{(jj')} = M^{-1}P_{kk'}\exp\left(ik'(\vartheta_j - \vartheta_{j'})\right)$ in $\mathrm{C}_{N\cdot M}$,

$$\lambda_p = \xi\mathcal{R}, \;\; \xi = \frac{\sum_{k,j} w_k |x_k^{(j)}|^2}{\sum_{k,j} w_k |x_k^{(j)}|^2/\nu_j}, \;\; \mathcal{R} = \frac{\langle \widehat{Q}\breve{x}, \breve{x}\rangle}{\langle \breve{x}, \breve{x}\rangle}. \qquad (15)$$

Eq.2 shows that $|x_k^{(j)}| \propto \nu_j$. Hence $|x_k^{(j)}|^2/\nu_j \propto \nu_j$ and tending to a limit $\nu_j \to 0$ (empty bunch) in ξ inflicts no problems. As $\nu_j \geq 0$ by definition, it is easy to see that $\xi \in \mathrm{Co}(\nu_j)$. For real ν_j it entails

$$\min\nolimits_j \nu_j \leq \xi \leq \max\nolimits_j \nu_j = 1. \qquad (16)$$

Now use $\vartheta_j = -2\pi j/M$ in $\widehat{Q}$ and DFT from Eq.5 to get

$$\langle \widehat{Q}\breve{x}, \breve{x}\rangle = \sum\nolimits_n \left(\widehat{P}\widehat{I}_n\, \vec{x}^{(n)}, \vec{x}^{(n)}\right), \tag{17}$$

$$\langle \breve{x}, \breve{x}\rangle = \sum\nolimits_n \left(\vec{x}^{(n)}, \vec{x}^{(n)}\right) > 0. \tag{18}$$

The last Eq. is but the Parseval sum due to orthogonality of CB modes in $\mathrm{C}_{N\cdot M}$.

For each n, eigenvectors $\vec{x}^{(n)}_{\bullet\ell}$ of $\widehat{P}\widehat{I}_n$ — modes of the basic beam — construct a complete countable skew basis in C_N. It can be used for coordinate representation of modes $\vec{x}^{(n)}_p$ of a subset beam,

$$\vec{x}^{(n)}_p = \sum\nolimits_m c_\ell\, \vec{x}^{(n)}_{\bullet\ell}. \tag{19}$$

Inserting this decomposition into Eqs.17,18 yields

$$\langle \widehat{Q}\breve{x}, \breve{x}\rangle = \sum\nolimits_{n,m,m'} \lambda_{\bullet\ell}\, c_\ell\, G^{(n)}_{mm'}\, c^*_{\ell'}, \tag{20}$$

$$\langle \breve{x}, \breve{x}\rangle = \sum\nolimits_{n,m,m'} c_\ell\, G^{(n)}_{mm'}\, c^*_{\ell'} > 0 \tag{21}$$

where $\ell' = (n, m')$. One does not know coordinates c_ℓ a priori. (Otherwise, spectral estimates in question would have not been required.) Let us allow c_ℓ be arbitrary complex numbers, and study the so called numerical field of $\widehat{Q}$ — a set $\overline{\mathcal{R}}$ of possible values of Rayleigh-Ritz ratios $\mathcal{R}$:

$$\mathcal{R} = \sum\nolimits_{n,m} a_\ell\, \lambda_{\bullet\ell}, \quad \sum\nolimits_{n,m} a_\ell = 1, \tag{22}$$

$$a_\ell = \frac{c_\ell \sum_{m'} G^{(n)}_{mm'}\, c^*_{\ell'}}{\sum_{n,m} c_\ell \sum_{m'} G^{(n)}_{mm'}\, c^*_{\ell'}}. \tag{23}$$

Then, majorizing the leftmost of Eqs.15, one gets the 'upper' estimate of a subset-beam spectrum locus,

$$\lambda_p \in \overline{\mathcal{R}} \subset \mathrm{C}_1. \tag{24}$$

Linear algebra tells that $\overline{\mathcal{R}}$, being a numerical field of a linear operator in $\mathrm{C}_{N\cdot M}$, is: (i) a bounded closed set in C_1, (ii) $\forall\lambda_{\bullet\ell} \in \overline{\mathcal{R}}$, and (iii) $\overline{\mathcal{R}}$ is a convex set that contains inside every straight-line segment which connects its elements pairwise.

Should matrix $G^{(n)}_{mm'}$ be diagonal, one would have got $a_\ell = a^*_\ell$, $0 \le a_\ell \le 1$ and, hence, $\overline{\mathcal{R}} = \mathrm{Co}\left(\lambda_{\bullet(n,m)}\right)$ which would have reproduced the result quoted in Sect.1. However, generally $G^{(n)}_{mm'} \ne \delta_{mm'}$. Only on adopting a *single-mode* model when a bunch is deprived of all but the m_1-th degree of freedom, i.e. $\lambda_{\bullet\ell} = \lambda_{\bullet\ell}\delta_{mm_1}$ and $c_\ell = c_\ell\delta_{mm_1}$, one virtually gets a diagonal 1×1 matrix $G^{(n)}_{mm'}$. This kind of assumption is tacitly implied in [1, 2]. It is well adequate when beam interacts with a band-pass high-Q HOM impedance ((L) or (T), chromaticity off), or with a low-pass narrow-band resistive-wall impedance ((T), chromaticity on and off).

In a *multi-mode* model, put Eq.19 into Eq.6 to reveal structure of subset-beam observable ($\lambda_p \not\equiv 0$) motion,

$$\vec{x}^{(n)}_p = \sum_{\{\ell'=(n',m'):\lambda_{\bullet\ell'}\not\equiv 0\}} \Delta_{nn'}\, c_{\ell'}\, (\lambda_{\bullet\ell'}/\lambda_p)\, \vec{x}^{(n')}_{\bullet\ell'}. \tag{25}$$

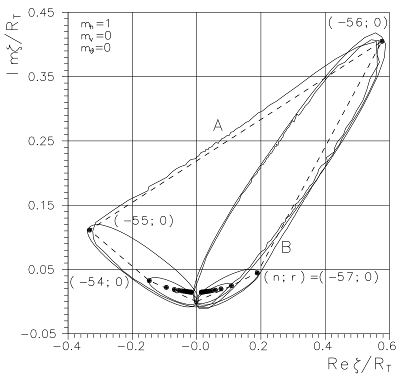
The span thus obtained over observable $\vec{x}^{(n')}_{\bullet\ell'}$ does not necessarily originate the entire C_N. Calculations show that for a given inside-bunch mode m eigenvector $\vec{x}^{(n)}_{\bullet(n,m)}$ is nearly independent of CB mode n. Mostly, it carries data on spacial localization of perturbation inside bunch. Comparison of Eqs.19,25 with $\vec{x}^{(n)}_{\bullet(n,m)} \approx \vec{x}^{(n')}_{\bullet(n',m)}$ in mind shows that a reasonable trial guess of subset-beam motion is arrived at with a truncated series

$$\vec{x}^{(n)}_p \simeq \sum\nolimits_{\{m:\lambda_{\bullet\ell}\not\equiv 0\}} c_\ell\, \vec{x}^{(n)}_{\bullet\ell} \tag{26}$$

where residuum $\Delta\vec{x}^{(n)}$ from a null space of $\widehat{P}\widehat{I}_n$ with $\widehat{I}_n\Delta\vec{x}^{(n)} = \vec{0}$ is disregarded due to a negligible value expected. In practice, use of Eq.26 entails that summation in Eqs.22,23 should go over $n = 0, 1, \ldots, M-1$ and the observable subset $\{m, m' :\ \lambda_{\bullet(n,m)}, \lambda_{\bullet(n,m')} \not\equiv 0\}$ only.

The last step is to search with computer for boundary of $\overline{\mathcal{R}}$ by, say, a Monte-Carlo scan over coordinates c_ℓ. By the way, due to convexity of $\overline{\mathcal{R}}$, any partial image of a subset in $\mathrm{C}_{N\cdot M}$ can be lawfully diluted to the nearest convex set.

The figure shows this approach applied to study resistive-wall instability in the UNK (betatron tune is $Q_T = 55.7$). Eigenvalues are interpreted as instability driving impedances ζ to be plotted in (Z)-plane of a threshold map. Eigenmode index is $\ell = (n, m_\vartheta, r)$ where head-tail mode is $m_\vartheta = 0$ throughout. The impedance is sampled with Eq.12 at frequency lines separated by $M\omega_0$ in $|\omega| \lesssim 3.5M\omega_0$. Thus, at most 7 radial modes $r = 0, 1, \ldots, 6$ are involved. Markers plot eigenvalues for $(n = -76, -75, \ldots, -36; r = 0)$. Dashed broken line is a convex hull over $(n = -M/2, \ldots, M/2; r = 0)$ which stands for a *single-mode* model. Ellipses encircling the origin are partial boundaries to image a span over radial modes $r = 0, 1, 2$ at a CB mode $n = -58, -57, \ldots, -54$ nearest to $n + Q_T \simeq 0$. Curve A images a span over $(n = -56, -55; r = 0, 1)$, curve B — that over $(n = -57, -56; r = 0, 1)$. Thus, one can assess quantitatively stability of a subset w.r.t. the basic beams.



5 REFERENCES

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