POLARIZABILITIES OF AN ANNULAR CUT IN THE THICK WALL

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Abstract

The polarizabilities of a ring-shaped cut in the wall of an arbitrary thickness are calculated using a combination of analytical, variational and numerical methods. The results are applied to estimate the coupling impedances of button-type beam position monitors.

1 INTRODUCTION

The coupling impedances of a small discontinuity on the wall of the vacuum chamber of an accelerator have been calculated in terms of the polarizabilities of the discontinuity [1]. The fields scattered by an aperture in the wall can be approximated by those due to effective electric P and magnetic M dipoles which are induced by an incident fields E^h_ν, H^h_τ [2]: $P_\nu = -\chi\varepsilon_0 E^h_\nu/2; \quad M_\tau = \psi H^h_\tau/2$, where χ is the electric polarizability and ψ is the magnetic susceptibility of the aperture, $\hat{\nu}$ is the normal vector to its plane, and $\hat{\tau}$ is the tangential one. When the wavelength of an incident field is large compared to the aperture size, the polarizabilities can be found by solving a static problem [3]. The solutions are known for a circular hole of radius b in a zero-thickness wall, $\psi = 8b^3/3$ and $\chi = 4b^3/3$ [2], and for elliptic holes in a thin wall [3]. In the case of a thick wall the polarizabilities have been studied using a variational technique for circular [4] and for elliptic [5] holes. Some approximate formulas for slots are compiled in [6].

In this paper, we present results for the polarizabilities of an annular cut in the perfectly conducting planar wall of an arbitrary thickness. Such an aperture can serve as a model of a coax attached to the waveguide, when the wall thickness is large. In the case of a thin or finite-thickness wall it is an approximation of an electrode of the button-type beam position monitors (BPMs). More details on derivation and solutions can be found in Ref. [7].

2 INTEGRAL EQUATIONS

We are looking for the field distribution produced by the aperture (hole) in a conducting wall of thickness t with its midplane at $z = 0$ when it is illuminated by a homogeneous static (normal electric or tangential magnetic) field from one ($z > 0$) side. Following [8, 4], we split the problem into two parts by decomposing the far field as $E_0/2 + E_0/2 = E_0$ for $z > 0$, and $E_0/2 - E_0/2 = 0$ for $z < 0$, and consider two separate problems: (i) the wall with the aperture is immersed into homogeneous field $E_0/2$ — the antisymmetric problem for the potential w.r.t. $z \leftrightarrow -z$; and (ii) the far field is directed to the wall from both sides, $E_0/2$ for $z > 0$ and $-E_0/2$ for $z < 0$, in which case the potential is symmetric. Solving these two problems yields χ_s and χ_a, and gives us the inside polarizability $\chi_{in} = \chi_s + \chi_a$, which defines the effective dipole for the illuminated side of the wall, $z > t/2$, and the outside one, $\chi_{out} = \chi_s - \chi_a$, for the shadow side. Likewise, the magnetic polarizabilities are $\psi_{in,out} = \psi_s \pm \psi_a$.

Consider the magnetic problem for an annular cut with inner radius a and outer radius b. It can be reduced to the integral equation for function $g(r)$, defined by $2H_z(\vec{r}, t/2)/H_0 = g(r)\cos\varphi$, (cf. [4] for a circular hole):

$$\int_a^b dr' r' g(r')[K_m(r, r') + K_{mt}(r, r')] = r \ , \qquad (1)$$

where the thin-wall part of the kernel

$$K_m(x, y) = \theta(y - x)\frac{x}{2y^2}\, {}_2F_1\left(\frac{3}{2}, \frac{1}{2}; 2; \frac{x^2}{y^2}\right) + \{x \leftrightarrow y\},$$

has a ln-singularity at $x = y$, and the thickness-dependent part K_{mt} is related to the field expansion inside the aperture, $a < r, r' < b$, $|z| < t/2$,

$$K_{mt}(r, r') = \sum_{n=1}^{\infty} F_n(r) F_n(r') \lambda_n^{-1} \left\{ \begin{array}{c} \tanh \\ \coth \end{array} \right\} \left(\frac{\lambda_n t}{2}\right), \qquad (2)$$

with the upper (lower) line corresponding to the (anti) symmetric problem. Here ${}_2F_1$ is the Gauss hypergeometric function, λ_n are subsequent positive roots of the equation $J_1'(\lambda_n a)Y_1'(\lambda_n b) - Y_1'(\lambda_n a)J_1'(\lambda_n b) = 0$, $J_n(x), Y_n(x)$ are the n-th order Bessel functions of the first and second kind, and the expansion functions F_n are

$$F_n(r) = C_n \left[J_1(\lambda_n r) - Y_1(\lambda_n r)\, J_1'(\lambda_n a)/Y_1'(\lambda_n a)\right] \ .$$

Normalization condition $\int_a^b r F_n^2(r) = 1$ defines

$$C_n = \frac{\pi\lambda_n}{\sqrt{2}} \left\{\left[Y_1'(\lambda_n b)\right]^{-2} \left(1 - \lambda_n^{-2} b^{-2}\right) - \{b \to a\}\right\}^{-1/2}.$$

The magnetic susceptibility is expressed in terms of $g(r)$ [4] as $\psi = \pi \int_a^b dr r^2 g(r)$. A solution $g(r)$ of Eq. (1) must have the correct singular behavior near the metal edge: $g(r) \propto \Delta^{-\alpha}$ when distance from the edge $\Delta = b - r \to 0$ or $\Delta = r - a \to 0$, where $\alpha = 1/2$ for $t = 0$, and $\alpha = 1/3$ for a thick wall, assuming 90° edge. The electric problem can be reduced to an integral equation in a similar way.

3 MAGNETIC PROBLEM

For a narrow annular cut (gap width $w = b - a$ is small, $w \ll b$) in a thin wall the integral equation has been solved analytically [9], and the magnetic polarizability is

$$\psi = \pi^2 b^2 a \left[\ln(32b/w) - 2\right]^{-1} . \tag{3}$$

It becomes large and close to that of a circular hole for relatively narrow gaps, $w/b \geq 0.1$. The physical reason for this surprising result is that an incident tangential magnetic field deeply penetrates even through a narrow annular gap in the thin wall, and this distortion creates a large effective magnetic dipole, comparable to that due to the open hole with the same radius.

For a wide cut, we apply a variational technique developed in [4] converting (1) into the variational form

$$\frac{\pi b^3}{\psi} = \frac{\int_\rho^1 x dx \int_\rho^1 y dy\, g(x) K(x,y) g(y)}{\left[\int_\rho^1 x^2\, dx\, g(x)\right]^2} , \tag{4}$$

where $x = r'/b$ and $y = r/b$, and $\rho = a/b$, and kernel $K = K_m + K_{mt}$. Solution $g(x)$ of Eq. (1) minimizes the RHS of Eq. (4). We are looking for a solution in the form of a series $g(x) = \sum_{n=0}^{\infty} c_n g_n(x)$ with unknown coefficients c_n. The choice of functions $g_n(x)$ is defined by the near-edge behavior of the solution:

$$\begin{aligned} g_0(x) &= \left[(1-x)(x-\rho)\right]^{-\alpha} , & (5)\\ g_k(x) &= P_{k-1}\left[(2x-\rho-1)/(1-\rho)\right] \text{ for } k \geq 1 , \end{aligned}$$

where $\alpha = 1/2$ and $P_n(x) = T_n(x)$ are Chebyshev's polynomials of the first kind for a zero-thickness wall, while for a thick wall $\alpha = 1/3$ and $P_n(x) = C_n^{1/6}(x)$ are Gegenbauer's polynomials. The choice of the polynomials is related to their orthogonality to the singular part $g_0(x)$ of the solution. Denoting $d_n = \int_\rho^1 dx x^2 g_n(x)$ and $a_n = c_n d_n$, we define matrix

$$K_{kn} = \int_\rho^1 x dx \int_\rho^1 y dy\, g_k(x) K(x,y) g_n(y)/(d_k d_n) , \tag{6}$$

and convert Eq. (4) into the following form

$$\frac{\pi b^3}{\psi} = \frac{\sum_{k,n} a_k K_{kn} a_n}{\left(\sum_n a_n\right)^2} . \tag{7}$$

Minimizing the RHS yields $\psi = \pi b^3 \sum_{k,n} \left(K^{-1}\right)_{kn}$, where matrix K^{-1} is the inverse of the matrix K, Eq. (6). The further procedure is straightforward: nth iteration ($n = 0, 1, 2, \ldots$) corresponds to the matrix (6) truncated to the size $(n+1) \times (n+1)$. Integrations and matrix inversions have been carried out using *Mathematica*.

For the thin wall, polarizability $\psi = \psi_{in} = \psi_{out}$ versus the cut width is shown in Fig. 1 (dashed line). The analytical solution (3) works well for narrow gaps, $w/b \leq 0.15$. The process converges in three iterations for the whole range of the cut width $0 \leq w/b \leq 1$.

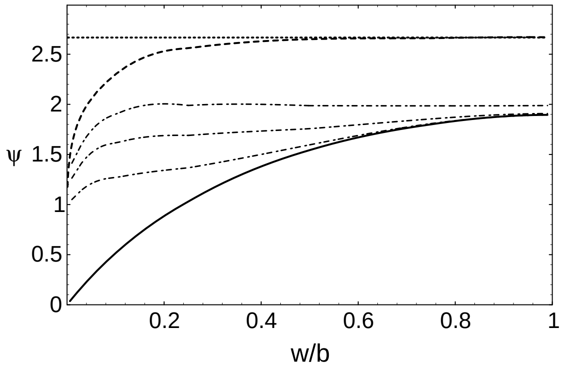

Figure 1: Inside magnetic polarizability (in units of b^3) of annular cut versus its relative width w/b for thin (dashed) and thick (solid) wall. Three dash-dotted curves are for fixed ratio $t/w = 0.5$; 1; 2 (from top to bottom). The dotted line corresponds to the circular hole in a thin wall.

For the case of a thick wall an asymptotic of ψ for a narrow gap can be obtained analytically using properties of eigenvalues: $\lambda_n b \to \pi(n-1)/\delta$ for $n \geq 2$, and $\lambda_1 b \simeq 1 + \delta/2$ when $\delta = w/b \to 0$. Keeping only $g_0(x)$ and leading term $\left[\int g(x) F_1(x)\right]^2 \propto \delta^{-1/3}$ in Eq. (4), we get $\psi_{in} = 2\pi b^2 w$. Comparison to the results of direct variational calculations for the thick wall in Fig. 1 (solid line) shows that this asymptotic works only for very small w/b. Variational calculations are similar to those for the zero-thickness case, except that one has to truncate the series (2). We have kept up to 6 terms in this series, and convergence was fast enough, requiring only up to 3 to 4 iterations. The inside magnetic polarizabilities reach their "thick-wall" asymptotic values approximately at $t/b = 2$, while the outside ones decrease exponentially with thickness increase, see pictures in [7]. In the limit $w/b \to 1$ our results coincide with those for a circular hole [4].

4 ELECTRIC PROBLEM

For a narrow annular cut $w \ll b$, the electric polarizability can be approximated by that of a narrow (yet bent) slot of width w and length $\pi(b+a) \gg w$, i.e., $\chi \simeq \tilde{\chi}\pi(b+a)$, where $\tilde{\chi}$ denotes the electric polarizability per unit length of the slot. The value of $\tilde{\chi}$ can be obtained using conformal mapping for a 2-D electrostatic problem: $\tilde{\chi} = \pi w^2/8$ for zero wall thickness, and $\tilde{\chi} = w^2/\pi$ for a thick wall, $t \gg w$, see [6]. In this way, we have two analytical estimates for the electric polarizability of a narrow annular cut:

$$\begin{aligned} \chi &\simeq \pi^2 w^2 (b+a)/8 \quad \text{for thin wall} , & (8)\\ \chi_{in} &\simeq w^2 (b+a) \quad \text{for thick wall} . & (9) \end{aligned}$$

For narrow gaps the electric polarizability is small compared to the magnetic one. The reason is that the normal electric field does not penetrate far enough through the narrow gap, unlike the tangential magnetic field on the parts of the annular cut which are parallel to its direction. The outside elec-

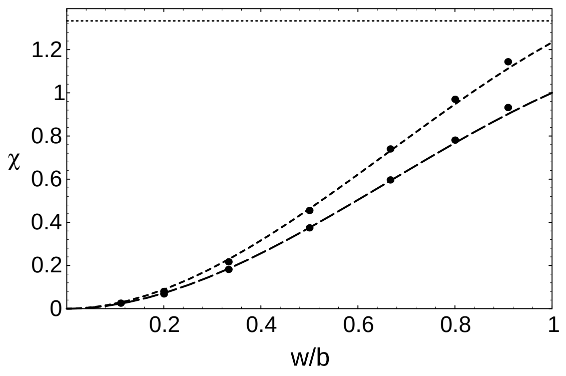

Figure 2: Inside electric polarizability (in units of b^3) of annular cut versus its relative width w/b: analytical estimates (8) for thin (short-dashed) and (9) for thick wall (long-dashed) and corresponding numerical results (thick dots). The dotted line is for the circular hole in a thin wall.

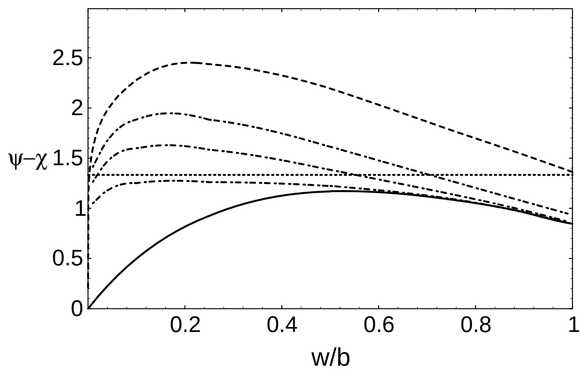

Figure 3: Difference of inside polarizabilities (in units of b^3) of annular cut versus its relative width w/b for different wall thicknesses $t = 0$; $w/2$; w; $2w$, and $t \gg w$ (from top to bottom). The dotted line corresponds to the circular hole in a thin wall, $(\psi - \chi)/b^3 = 4/3$.

tric polarizability of the gap in a thick wall can be estimated as $\chi_{out} \simeq w^2(b + a)\exp(-\pi t/w)$.

Both the electro- and magnetostatic problems under consideration can be solved numerically. With boundary conditions which ensure a given homogeneous field far from the aperture plane, an electric or magnetic potential could be computed. Unfortunately, for the magnetic problem, as well as for an arbitrary-shaped aperture, this approach requires 3-D codes. However, our electric problem is effectively a 2-D one due to its axial symmetry. On the other hand, an application of the variational technique to the electric problem is complicated since its zero-thickness kernel is more singular. That is why we choose the numerical approach for the wide gap applying the *POISSON* code.

The results are shown in Fig. 2. Analytical estimates (8) and (9) work amazingly well even for very wide gaps. We intentionly did not interpolate the numerical dots in Fig. 2, otherwise it would be difficult to distinguish the numerical curves from those given by formulas (8)-(9); they overlap except in the region $w/b \geq 0.85$. Numerical results for finite wall thickness $t/w = 1$ and even $t/w = 0.5$ are very close to those for a very thick wall (the lower curve).

5 BEAM COUPLING IMPEDANCES

The beam-chamber coupling impedances can be obtained using formulas from [1] and polarizabilities found above. An annular cut of radius b and width w on the wall of a circular pipe of radius $r \gg b$ produces the longitudinal impedance

$$Z(\omega) = -iZ_0\omega\left(\psi_{in} - \chi_{in}\right)\left(8\pi^2 cr^2\right)^{-1} , \qquad (10)$$

where $(\psi_{in} - \chi_{in})/b^3$ is plotted in Fig. 3. For other cross sections of the vacuum chamber, the transverse impedance, and $Re\, Z$, see [10] and references therein. As seen from Fig. 3, the impedance of a narrow cut in a thin wall is larger than (but less than twice) that of a circular hole with radius b, and tends to the last one when $w \to b$.

As an example, we estimate the broad-band impedance for BPMs of the PEP-II B-factory at SLAC and compare it with 3-D numerical simulations [11]. The BPM has 4 buttons of inner radius $a = 7.5$ mm, gap width $w = 1$ mm, at the distance $r = 30$ mm from the chamber axis. In fact, the PEP-II chamber has an octagonal cross section, but we approximate it by a circular pipe with radius 30 mm. While the wall thickness is not specified in [11], it is usually a few times larger than the gap width. The estimate (10) gives the inductance $L = 0.06$ nH per BPM ($Z = -i\omega L$), when the thickness is taken $t = 2w = 2$ mm, and $L = 0.032$ nH for a very thick wall, $t \gg w$. The numerical result [11] is $L = 0.04$ nH per BPM, in an agreement with our estimate.

6 REFERENCES

[1] S.S. Kurennoy, *Part. Acc.*, **39**, 1 (1992); R.L. Gluckstern, *Phys. Rev. A*, **46**, 1106 (1992).

[2] H.A. Bethe, *Phys. Rev.*, **66**, 163 (1944).

[3] R.E. Collin, *Field Theory of Guided Waves*. New York: McGraw-Hill, 1960.

[4] R.L. Gluckstern and J.A. Diamond, *IEEE Trans. MTT*, **39**, 274 (1991).

[5] B. Radak and R.L. Gluckstern, *ibid.*, **43**, 194 (1995).

[6] S.S. Kurennoy, "Pumping slots: coupling impedance calculations and estimates," Report SSCL-636, Dallas (1993).

[7] S.S. Kurennoy, *IEEE Trans. MTT*, **44**, No. 7, July 1996.

[8] R.L. Gluckstern, R.Li and R.K. Cooper, *ibid.*, **38**, 186 (1990).

[9] S.S. Kurennoy, "Polarizabilities of an annular cut," in *Proc. Part. Accel. Conf.*, Dallas, TX (1995), p.3379.

[10] S.S. Kurennoy, R.L. Gluckstern and G.V. Stupakov, *Phys. Rev. E*, **52**, 4354 (1995).

[11] C.-K. Ng, et al. "Simulation of PEP-II beam position monitors", in *Proc. Part. Accel. Conf.*, Dallas, TX (1995), p.2485.