THE ELECTROMAGNETIC FIELD AS CONSTRAINED SYSTEM

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Abstract

The gauge invariance of the electromagnetic field, as well as the geometrical restrictions imposed by construction for various accelerating devices, transform the field configuration in a constrained system and require supplementary care in the study of its dynamics. Using the BRST rules, we shall construct a suitable phase space and an extended hamiltonian for such a special configuration of the electromagnetic field.

1 INTRODUCTION

The process of acceleration of charged particles imposes the use of various electromagnetic field configurations. For example, in the method of plasmoids acceleration [1] charged and neutral particles-plasmoids-are localized in the nodes of an electric field and another electromagnetic field can determine an accelerating motion of the plasmoids or can produce forced oscillations of these structure, giving rise to e.m. waves. The waves propagate and could accelerate on its turn other charged particles. Between the possible configurations of the electromagnetic field used in order to obtain such accelerating motion we note the cylindrical axially symmetric field $E\,(z, r, \phi)$:

$$\begin{array}{c} E_z = E_0\, J_0(k_r r) \sin k_z z \sin \omega t \\ E_r = -(\frac{k_z}{k_r}) E_0 J_1(k_r r) \cos k_z z \sin \omega t \\ B_\phi = (\frac{k}{c k_r}) E_0 J_1(k_r r) \cos k_z z \sin \omega t \end{array} \tag{1}$$

where J_n represents the Bessel function of order n and $k_r^2 + k_z^2 = k^2 = \omega^2/c^2$. In this case it has been demonstrate that , under supplementary conditions, stable ellipsoidal and thoroidal bunches may occur and the limitations in the accelerating gradient have been calculated [2].

From another point of view, to consider conditions of the type (1) is similar with the introduction for the electromagnetic field of some geometrical constraints. As even the free electromagnetic field has its own dynamical constraints, it is clear that a coherent description of such configurations impose the using of the specific methods of the constrained dynamical systems. The presentation of one of these methods, the BRST technique, is the main goal of our paper. In the next section we shall construct the extended phase space for an irreducible theory as the electromagnetism is. The construction will be achieved in the enlarged context of a multiple BRST symmetry[3]. In the last section of the paper will be obtained the extended hamiltonian of the electromagnetic field that put together the dynamical constraints and the geometrical constraints of the type (1). This is the hamiltonian that must be used in the description of the field configurations used in the various accelerator devices.

2 IRREDUCIBLE GAUGE THEORIES

We consider a dynamical system that in the initial phase space with the canonical variables (q^i, p_i), $i = 1, ..., n$ is described by the hamiltonian $H_0(q, p)$ and the set of linearly independent first-class constraints $G_\alpha = G_\alpha(q, p)$ with the Grassmann parities $\epsilon(H_0) = 0\,;\, \epsilon(G_\alpha) = \epsilon_\alpha$

The first class for the constraints means, in the Dirac terminology, that they satisfy the relations

$$[G_\alpha, G_\beta] = f^\gamma_{\alpha\beta} G_\gamma\,;\, [H_0, G_\alpha] = V^\beta_\alpha G_\beta \tag{2}$$

where [,] means the Poisson bracket with respect the variables (q,p). The existence of the constraints determines some relations between the canonical variables, such that they will be not independent. In order to built up a coherent phase-space for the system under consideration we shall apply the BRST technique and we shall add to the initial coordinates new sets of variables, without physical significance, called **ghost-variables.** One obtain an extended phase space and it can be organized as a differential complex. This task is accomplished by construction of the Koszul-Tate complex and its acyclic differential δ_K. The structure of this complex is basically determined by the constraints and the relations satisfied by them [4]. We shall use hear a strategy similar with those used in [5] for the construction of a sp(2) BRST symmetry. As in [3] we shall be interested in the construction of a sp(3) symmetry, and from this reason we shall treble the constraints:

$$G_A = \{G^{(a)}_\alpha, a = 1, 2, 3; \alpha = 1, ..., m\} \tag{3}$$

In addition we shall suppose that the new constraints are in fact the old ones, that is:

$$G^{(1)}_\alpha = G^{(2)}_\alpha = G^{(3)}_\alpha \equiv G_\alpha \;\; ; \alpha = 1, ..., m \tag{4}$$

For each constraint we shall attach a ghost-momenta $P_A = P^{(a)}_\alpha$ having opposite parity as G_A and ghost number -1.

$$\epsilon(P_A) = \epsilon_A + 1\,;\, gh(P_A) = -1 \tag{5}$$

The Koszul-Tate differentials δ_K acts as a derivation on the polynomials of $P^{(a)}_\alpha$ and its defining properties are:

$$\delta_K P^{(a)}_\alpha = G^{(a)}_\alpha \tag{6}$$

$$\delta_K(q,p)=0 \tag{7}$$

$$\delta_K(AB)=A(\delta_K B)+(-1)^{\epsilon_B}(\delta_K A)B \tag{8}$$

The crucial property of δ_K is its acyclicity which means nonexistence of the real cycles i.e. nonexistence of the solutions of equation $\delta_K\omega=0$ which are not exact: $\omega\neq\delta_K(\eta)$ In order to assure this acyclicity we must introduce new momenta to kill all non-exact closed forms. From equations (**??**) and (**??**) one obtain:

$$\begin{aligned}\delta_K(P_\alpha^{(1)}-P_\alpha^{(2)})=\delta_K(P_\alpha^{(2)}-P_\alpha^{(3)})=\\=\delta_K(P_\alpha^{(3)}-P_\alpha^{(1)})=0\end{aligned}$$

These cocycles are killed if we introduce the new momenta $\pi_\alpha^{(a)}$ such that

$$\epsilon^{abc}\delta_K\pi_\alpha^{(c)}=P_\alpha^{(a)}-P_\alpha^{(b)} \tag{9}$$

i.e.

$$\begin{array}{c}\delta_K\pi_\alpha^{(1)}=P_\alpha^{(2)}-P_\alpha^{(3)}\\ \delta_K\pi_\alpha^{(2)}=P_\alpha^{(3)}-P_\alpha^{(1)}\\ \delta_K\pi_\alpha^{(3)}=P_\alpha^{(1)}-P_\alpha^{(2)}\end{array} \tag{10}$$

Again from equations (**??**) one can obtain one nonexact closed form

$$\delta_K(\pi_\alpha^{(1)}+\pi_\alpha^{(2)}+\pi_\alpha^{(3)})=0$$

which again must be killed by new momenta τ_α:

$$\pi_\alpha^{(1)}+\pi_\alpha^{(2)}+\pi_\alpha^{(3)}=\delta_K\ (\tau_\alpha) \tag{11}$$

The remarkable point in these definitions is the fact that the Koszul-Tate differential δ_K can be decomposed in three parts

$$\delta_K=\delta^1+\delta^2+\delta^3 \tag{12}$$

with

$$\begin{array}{c}\delta^a P_\alpha^{(b)}=\delta^{ab}G_\alpha;\delta^a\pi_\alpha^{(b)}=\epsilon^{abc}P_\alpha^{(c)}\\ \delta^a\tau_\alpha=\pi_\alpha^{(a)};a,b,c=1,2,3\end{array} \tag{13}$$

It is easy to see that these new differentials fulfill the anticommutation relations:

$$\delta^a\delta^b+\delta^b\delta^a=0\,;\,a,b=1,2,3 \tag{14}$$

The canonical description impose to define for each momentum a canonical pair, conjugated with the momentum in respect with a generalized Poisson bracket. So, one can show [3] that for an irreducible gauge theory the whole spectrum of ghost and ghost-momenta will be:

$$\begin{array}{c}Q^A=\{q^i,Q^{\alpha a},\lambda^{\alpha a},\eta^\alpha\}\\ P_A=\{p_i,P_{\alpha a},\pi_{\alpha a},\tau_\alpha\}\end{array} \tag{15}$$

The Grassmann parity of the variables will be given by:

$$\begin{array}{c}\epsilon(G_\alpha)=\epsilon(\lambda^\alpha)=\epsilon(\pi_\alpha)\equiv\epsilon_\alpha\\ \epsilon(Q^\alpha)=\epsilon(P_\alpha)=\epsilon(\eta^\alpha)=\epsilon(\tau_\alpha)\equiv\epsilon_\alpha+1\end{array} \tag{16}$$

For two arbitrary quantity, F and G, with the Grassmann parities $\epsilon(F)=\epsilon_F$ and $\epsilon(G)=\epsilon_G$, we shall adopt for the generalized Poisson bracket the form:

$$[F,G]=\frac{\partial F}{\partial Q^A}\frac{\partial G}{\partial P_A}-(-)^{\epsilon_F\epsilon_G}\frac{\partial G}{\partial Q^A}\frac{\partial F}{\partial P_A} \tag{17}$$

3 THE EXTENDED HAMILTONIAN

The electromagnetic field is described by the action

$$S\left[\mathbf{A}\right]=-k\int d^4xF_{\alpha\beta}F^{\alpha\beta} \tag{18}$$

where k is a constant, A is the four dimensional potential of the electromagnetic field, and $F_{\alpha\beta}\equiv\partial_\alpha A_\beta-\partial_\beta A_\alpha$.

The components of the four momenta vector have the expressions:

$$\Pi_0=0\ ;\ \Pi_k=F_{k0}\ \ ;k=1,2,3 \tag{19}$$

The relation $\Pi_0=0$ represents a primary constraint of the theory and, looking for its validity at any moment of time, we generate a secondary constraint:

$$0=[H_0\ \ ,\ \ \Pi_0]=\partial^k\Pi_k \tag{20}$$

This last relation is nothing else that the Gauss low. The two above mentioned constraints are the unique constraints appearing when the dynamical equations of the free electromagnetic field are written down. If the field is involved in some special device, that is with a special geometry, the peculiar feature of the field must be added as supplementary **geometrical constraints**. Let us note by $\phi^{(m)}$ these constraints. They have to be added to the previous ones and will be put together in the canonical hamiltonian. One obtains the so-called **extended hamiltonian** and it will be the adequate hamiltonian to describe the studied field configuration .

We start from the expression of the canonical hamiltonian:

$$H_0=\int d^3\mathbf{x}\ (\frac{1}{2}\Pi^k\Pi_k+\frac{1}{4}F_{ij}F^{ij}-A_0\partial^i\Pi_i)$$

The extended BRST hamiltonian for the electromagnetic field will be constructed on the basis of the ghost spectrum (15) in the form of a sum:

$$H=H_0+\sum_{n\geq1}H^{(n)}$$

The effective expression is chosen so that the hamiltonian has to satisfy the treble global symmetry:

$$s^aH=0\ ;\ a=1,2,3$$

where the operators $s^a=\delta^a+...$ are the BRST differentials.

4 REFERENCES

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