On the Influence of some Discretisation Methods in the Study of the Dynamical Systems

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Abstract

It is already known that, in the case of simple dynamics, the discretisation process for the dynamical systems moulded by nonlinear differential equations has no significant effect. But when the flow is not stable - it appears the sensitive dependence of the initial condition- or when the system is not structurally stable, the numerical dynamics is not always the same with the dynamics of the original system.

The aim of this paper is to study, by different numerical methods, some differential equations and to compare qualitatively and quantitatively the results. We make some remarks about the optimal choice of the method and of the discretisation step in order to obtain a faithful description of the initial system .

We are also interested in the occurrence of the chaotic behavior due to the discretization method. We analyze a system with discontinuous control in order to observe that the discretization introduce a new degree of freedom (the discretization step) whose value is crucial in the occurrence of the chaotic behavior.

1 INTRODUCTION

Let $M \subset R^n$ be a manifold, let $f : M \to R^n$ be a differentiable function. We consider the continuous dynamical system described by

$$\begin{cases} \dot{x}= f(x) \\ x(t_0) = a \end{cases} \tag{1}$$

If (1) can not be solved by analytical methods, we can use some numerical methods in order to approach the solution. There are some classical numerical methods (the Euler method, the Runge-Kutta method of different orders) which are used even in the most performed soft.

The discrete systems associated to (1) by Euler method and by Runge-Kutta method of the second order are respectively

$$(M, f_h^E(x)) \tag{2}$$

and

$$(M, f_h^{RK}(x)) \tag{3}$$

where $f_h^E(x) = x + hf(x)$ and $f_h^{RK}(x) = x + \alpha hf(x) + (1-\alpha) hf\left(x + \frac{h}{2(1-\alpha)} f(x)\right)$

The dynamics of (1) is governed by the presence of the attractors: a simple attractor induce a simple dynamics and a strange attractor is the signature of the complex, chaotic behavior of the flow. Sometimes we can not study the continuous system (1) and than we study the associated discrete system. Now the problem is to know the correlations between the two dynamics, namely the connections between the attractor of the continuous system and the attractor of the discrete system.

We present here some considerations about this and also some anomalies which can appear in the discretisation process.

2 NUMERICS AND DYNAMICS

The presence of an equilibrium point of (1), respectively a fixed point of (2) or (3) is very important in the dynamics of the flow. The following results assure that the identification of the fixed points of (2) or (3) guaranties the presence of an equilibrium point of (1).

Lemma : $\bar{x} \in R^n$ is an equilibrium point of (1) if and only if $\bar{x}$ is a fixed point of (2), respectively (3).

But the properties of the equilibrium are not always preserved:

Theorem 1: Let $\bar{x}$ be an equilibrium of (1).Then:

a) If $\bar{x}$ is a repelling point then it is a repelling fixed point of (2) respectively (3) for any value of h.

b) $\bar{x}$ is attractor (or a node) if and only if $\bar{x}$ is an attracting fixed point (or a node) of (2), respectively (3) for all enough small values of h.

Theorem 2: Let $\bar{x}$ a fixed point of (2) respectively (3). Then:

a) If $\bar{x}$ is a repelling point then $\bar{x}$ is a repelling equilibrium of (1) or it is not hyperbolic.

b) if $\bar{x}$ is a node for all enough small values of h, then it is a node of (1) or it is not hyperbolic.

This results show that if we know the properties of the equilibrium $\bar{x}$ of (1) we can precise the properties of the fixed point $\bar{x}$ of (2) or (3), but the converse is not true. the following examples clarify this assertion.

Example 1:

For the system $\begin{cases} \dot{x}= y \\ \dot{y}= -x \end{cases}$ the equilibrium $(0, 0)$ is not hyperbolic because the characteristic exponents are $\pm i$.It is a center. The flow is stable but it is not asymptotically stable. The system is not structurally stable (because the equilibrium is not hyperbolic).

For the discrete systems obtained by the Euler method and by the Runge-Kutta method the fixed point $(0, 0)$ is a repelling point for any value of the discretisation step.

Only the Runge-Kutta method of the fourth order give a good description of the center $(0,0)$.

An other classical method which gives a correct description of the system is the centered difference method.

Example 2: For the system

$$\begin{cases} \dot{x} = y + ax\left(x^2+y^2\right) \\ \dot{y} = -x + ay\left(x^2+y^2\right) \end{cases}$$

the equilibrium $(0,0)$ is not hyperbolic. Using polar coordinates $x = r\cos\theta$, $y = r\sin\theta$ we obtain the system

$$\begin{cases} \dot{r} = ar^3 \\ \dot{\theta} = -1 \end{cases}$$

.

We can observe that:for $a \prec 0$, $(0,0)$ is an attractor, for $a = 0$, $(0,0)$ is a center and for $a \succ 0$, $(0,0)$ is a repelling point.

If we proceed to the discretisation of the system by the Runge-Kutta method of the second order we obtain a discrete system whose characteristic exponents of $(0,0)$ have the absolute value greater than 1, so $(0,0)$ is a repelling point for any positive value of the discretisation step. Anyone can observe that the dynamics of the real system is deformed by the discretisation process,when $a \le 0$.A more complicate problem is that it is happening for any (small) discretisation step, so, if we do not have an analytical control of the flow, we can be wrong by the numerical results.

For the discrete system associate by the Euler method, $(0,0)$ is also a repelling fixed point, but the system has also an attracting limit circle, whose radius depends on a and h.The real system has not such an orbit.

The Runge-Kutta method of the fourth order gives a faithful description of the initial system.

3 THE OCCURRENCE OF THE CHAOS DUE TO THE NUMERICAL METHODS

More complicated situations appear when the discrete system associated to (1) has complex, chaotic behavior.

Usually we do not have the analytical control of the system ant we trust the numerical results, Which can be dramatically different from the original ones. We shall see some examples.

Example 3 Let consider the logistic system $\begin{cases} \dot{x} = kx\left(L-x\right) \\ x\left(0\right) = x_0 \end{cases}$. The flow $\Phi_t\left(x_0\right)$ has the property that $\lim_{t\to\infty} \Phi_t\left(x_0\right) = L$, for any x_0, so the system has a simple dynamics.

Using the Euler method we obtain the discrete system generated by $F_a\left(x\right) = ax\left(1-x\right)$, where $a = 1 + Lkh$.

This application is the signature of the chaos. For different values of the parameter a (so for different values of the discretisation step h, the system has different asymptotic behavior:

-if $h \in \left(0, \frac{2}{Lk}\right)$ then 0 is a repelling point and $x = \frac{Lkh}{1+Lkh}$ is an attractor In this case the dynamics of the two systems are qualitatively the same.

-when the parameter a is increasing, it appears a chain of doubling period of the global attracting orbit.

- for others values of h , as $h = \frac{2.56994}{Lk}$ or $h = \frac{3}{Lk}$ the system is chaotic

So , in order to obtain a faithful description of the system, we must choose $h \le \frac{2}{Lk}$.

From the same equation, discretisated by the mixed differences method, we obtain a discrete system which is topologically conjugated with the Henon system. So for some values of h this discrete system is chaotic and the original system is not.

Let see the discretisation effect on a nonlinear control system with discontinuity.

Let consider the Van der Pol equation:

$$\begin{cases} \dot{x} = y \\ \dot{y} = 2a\omega\left(1-\mu x^2\right)yu - \omega^2 x \end{cases} \tag{4}$$

If $u = 1$, (4) is a damped system in which the negative damping occurs in the strip$|x| < \frac{1}{\sqrt{\mu}}$ and positive damping occurs for$|x| > \frac{1}{\sqrt{\mu}}$. The system has a stable limit cycle denoted by $C\left(x,y\right) = 0$.

If $u = -1$, the system (4) has a reverse time and rotated limit cycle which is unstable.

Let consider $S = \left\{(x,y)\,/\,s = x^2 + \frac{y^2}{\omega^2} - r^2 = 0\right\}$ with $r < \frac{1}{\sqrt{\mu}}$ as a switching manifold and $u\left(x\right) = \begin{cases} -1, s\left(x\right) > 0 \\ 1, s\left(x\right) < 0 \end{cases}$ the control law, the oscillator approaches the sinusoidal response with the radius r.Using the Runge-Kutta algorithm we obtain the discrete system$\begin{cases} x_{n+1} = f_1\left(x_n, y_n, h\right) \\ y_{n+1} = f_2\left(x_n, y_n, h\right) \end{cases}$

The choice of the discretisation step h is very important in order to obtain a control of the system.

The sampling period H is given by

$$H = \sup\{h > 0 \;\; / \;\; C(f_1\left(x_n, y_n, h\right), \\ f_2\left(x_n, y_n, h\right)) \le 0 \text{ and } x_n^2 + \frac{y_n^2}{\omega^2} < r^2\}$$

If $h < H$ the behavior of the flow is quasiperiodic and the system trajectory zigzags along the switching manifold indicating the existence of a pseudo sliding mode.

If $h > H$ the trajectory is moving away from the switching circle towards infinity, indicating instability.

It is clear that the choice of the discretisation step is very important in order to obtain a correct description of the system (4) .

4 REFERENCES

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