EFFECTS OF THE TRANSVERSE-LONGITUDINAL COUPLING ON THE STRONG FOCUSING OF CHARGED BEAMS FOR ICF

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1 INTRODUCTION

A key issue in the conceptual design of a ion driven ICF reactor is the final focusing system [1,2]. In fact the possibility of a strong focusing of the beams in very small spots is a condition for achieving the required energy deposition power to implode the pellet. On the other hand the capability of concentrating the beam relaxes the need for high intensity, allowing a more realistic accelerator design.

This requirement has stimulated new effort in the magnetic lenses design and some solutions have been proposed in order to gain large concentration factors. Among these the plasma lens solution seems to be particularly appropriate for ICF for various reasons [3,4]. In particular it has been recently proposed a new attractive scheme, called adiabatic plasma lens focusing [5], for the collection of multiple converging beams in the reactor chamber.

Classical fundamental limitations to the beam focusing are, as it is well known, the so called transverse emittance of the beam, the space charge effects and the beam momentum spread (or longitudinal emittance). A different kind of limitations could arise from non-linearity in the field profile [6]. In the strong focusing area, when large convergence angles and small spot sizes have to be reached, another fundamental limitation arises from the effects of transverse-longitudinal motion coupling. They have, in fact, to be carefully considered in order to establish the related limitations in the focusing process. This point has been often disregarded because of minor importance in traditional focusing systems, as for example quadrupoles multiplets, where the so called para-axial approximation holds. On the contrary, as long as the converging angle increases, the condition for this approximation weakens and the arise of some kind of aberration is expected [7].

In this work we develop a model, starting from the complete equation of motion for the beam particle in an axial symmetric azimuthal magnetic field (the one produced in a plasma lens), which takes into account in the exact way the transverse-longitudinal coupling effects. The analysis can be easily extended to more general situations, where no axial symmetry is supposed.

After a brief description of the mathematical model for fully coupled motion we obtain the exact solution by means of elliptic integrals. Moreover we discuss some general properties of the coupled dynamic. A simpler model, based on the thin lens approximation is also proposed and its results compared to the general ones. Finally dimensionless parameters are related to cases of practical interest and the aberration effects are evaluated in those cases.

2 THE MATHEMATICAL MODEL

We consider the trajectory equation in the case of axis-symmetric magnetic field. Because of the symmetry it is possible to study the motion in a generic transverse plane x,z. Let $\boldsymbol{v} = v_x \boldsymbol{i}_x + v_z \boldsymbol{i}_z$ the velocity of a particle in this plane. The equation of motion is $m\, d\mathbf{v}/dt = qv \times \boldsymbol{B}$, where m is the particle mass, q its charge and $\boldsymbol{B}$ the magnetic induction field. The conservation of energy gives $v_x^2 + v_z^2 = const = k^2$, where we assume v_z always positve. From these equations is straightforward to obtain the trajectory equation for the transverse motion. Let us first introduce dimensionless coordinate $x=x/R$, $z=z/R$, being R the lens radius (or aperture). If we denote with x' the derivative of x respect to z the trajectory equation takes the form:

$$x'' + \alpha^2 \left(1 + x'^2\right)^{3/2} f(x) = 0, \tag{1}$$

where $B = B_0 f(x)$, with f a proper dimensionless "shape" function, and

$$\alpha^2 = \frac{qB_0 R}{mk}. \tag{2}$$

The equation (1) may be rewritten in a normal form:

$$\begin{aligned} x_1' &= \alpha^2 \left(1 + x_1^2\right)^{3/2} f(x_2), \\ x_2' &= x_1 \end{aligned} \tag{3}$$

where $x_1 = x', x_2 = x$.

It is straightforward to verify that Liouville's theorem no longer holds for system (3), as expected.

3 THE "THICK LENS" SOLUTION

By multiplying the autonomous equation (1) by x' it is easy to obtain its first integral:

$$\alpha^2 g(x) - \left(1 + x'^2\right)^{-1/2} = \Gamma, \tag{4}$$

where $f(x) = dg(x)/dx$ and Γ has to be regarded as a constant of the motion, satisfying the condition $-1 \le \Gamma < 0$. Solving equation (4) for $(x')^2$ we obtain:

$$(x')^2 = \frac{1}{\left[\alpha^2 g(x) - \Gamma\right]^2} - 1 = h(y). \tag{5}$$

The function $h(y)$ has zeros where $\alpha^2 g(x) - \Gamma = \pm 1$. Because of the assumption $v_z > 0$, it follows

$\alpha^2 g(x) - \Gamma \geq 0$, so the only zeros of physical interest are where $\alpha^2 g(x) - \Gamma = 1$. As f is an odd function, g must be an even function with two symmetric zeros in $\pm\bar{x}$, that we assume to be simple zeros. For example, for linearly growing magnetic field, $f(x) = x$ and $g(x) = x^2/2$, so that $\bar{x} = \sqrt{2(\Gamma+1)}\ \alpha$.

Finally, the solution of equation (5) is given by:

$$z = \int_{x_0}^{x} \frac{d\chi}{\sqrt{h(\chi)}}. \qquad (6)$$

Because of the structure of equation (5) we may foresee oscillations between $\pm\bar{x}$. In fact, let $-\bar{x} \leq x_0 \leq x$, then if x'_0 is positive it remains positive till x reaches the point $\bar{x}$. A formally possible solution henceforward would be x = constant = $\bar{x}$, but this is excluded by the original equation (1). Hence x' must reverse its sign and x decreases steadily to $-\bar{x}$, where x' again reverses. So x is periodic of period:

$$\Omega = 2\int_{-\bar{x}}^{\bar{x}} \frac{d\chi}{\sqrt{h(\chi)}} = 4\int_{0}^{\bar{x}} \frac{d\chi}{\sqrt{h(\chi)}}. \qquad (7)$$

Moreover, as x bounces between the two zeros of $h(x)$, if x_0 belongs to this interval, we have to assume $\bar{x} \leq 1$, otherwise the particle would run out of the lens. This limits the acceptance of the lens. For an elliptical shape of the beam in the phase plane we have $x^2 + x'^2 R^2/\varepsilon^2 = 1$ and for a particle initially on the axis ($x_0 = 0$) we may find the maximum value of Γ and, consequently, the maximum of ε. For a linearly growing magnetic field, for instance, we have:

$$\varepsilon_M = R\left[1\Big/\left(1-\alpha^2/2\right) - 1\right]^{1/2}. \qquad (8)$$

The integral in equation (6) can be analytically carried out, resulting:

$$z = \frac{1}{\alpha}\left\{2E(\beta,m) - F(\beta,m) - 2E(\beta_0,m) + F(\beta_0,m)\right\}. \qquad (9)$$

where $\beta = cos^{-1}\left(\frac{\alpha x}{\sqrt{2(\Gamma+1)}}\right)$, $\beta_0 = cos^{-1}\left(\frac{\alpha x_0}{\sqrt{2(\Gamma+1)}}\right)$, $m = (\Gamma+1)/2$ and F and E are respectively the elliptic integrals of first and second kind.

The equation (10) for fixed length of the lens $z=l$, gives, in an implicit form, the transformation map $\boldsymbol{M}$ of the focusing system in the phase space coordinates

$$\boldsymbol{x}(l) = \boldsymbol{M}_l[\boldsymbol{x}(0)], \text{ where } \boldsymbol{x} = |x, x'|^T. \qquad (10)$$

4 "THIN LENS" APPROXIMATION

A method that allows the a simple construction of an analytical approximation of the map (10) is the *thin lens approximation* (or *one kick approximation*). In this case one considers the magnetic field concentrated in one point. In this model, a different definition of f in equation (1) is needed:

$$f(x;z) = \Pi_\Delta(z) f(x) \quad \text{for} \quad \Delta \to 0, \qquad (11)$$

where

$$\Pi(z) = \begin{cases} 1/\Delta & \text{for } 0 < z < \Delta \\ 0 & \text{for } z < 0,\ z > \Delta \end{cases}, \qquad (12)$$

and $z = \Delta$ is the lens end. From the physical point of view we are considering a lens for which $\Delta/\Omega << 1$, that is, the lens length is much smaller than the characteristic wavelength Ω of the betatron oscillation.

The one kick approximation consists in solving equation (1) assuming the particle position $x(z)$ constant within the lens region ($0 < z < \Delta$), that is:

$$x(z) \cong x(0) \quad \text{for} \quad 0 < z < \Delta. \qquad (13)$$

Under this assumption, the simplified equation:

$$x'' + \frac{\alpha^2}{\Delta}\left(1 + x'^2\right)^{3/2} f[x(0)] = 0, \quad 0 < z < \Delta, \qquad (14)$$

have to be solved with the initial condition on $x'(0)$, to get an approximation of $x'(z)$ at z=Δ (a thin lens acts on the transverse momentum of the particle and leaves unchanged its position). Its solution gives directly:

$$x'(\Delta) = x'(0) + F[x(0), x'(0)], \qquad (15)$$

where

$$F[x(0), x'(0)] = \frac{c[x'(0)] - \alpha f[x(0)]}{\sqrt{1 - \left\{c[x'(0)] - \alpha f[x(0)]\right\}^2}} \qquad (16)$$

with $\quad c[x'(0)] = x'(0)\Big/\sqrt{1 + x'^2(0)}. \qquad (17)$

Let us consider a focusing system consisting of a thin lens and drift section of length d. The total transfer map $\boldsymbol{M}(d)$ of this system is then given by

$$\begin{aligned} x(d) &= x(0) + \left\{x'(0) + F[x(0), x'(0)]\right\} d \\ x'(d) &= x'(0)) + F[x(0), x'(0)] \end{aligned} \qquad (18)$$

The equations (18) allow us to describe analytically the behaviour of the trace-space contour (the trace-space contour is related to the projection of the phase-space boundary into the $x - x'$plane), as function of d and of the other beam parameters.

Due to the coupling between the $x-$motion and the z-motion, the area enclosed by the $x - x'$ contour is not conserved across the lens. The change of this area depends on the determinant of the Jacobian matrix of the non-linear mapping (18).

5 THE EFFECT OF NON LINEARITY

In the following we assume, for the shape function f the linear function $f(x)=x$. In fact, if the transverse to longitudinal coupling is neglected, this choice leads to a linear map for the focusing element. Therefore in this way we are able to consider separately only the non-linear effect due to the coupling.

In order to show the aberration effects on the beam contour we report, in figure 1, the results, for an elliptic phase space contour beam of dimensionless emittance $\varepsilon = 0.2$, focused according to the map (11) for $\alpha = 1$; figure 1a shows the trace space particle trajectories, figure 1b the corresponding entrance and exit phase space beam contours.

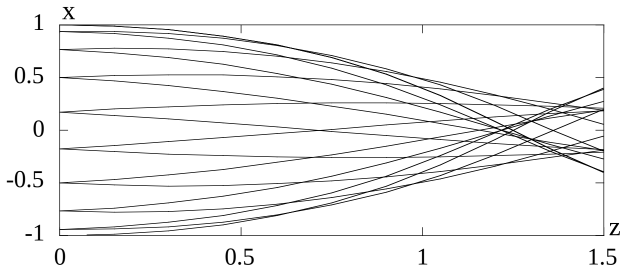

Figure 1a: trace space plot of the trajectories of particles in a lens described by the map (11) for $\varepsilon = 0.2$, $\alpha = 1$

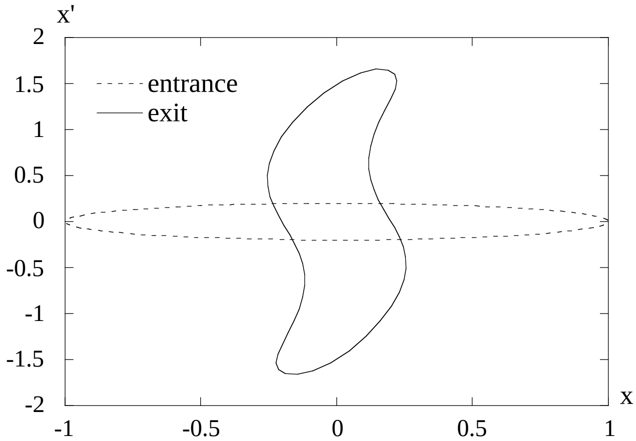

Figure 1b: phase space plot for the entrance and exit beam contours for the same parameters of fig. 1a.

The distortion both in the trace space and in the phase space contour is evident. This distortion determines an increase in the spot size at the focal plane compared to the case of negligible coupling. This phenomenon is due to the dependence of the focal distance by the initial condition of the particle.

The aberrations due to longitudinal-transverse coupling can be conveniently described by the increase of the spot dimension at the focal plane of an ideal parallel beam. In fact, in the case of a parallel beam ($x'=0$) the focal distance depends only on the initial position x_0.

$$z_f = \left[2E(m) - K(m)\right]/\alpha. \tag{19}$$

By approximating the elliptic integrals a simple explicit relation for the focal distance z_f can be obtained for small values of α :

$$z_f \cong \frac{\pi}{2\alpha}\left[1 - \frac{3\alpha^2 x_0^2}{16}\right]. \tag{20}$$

The focal length is a decreasing function of x_0. This can be easily explained once one consider that a more external particle suffers a larger deflection from the lens, and consequently a larger decrement of the longitudinal velocity. This gives the idea, to be further investigated, that it is possible with a suitable shape of the magnetic field to compensate this kind of aberration.

From equation (20) is then possible to directly express the corresponding transverse spread Δx_f:

$$\Delta x_f \cong \frac{3\pi}{64}\frac{\alpha^2}{\sqrt{1-\alpha^2}}. \tag{21}$$

It is worth noticing that the scaling expressed by (21) is in very good agreement to the results one obtains from the thin lens model previously discussed. For $\alpha = 0.1$, for instance, the latter gives $\Delta x_f = 0.0013$, that is less than 10% smaller than the result given by (21).

6 CONCLUDING REMARKS

The concrete consequences of equation (21) are clear once the dimensionless parameter α is related to the physical parameters. For cases of practical interest in the context of ICF applications we consider, as focusing element, a plasma lens with a current in the range 10 kA to 1 MA, that is at the present time a realistic range. It is straightforward to relate α to the lens current and to the beam magnetic rigidity. For a magnetic rigidity ranging between 1 to 10 Tm the α parameter ranges between 0.01 and 1. Larger values of α are related to large convergence angles in the focused beam, so that the aberration is stronger as the angle increases, as expected. In the dimensionless model the convergence angle directly corresponds, both for the thick and thin lenses, to the ratio of the lens aperture to the focal length. In common applications of plasma lenses this ratio rarely exceed the value of 1/10, corresponding to distortions of the order of 10^{-3} of the initial beam radius. On the contrary, in the adiabatic focusing, for which very large converging angles have to be considered because the multiple beam design, the effects of the coupling have to be carefully considered.

A perspective for future work is to consider the possible correction of the considered aberration with a proper non-linear profile for the magnetic field in the lens.

REFERENCES

[1] C. Rubbia, *Il Nuovo Cimento A* , Vol. 106 A, N. 11, 1993.

[2] R.O. Bangerter, *Il Nuovo Cimento A* , Vol. 106 A, N. 11, 1993.

[3] A. Tauschwitz et al., *Las. and Part. Beams*, Vol. 13, N.2, 1995.

[4] E. Boggasch et al., *Nucl. Instr. Meth. in Phys. Res. A*, 336, 1993.

[5] A. Tauschwitz et al., *GSI Report* GSI-96-02, 1996.

[6] M. de Magistris et al., *Fusion Eng. Des.*, to appear.

[7] C.M Celata et al., *Il Nuovo Cimento A* , Vol. 106 A, N. 11, 1993.