# NORMAL MODE LASLETT COEFFICIENTS

S. Petracca, Acc. Th. Group, KEK, Tsukuba, Japan*
I.M. Pinto, D.I.[3]E., Univ. of Salerno, Italy
F. Ruggiero, SL Div., CERN, Geneva, Switzerland

*Abstract*

General formulae are given for computing the normal mode incoherent and coherent Laslett coefficients for beam liners surrounded by a (coaxial) magnetic yoke, in terms of complex potentials. Applications to the circular and square geometries are discussed.

## 1 INTRODUCTION

In a previous paper one of the Authors (S.P.) has shown that vertical and radial betatron oscillations are coupled in general, so that Laslett coefficients [1] form a non-diagonal tensor. It is thus possible to compare different pipe geometries in terms of Laslett coefficients in a meaningful and non-ambiguous way only *after* introducing betatron normal modes [2]. In this communication the normal mode Laslett coefficient computational framework formulated in [2] is extended to the more general case where the beam pipe is encircled by a magnetic yoke.

## 2 THEORY

The transverse motion of a particle in a beam is driven by space-charge, image and guiding forces:

$$\vec{f} = \vec{f}^{(sp.ch.)} + \vec{f}^{(im.)} + \vec{f}^{(g.f.)}. \qquad (1)$$

The *space charge* force $\vec{f}^{(sp.ch.)}$, related to the beam charge distribution [3], is the same as in *free space*, and will be neglected here, for simplicity. The *image* force $\vec{f}^{(im.)}$, is due to the conducting and magnetic boundaries, and is computed as if the beam were a line charge through the (transverse) center of charge $\vec{\rho}_b$. The equilibrium condition is defined by[1]:

$$\vec{f}^{(g.f.)}|_{\vec{\rho}=\vec{\rho}_{eq.}} + \vec{f}^{(im.)}|_{\vec{\rho}=\vec{\rho}_b=\vec{\rho}_{eq.}} = 0. \qquad (2)$$

For small displacements, thereof two regimes are possible:

i) $\vec{\rho}_b = \vec{\rho}_{eq.}, \vec{\rho} \neq \vec{\rho}_{eq.}$, *incoherent*, single particle regime:

$$\vec{f} = (\vec{\rho} - \vec{\rho}_{eq.}) \cdot \nabla_{\vec{\rho}} \left[ \vec{f}^{(im.)} + \vec{f}^{(g.f.)} \right], \qquad (3)$$

ii) $\vec{\rho} = \vec{\rho}_b \neq \vec{\rho}_{eq.}$, *coherent*, whole beam regime:

$$\vec{f} = (\vec{\rho} - \vec{\rho}_{eq.}) \cdot \left[ (\nabla_{\vec{\rho}} + \nabla_{\vec{\rho}_b}) \vec{f}^{(im.)} + \nabla_{\vec{\rho}} \vec{f}^{(g.f.)} \right], \qquad (4)$$

where $\nabla_{\vec{\rho}}$, $\nabla_{\vec{\rho}_b}$ is the gradient taken w.r.t. the suffix coordinate, and all derivatives are taken at $\vec{\rho} = \vec{\rho}_b = \vec{\rho}_{eq.}$.

For both cases, the linearized Lorentz force equation reads:

$$\frac{d^2 \vec{\delta}}{d\tau^2} + \Omega_c^2 \nu_0^2 \, \bar{\bar{U}} \cdot \vec{\delta} = 0, \qquad (5)$$

where $\vec{\delta} = \vec{\rho} - \vec{\rho}_{eq.}, \tau = s/c$, $\Omega_c$ is the circulation frequency, $\nu_0$ the unperturbed tune, $\nu_0^2 \Omega_c^2 = (m_0\gamma_0)^{-1} \partial_x f_x^{(g.f.)} = (m_0\gamma_0)^{-1} \partial_y f_y^{(g.f.)}$ and $\bar{\bar{U}}$ is a $2nd$-rank tensor[2]:

$$\bar{\bar{U}} = \bar{\bar{I}} + \frac{2}{\nu_0} \bar{\bar{\Delta}}\nu, \qquad (6)$$

where the tune-shift tensor $\bar{\bar{\Delta}}\nu$ can be further factored as:

$$\bar{\bar{\Delta}}\nu = -\frac{N \, R \, r_0}{\pi \nu_0 \beta_0^2 \gamma_0 L^2} \bar{\bar{\epsilon}}. \qquad (7)$$

Here the first factor depends only on the gross machine features, ($N$ is the total # of particles in the beam, $R$ the ring radius, $r_0$ the classical particle radius, $L$ the transverse dimension of the chamber) while the Laslett *tensor*:

$$\bar{\bar{\epsilon}} = \frac{L^2}{4\Lambda} q^{-1} \begin{cases} \nabla_{\vec{\rho}} \vec{f}^{(im.)} \Big|_{\vec{\rho}=\vec{\rho}_b=\vec{\rho}_{eq}}, incoh.; \\ (\nabla_{\vec{\rho}} + \nabla_{\vec{\rho}_b}) \vec{f}^{(im.)} \Big|_{\vec{\rho}=\vec{\rho}_b=\vec{\rho}_{eq.}}, coh., \end{cases} \qquad (8)$$

depends only on the transverse pipe geometry $\Lambda = Nq/2\pi R$ being the beam linear charge density. Introducing the betatron normal modes diagonalizes the Laslett tensor, yielding the normal-mode Laslett coefficients [2]:

$$\epsilon_{1,2} = \frac{\epsilon_{11} + \epsilon_{22}}{2} \pm \left[ \left( \frac{\epsilon_{11} - \epsilon_{22}}{2} \right)^2 + \epsilon_{12}\epsilon_{21} \right]^{1/2}. \qquad (9)$$

### 2.1 Image Force Potential

For coasting or relativistic bunched beams running parallel to the $z$-axis, $\vec{f}^{(im.)}$ can be computed in terms of electric and magnetic image potentials $\phi^{(im.)}$ and $\vec{A}^{(im.)} = A^{(im.)} \hat{u}_z$ as follows:

$$q^{-1} \vec{f}^{(im.)} = \nabla_t \left[ -\phi^{(im.)} + \beta_0 A^{(im.)} \right], \qquad (10)$$

* On leave of absence from Dip. Fis. Teor. e S.M.S.A., Univ. of Salerno, ITA.

[1] The equilibrium position $\vec{\rho}_{eq.}$ coincides with the chamber center of symmetry only in the absence of guiding fields.

[2] Here we assume for simplicity no H-V betatron coupling, in the absence of space-charge and image effects, as well as H-V symmetry.

Where $\phi$, $A$ are found by solving:

$$\nabla^2_{\vec{\rho}} \left\{ \begin{array}{l} \phi \\ A \end{array} \right. = 2\pi \left\{ \begin{array}{l} 1 \\ \beta_0 \end{array} \right. \Lambda\delta(\vec{\rho}-\vec{\rho}_b). \tag{11}$$

In general both image potentials contain static ($=$) as well as dynamic ($\sim$) terms. The boundary conditions to be imposed on the *static* and *dynamic* components of $\phi$ and $A$ are different. For the static components, the boundary conditions are (continuity of the tangential fields across the conducting liner and magnetic yoke surfaces $S_\ell$ and $S_Y$, respectively):

$$\left\{ \begin{array}{l} \phi_{=}|_{S_\ell} = const., \\ \left.\dfrac{\partial A_{=}}{\partial n}\right|_{S_{Y-}} = \mu_R^{-1} \left.\dfrac{\partial A_{=}}{\partial n}\right|_{S_{Y+}} . \end{array} \right. \tag{12}$$

For a liner made of *good* conductor of finite thickness, the *high frequency* spectral components of the (dynamic) potentials will *not* penetrate beyond the liner's wall, and the b.c. will be:

$$\phi_\sim|_{S_\ell} = const., \quad A_\sim|_{S_\ell} = const., \tag{13}$$

viz., $\hat{n} \times \vec{e}_\sim = \hat{n} \cdot \vec{b}_\sim = 0$ at $S_\ell$, respectively, whence, in view of (11):

$$A_\sim = \beta_0 \phi_\sim = \beta_0 \left[\phi(\vec{\rho}, \vec{\rho}_b) - \phi_{=}(\vec{\rho}, \vec{\rho}_{eq.})\right]. \tag{14}$$

The spectral components of the magnetic field *below* some critical frequency will penetrate beyond the liner's wall, and for these *penetrating* AC components, the b.c. will be the same as for the static term, viz. (12)[3]. In the incoherent regime, the (transverse) beam center of charge is *fixed* at $\vec{\rho}_b = \vec{\rho}_{eq.}$, so that both $\phi$ and $A$ are *static*. In the coherent regime the beam undergoes coherent (rigid, collective) transverse oscillations, and the fields contain *both* static and dynamic terms. It is seen from (14) that in the non-penetrating (high frequency) regime the image force is the same as in the incoherent regime in the limit of $\beta_0 \longrightarrow 1$.

## 2.2 Auxiliary Complex Potentials

The force can be conveniently derived from a complex potential, where, in general [2]:

$$\phi^{(im.)} - \beta_0 A^{(im.)} = 2\Lambda\ Re\ \bar{\Psi}(\bar{z}, \bar{z}_b, \bar{z}_b^*), \tag{15}$$

where $\bar{z} = (x+iy)/L$ is the (scaled) field-point, and $\bar{z}_b = (x_b + iy_b)/L$ the (scaled) source-point, $L$ being a problem-dependent scaling length (e.g., the pipe size). Let further:

$$\dot{\bar{\Psi}} = \bar{U}(\bar{z}_b) + \bar{V}(\bar{z}_b^*), \tag{16}$$

[3] In order to decide between the penetrating and non penetrating field regime at a given frequency $f$, we should compare the liner's wall thickness to the skin depth at that frequency $\delta = (\pi f \mu_w \sigma_w)^{-1/2}$, $\sigma_w$ and $\mu_w$ being the electrical conductivity and magnetic permeability of the liner's wall. A possible refinement would be to consider *partial* penetration of the fields through the pipe walls [4], [5].

where dots mean derivation w.r.t. the argument. Then, from eq.s (9), the following formulae are readily established [2]:

$$\epsilon_{1,2}^{(inc.)} = \pm\frac{1}{2}\left|\ddot{\bar{\Psi}}\right| \tag{17}$$

for the incoherent regime;

$$\epsilon_{1,2}^{(coh.,P)} = \frac{1}{2}\left\{-Re\dot{\bar{V}} \pm \left[\left|\ddot{\bar{\Psi}} + \dot{\bar{U}}\right|^2 - Im^2\dot{\bar{V}}\right]^{1/2}\right\} \tag{18}$$

for the (low frequency) coherent penetrating regime, and:

$$\epsilon_{1,2}^{(coh.,NP)} = \frac{1}{2}\left\{-(1-\beta_0^2)Re\dot{\bar{V}}_{el.} \pm \left[\left|\ddot{\bar{\Psi}} + (1-\beta_0^2)\dot{\bar{U}}_{el.}\right|^2 - (1-\beta_0^2)Im^2\dot{\bar{V}}_{el.}\right]^{1/2}\right\} \tag{19}$$

for the (high frequency) coherent non-penetrating regime, where $\bar{U}_{el.}$, $\bar{V}_{el.}$ denote the *electric* parts of $\bar{U}$, $\bar{V}$, and all derivatives are evaluated at $\bar{z} = \bar{z}_b = \bar{z}_{eq.}$. Equations (17) to (19) are not restricted to any special geometry, and thus provide a general framework for computing the normal mode (incoherent as well as coherent) Laslett coefficients for beam liners surrounded by a (coaxial) magnetic yoke.

# 3 RESULTS

The above formalism has been applied to a variety of cases of practical interest, e.g. for the LHC. As an example, the incoherent and coherent Laslett coefficients for a circular pipe in a coaxial magnetic yoke are shown in *Fig.s 1* to *4*.

This work has been sponsored in part by INFN through the Salerno University group.

# 4 REFERENCES


[1] L.J. Laslett, Proc. 1963 Summer Sch. BNL 7534.

[2] S. Petracca, Part. Acc., 48, p. 181, 1994.

[3] B. Houssais, Thesis, Univ. de Rennes, 1967.

[4] B. Zotter, CERN ISR-TH/72-8.

[5] V.I. Balbekov, Proc. EPAC-92, Berlin DBR, 1992.

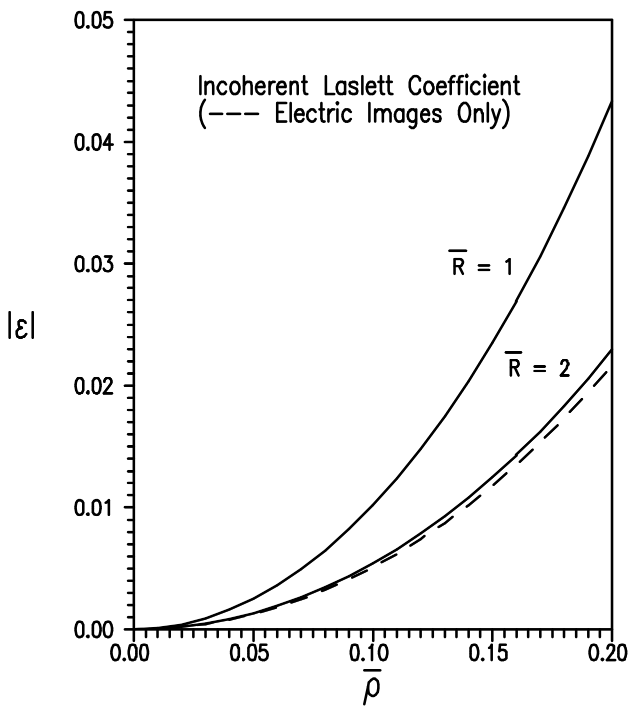


Figure 1: Incoherent Laslett Coefficient for circular liner within a circular bore magnetic yoke ($\mu_r$=5000); $\rho$ = scaled distance from axis. R = scaled yoke radius

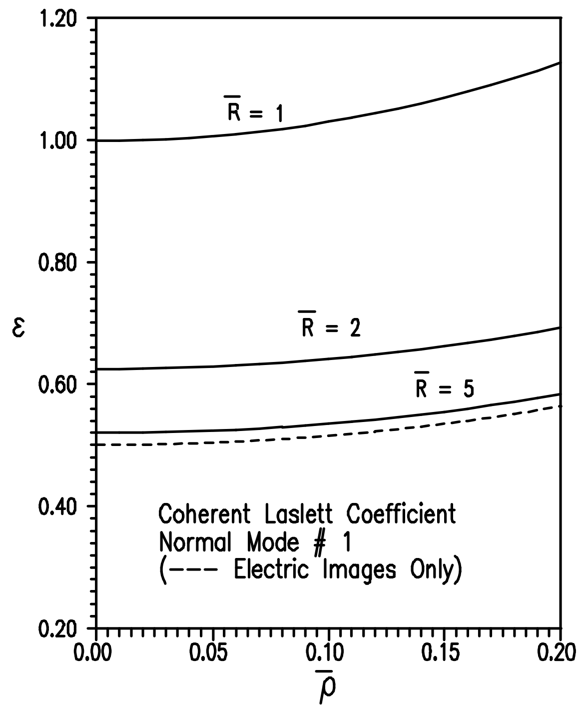


Figure 2: Coherent Laslett Coefficient, normal mode #1, for circular liner within a circular bore magnetic yoke ($\mu_r$=5000); $\rho$ = scaled distance from axis. R = scaled yoke radius

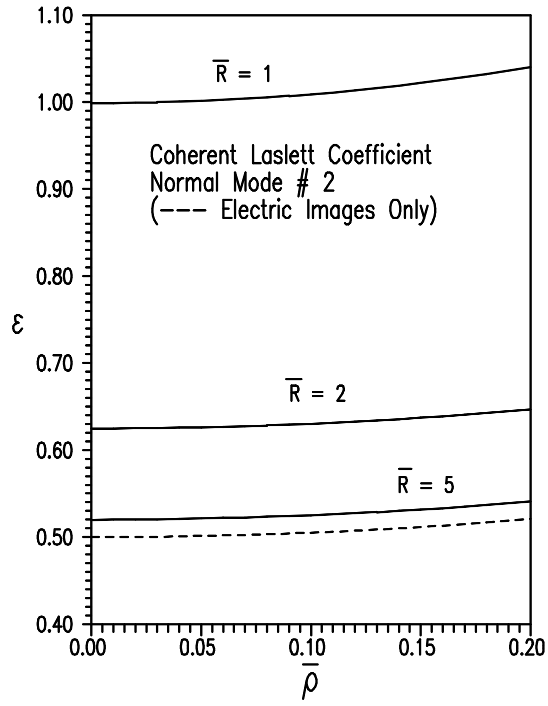


Figure 3: Coherent Laslett Coefficient, normal mode #2, for circular liner within a circular bore magnetic yoke ($\mu_r$=5000); $\rho$ = scaled distance from axis. R = scaled yoke radius

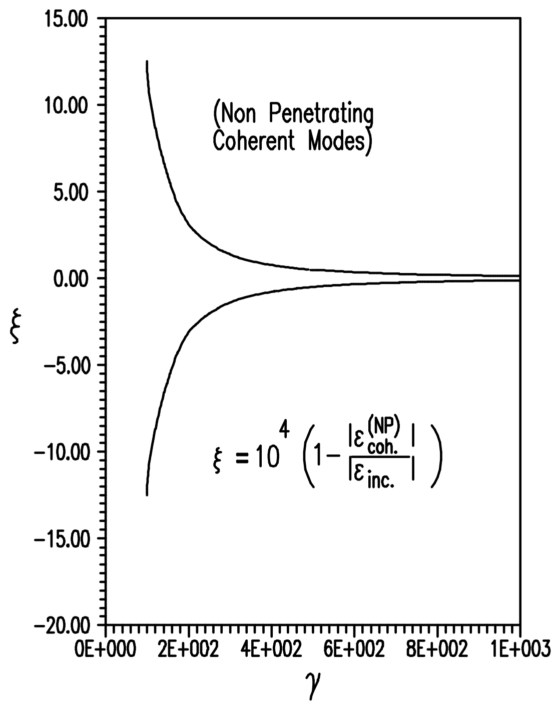


Figure 4: Coherent Laslett Coefficient, non-penetrating modes, for circular liner within a circular bore magnetic yoke ($\mu_r$=5000); $\rho$ = scaled distance from axis. R = scaled yoke radius