

MATHEMATICAL METHODS OF OPTIMIZATION OF CHARGED PARTICLE BEAMS DYNAMICS

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Abstract

At present, mathematical methods of modeling and optimization are extensively used in many fields of science and technology. Development of specialized software for various applications becomes of ever increasing importance. A special class of the problems attracting attention of numerous researches is represented by the problems associated with the beam dynamics formation in accelerators. The paper deals with optimization problems of charged particle beam in Linac. The theory of the optimal beam dynamics formation in both accelerating and focusing structures is suggested. Different mathematical control models describing beam dynamics are presented. Especially we consider the problems connected with taking into account of charged particles interaction. New classes of functionals are introduced for the estimation of beam dynamics. The optimization methods are developed for the presented functionals. They are used for the solving of various beam dynamics problems in different accelerating and focusing structures.¹

1 CONTROL PROBLEM

The problem of beam control of interacting particles, which dynamics is described by integro-differential equations, is considered. Let us assume that evolution of particle beam is described by equations [1, 2]

$$\frac{dx}{dt} = f(t, x, u), \quad (1)$$

$$\frac{\partial \rho}{\partial t} + \frac{\partial \rho}{\partial x} f(t, x, u) + \rho \operatorname{div}_x f(t, x, u) = 0 \quad (2)$$

with initial conditions

$$x(0) = x_0 \in M_0, \quad \rho(0, x) = \rho_0(x), \quad (3)$$

where

$$f = f_1(t, x, u) + \int_{M_{t,u}} f_2(t, x, y_t) \rho(t, y_t) dy_t.$$

Here t is the time; x is n -vector of phase coordinates; $u = u(t)$ is r -dimensional control vector-function; $\rho = \rho(t, x)$ is the particle distribution density in the phase space; f_1 is

n -dimensional vector-function determined by external electromagnetic fields; f_2 is n -dimensional vector-function associated with the particle interactions; the set $M_{t,u}$ is the cross-section of the trajectory set. It is obtained by time shift of the initial set M_0 through solutions of equation (1) with given control $u = u(t)$. The set M_0 is a given set in the phase space, which describes the set of initial states for a charged particle beam at the initial time moment. The function $\rho_0(x)$ is a given function describing the particle distribution density at the moment $t = 0$.

The equations (1) – (2) can be considered as Vlasov equations. We meet with these equations if interaction between particles, for example the Coulomb repulsion, is taken into account.

Let us introduce a functional

$$I(u) = \int_0^T \int_{M_{t,u}} \varphi(t, x_t, \rho(t, x_t)) dx_t dt + \int_{M_{T,u}} g(x_T, \rho(T, x_T)) dx_T \quad (4)$$

characterizing the dynamics of the process. Here φ and g are given non-negative functions, T is fixed.

Consider the minimization problem of functional (4). Analyzing various systems which are designed for acceleration, focusing and transporting of charged particle beams, it should be noticed that electrical and magnetic fields can be treated in a certain structural and parametric form. Thus certain components and parameters of electromagnetic fields and geometric systems of accelerating or focusing can be taken as control variables.

2 OPTIMIZATION METHODS

The directed methods of the optimal structures search are presented in [1, 2]. These methods are based on the following form of functional variation

$$\delta I = - \int_0^T \int_{M_{t,u}} (\psi^*(t, x_t) \Delta_u f_1(t, x_t, u) + \lambda(t, x_t) \Delta_u \operatorname{div}_x f_1(t, x_t, u)) dx_t dt, \quad (5)$$

where

$$\Delta_u f_1 = f_1(t, x_t, u(t) + \Delta(t)) - f_1(t, x_t, u(t)). \quad (6)$$

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Here and further * denotes transposition. The auxiliary functions ψ and λ satisfy the following equations

$$\begin{aligned} \frac{d\psi}{dt} = & - \left(\frac{\partial f(t, x(t), u(t))}{\partial x} + \right. \\ & E \operatorname{div}_x f(t, x(t), u(t)) \left. \right)^* \psi - \\ & \lambda \left(\frac{\partial \operatorname{div}_x f(t, x(t), u(t))}{\partial x} \right)^* + \\ & \left(\frac{\partial \varphi(t, x(t), \rho(t, x(t)))}{\partial x} \right)^* - \rho(t, x(t)) \times \\ & \int_{M_{t,u}} \left\{ \left(\frac{\partial f_2(t, y_t, x(t))}{\partial x} \right)^* \psi(t, y_t) + \right. \\ & \left. \lambda(t, y_t) \left(\frac{\partial \operatorname{div}_x f_2(t, y_t, x(t))}{\partial x} \right)^* \right\} dy_t, \end{aligned} \quad (7)$$

$$\begin{aligned} \frac{d\lambda}{dt} = & -\lambda \operatorname{div}_x f(t, x(t), u(t)) + \\ & \varphi(t, x(t), \rho(t, x(t))) - \\ & \frac{\partial \varphi(t, x(t), \rho(t, x(t)))}{\partial \rho} \rho(t, x(t)) \end{aligned} \quad (8)$$

along the trajectories of the system (1)–(2) with a given control $u = u(t)$ and terminal conditions

$$\psi^*(T, x(T)) = - \frac{\partial g(x(T), \rho(T, x(T)))}{\partial x}, \quad (9)$$

$$\begin{aligned} \lambda(T, x(T)) = & -g(x(T), \rho(T, x(T))) + \\ & \frac{\partial g(x(T), \rho(T, x(T)))}{\partial \rho} \rho(T, x(T)). \end{aligned} \quad (10)$$

Here

$$\operatorname{div}_x f = \sum_{i=1}^n \frac{\partial f}{\partial x_i}.$$

In the case when $\varphi = -\frac{\partial \varphi}{\rho} \rho \equiv 0$ and $g - \frac{\partial g}{\rho} \rho \equiv 0$ we have

$$\delta I = - \int_0^T \int_{M_{t,u}} (\psi^*(t, x_t) \Delta_u f_1(t, x_t, u) dx_t dt, \quad (11)$$

where $\psi(t, x(t))$ satisfies equation

$$\begin{aligned} \frac{d\psi(t, x(t))}{dt} = & - \left(\frac{\partial f(t, x(t), u(t))}{\partial x} + \right. \\ & E \operatorname{div}_x f(t, x(t), u(t)) \left. \right)^* \psi(t, x(t)) + \\ & \left(\frac{\partial \varphi(t, x(t), \rho(t, x(t)))}{\partial x} \right)^* - \rho(t, x(t)) \times \\ & \int_{M_{t,u}} \left(\frac{\partial f_2(t, y_t, x(t))}{\partial x} \right)^* \psi(t, y_t) dy_t. \end{aligned} \quad (12)$$

In the case $u = u(t, x)$ we can obtain analogous formulas.

As it was mentioned above formulas (5), (11) for the functional variation allow us to construct various methods of searching for optimal control. If the control u is defined by some set parameters, then we can obtain formula for gradient of the functional (4).

3 MODELING OF BEAM DYNAMICS

Consider several control problems related to the formation of required accelerating and bunching regimes for charged particle beams. First describe the problem of particles capture in the accelerating mode. Consider, for definiteness, the particle dynamics in the field of traveling-wave. Without taking into account the interaction between the particles the equations of the longitudinal motion have the following form

$$\begin{aligned} \frac{d\gamma}{d\xi} = & -\alpha(\xi) \sin \varphi, \\ \frac{d\varphi}{d\xi} = & 2\pi \left(\frac{1}{\beta_{ph}(\xi)} - \frac{1}{\sqrt{\gamma^2 - 1}} \right), \end{aligned} \quad (13)$$

where $\gamma = W/W_0$ is the reduced particle energy; φ is the particle phase; ξ is the reduced distance; $\alpha(\xi)$, $\beta_{ph}(\xi)$ are the reduced parameters of the accelerating wave.

Let \overline{M}_0 be the set of initial energies and phases (γ_0, φ_0) for the system (13). Assume that $\varphi \in [-\pi, \pi]$. Denote by $\gamma(\xi, \gamma_0, \varphi_0)$, $\varphi(\xi, \gamma_0, \varphi_0)$ the solution of the system (13) with the initial data (γ_0, φ_0) . Then the capture of the particles with the initial values from \overline{M}_0 into the acceleration mode means that for arbitrary $(\gamma_0, \varphi_0) \in \overline{M}_0$ the phase $\varphi(\xi, \gamma_0, \varphi_0) \in [-\pi, \pi]$ for all $\xi \in [0, L]$. Here L is the reduced length of the accelerator. Call \overline{M}_0 the set of capture into acceleration mode. Let the set \overline{M}_0 be such that any subset is the set of capture into acceleration mode and any set including \overline{M}_0 and not identical to it does not fit the requirement. Call \overline{M}_0 the maximal set of capture into acceleration mode. Its area is called the longitudinal acceptance of accelerating structure.

Assume

$$\begin{aligned} I_1(\alpha, \beta_{ph}) = & \int_0^L \int_{\overline{M}_{\xi, \alpha, \beta_{ph}}} \Phi(\xi, \gamma_\xi, \varphi_\xi) d\gamma_\xi d\varphi_\xi d\xi + \\ & \int_{\overline{M}_{L, \alpha, \beta_{ph}}} G(\gamma_L, \varphi_L) d\gamma_L d\varphi_L. \end{aligned} \quad (14)$$

The functions $\alpha(\xi)$ and $\beta_{ph}(\xi)$ are used as control. Let us introduce the function

$$g(\omega, \omega_1, \omega_2) = \begin{cases} (\omega - \omega_1)^2, & \omega < \omega_1; \\ 0, & \omega \in [\omega_1, \omega_2]; \\ (\omega - \omega_2)^2, & \omega > \omega_2. \end{cases} \quad (15)$$

Suppose

$$\begin{aligned} \Phi(\xi, \gamma, \varphi) &= C_1 g(\varphi, -\pi, \pi), \\ G(\gamma, \varphi) &= C_2 g(\gamma, \gamma_{\min}, \gamma_{\max}), \end{aligned} \quad (16)$$

where C_1 and C_2 are the weight constants.

Functional (14) with the functions Φ and G defined by (16) are non-negative. They are equal to zero in the case when \overline{M}_0 is the set of capture in acceleration mode and the output energies of particles are within the interval $[\gamma_{\min}, \gamma_{\max}]$. In this case $\Delta = \gamma_{\max} - \gamma_{\min}$ defines the width of the particle energy spectrum.

Thus minimizing the functional (14) one can provide the required particle capture into the acceleration mode with prescribed accelerating rate. Varying the values γ_{\min} and γ_{\max} one can change acceleration rate and energy spectrum of the output beam. Extending the set \overline{M}_0 one can solve longitudinal acceptance maximization problem.

To analyze a longitudinal motion with space charge, we consider the model of large-scale particles. We model the beam by assembly of thick disks of radius R and $2d$ in thickness. Let us write out equations for the traveling wave field

$$\frac{d\xi}{d\tau} = \frac{p}{\sqrt{1+p^2}}, \quad (17)$$

$$\frac{dp}{d\tau} = -\alpha(\xi) \sin \varphi + \int_{M_{\tau, \alpha, \beta_{ph}}} f_2(\xi, p, \xi'_\tau, p'_\tau) \rho(\tau, \xi'_\tau, p'_\tau) d\xi'_\tau dp'_\tau. \quad (18)$$

Here $p = \gamma\beta$ is the reduced impulse; $\tau = ct/\lambda$;

$$\varphi(\tau) = 2\pi \int_0^\tau \left[\frac{\beta(\tau)}{\beta_{ph}(\xi(\tau))} - 1 \right] d\tau + \varphi_0, \quad (19)$$

$$f_2(\xi, p, \xi'_\tau, p'_\tau) = \frac{e}{m_0 c^2} \frac{\lambda Q a^2}{2\epsilon_0 \pi R^2 d^2} \times$$

$$\sum_{m=1}^{\infty} \left\{ \frac{J_1^2(\lambda_m R/a)}{\lambda_m^4 J_1^2(\lambda_m)} [2f_m(\sqrt{1+p_\tau'^2}(\xi - \xi'_\tau)\lambda) -$$

$$f_m(\sqrt{1+p_\tau'^2}(\xi - \xi'_\tau)\lambda - 2d) -$$

$$f_m(\sqrt{1+p_\tau'^2}(\xi - \xi'_\tau)\lambda + 2d)] \right\},$$

$$f_m(y) = \text{sign}(y)(1 - e^{-\lambda_m |y|/a});$$

a is the channel aperture; Q is total beam charge; λ is the wave length in the free space; e is the charge of particle; m_0 is the rest mass of particle; c is the speed of light; ϵ_0 is the electric constant; J_1 is the Bessel function of the first order; λ_m are the roots of zero order Bessel function, i.e. $J_0(\lambda_m) = 0$.

Here we assumed that

$$\int_{M_0} \rho(\xi_0, p_0) d\xi_0 dp_0 = 1.$$

When integrating (17), (18) for computation of the phase φ it may be wise to consider the differential equation

$$\frac{d\varphi}{d\tau} = 2\pi \left(\frac{p}{\sqrt{1+p^2}} \frac{1}{\beta_{ph}(\xi)} - 1 \right)$$

instead of equality (19) and then integrate it together with the preceding equations.

By analogy with functional (14) we may introduce the corresponding functional in this case, too.

A number of different problems of beam physics can be solved with the help of mentioned above or analogous functionals. Some applications of this approach are considered in [1 – 4].

4 CONCLUSION

Problems of beam dynamics optimization are stated as problems of trajectory bundle control. An extensive class of dynamic systems and problems under study enables solutions of a variety of accelerator-theory problems, related to the forming of optimal beam dynamics in the accelerating and focusing structures.

5 REFERENCES

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