

SOLITARY WAVES IN PARTICLE BEAMS

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ABSTRACT

Since space charge waves on a particle beam exhibit both dispersive and nonlinear character, solitary waves or solitons are possible. Dispersive, nonlinear wave propagation in high current beams is found to be similar to ion-acoustic waves in plasmas with an analogy between Debye screening and beam pipe shielding. Exact longitudinal solitary wave propagation is found for potentials associated with certain transverse distributions which fill the beam pipe. For weak dispersion, the waves satisfy the Korteweg-deVries (KdV) equation, but for strong dispersion they exhibit breaking. More physically realizable distributions which do not fill the beam pipe are investigated and shown to also satisfy a KdV equation for weak dispersion if averaging over rapid transverse motion is physically justified. Scaling laws are presented to explore likely parameter regimes where these phenomena may be observed experimentally.

1 SOLITARY WAVES AND SOLITONS

Nonlinearity in wave propagation typically leads to steepening phenomena. For example, consider the simple wave equation

$$u_t + (1+u)u_x = 0 \quad (1)$$

which has the implicit solution [1]

$$u(x,t) = f(x - (1+u)t) \quad (2)$$

where f is an arbitrary differentiable function. Note that the velocity, $(1+u)$, depends on the amplitude, and in particular, higher amplitudes propagate faster. If f describes a localized distribution, the peak value will tend to overtake lower values, and steepening and breaking of the pulse and formation of a shock will result. On the other hand, if the velocity depends strongly on wavelength (dispersion), a localized distribution tends to spread. A solitary wave results when the nonlinear steepening is canceled by the dispersive spreading, yielding a localized disturbance which propagates without distortion. Since solitary waves of different heights will generally travel with different velocities collisions can occur. The term soliton describes solitary waves which maintain their identity and shape after collision.

2 SPACE CHARGE WAVES

Space charge forces can produce longitudinal density waves in low momentum spread, charged particle beams. For a uniform, non-relativistic beam of radius a in a uniform beam pipe of radius b , the propagation is nondispersive in the linear, long wavelength approximation. The wave velocity v_p is

$$v_p = \frac{\omega}{k} = \sqrt{\frac{e^2 \lambda_o g}{4\pi\epsilon_o m}} \quad (3)$$

where ω is the mode frequency for wave number k , e is the electron charge, λ_o is the unperturbed linear charge density, g is a geometric factor depending on particle distribution and pipe radius, ϵ_o is the permittivity of free space, and m is the mass of the beam particles. The associated force is given by

$$F = -\frac{ge^2}{4\pi\epsilon_o} \frac{\partial \lambda}{\partial z} \quad (4)$$

for density λ . In k -space, the spatial Fourier transform $\tilde{F} \propto ik\tilde{\lambda}$. More generally, the Green's function for a cylindrically symmetric pipe is

$$G(\rho, z; \rho', z') = \frac{1}{4\pi\epsilon_o} \frac{2}{\pi b^2} \int_{-\infty}^{\infty} dk \sum_{n=1}^{\infty} (-ik) \times e^{ik(z-z')} \frac{J_o\left(\frac{x_n \rho}{b}\right) J_o\left(\frac{x_n \rho'}{b}\right)}{\left(\left(\frac{x_n}{b}\right)^2 + k^2\right) J_1^2(x_n)} \quad (5)$$

where x_n is the n^{th} zero of the Bessel function J_o .

Consider a distribution of the form $J_o(x_1 \rho/b) \exp(ikz)$. In a linearized fluid model, this function describes a perturbation eigenmode of a uniform beam filling the beam pipe. The underlying force law is modified from

$$ik \rightarrow \frac{ik}{1 + \alpha k^2} \quad (6)$$

where $\alpha = (b/x_1)^2$. The phase velocity

$$v_{\text{phase}} = \frac{v_p}{\sqrt{1 + \alpha k^2}} \quad (7)$$

now shows wavenumber dependence or dispersion. Expansion of the denominator of (6) for small α generates a third derivative term ($-ik^3$) which is suggestive of the structure of the KdV equation,

$$u_t - 6uu_x + u_{xxx} = 0 \quad (8)$$

In fact the force law of (6) represents a simple exponential rolloff in position analogous to Debye screening in plasma ion-acoustic waves where solitary-wave behavior has been modeled and observed [2]. Additionally, simulation studies of water waves [3], which are driven by the same basic force law, show soliton-like behavior; i.e., preservation of identity after collision.

3 1-D NONLINEAR FLUID MODEL

As a first step in understanding the interplay of nonlinearity and dispersion for space-charge-dominated beams, a 1-D nonlinear cold fluid model [4] of a uniform beam with the force law given in relation (4) is described. Some possibly important transverse effects may be lost, but this solvable model provides a good footing for further considerations. The fluid equations are

$$\frac{\partial n}{\partial t} + \frac{\partial}{\partial x}(nv) = 0 \quad (9)$$

$$\frac{\partial v}{\partial t} + v \frac{\partial v}{\partial x} = -\frac{\partial \Phi}{\partial x} \quad (10)$$

$$\tilde{\Phi}(k) = \frac{\tilde{n}_1}{1 + \alpha k^2} \quad (11)$$

$$n(x, t) = n_o + n_1(x, t) \quad (12)$$

where n is the density; n_o , the unperturbed density; n_1 , the perturbed density; v , the velocity; and Φ , the potential. These equations are normalized to $v_p = \sqrt{n_o}$. At this point we can parallel Davidson's discussion of ion-acoustic solitary waves [5], and look for solutions of the form $n_1(qx - \omega t)$, $v(qx - \omega t)$, etc. which rolloff at $\pm\infty$. Equations (9)-(11) imply that

$$n = \frac{n_o}{1 - \frac{q}{\omega}v} \quad (13)$$

$$\left(\frac{\omega}{q}\right)^2 = \left(\frac{\omega}{q} - v\right)^2 + 2\Phi \quad (14)$$

Bringing the denominator of the right hand side of equation (11) to the left hand side yields the second order differential equation

$$\Phi - \alpha q^2 \Phi'' = n_1 = \frac{n_o}{\sqrt{1 - 2\left(\frac{q}{\omega}\right)^2 \Phi}} - n_o \quad (15)$$

On integrating, we finally have

$$\frac{\alpha q^2 (\Phi')^2}{2} - \frac{\Phi^2}{2} - n_o \left(\frac{\omega}{q}\right)^2 \left(1 - 2\left(\frac{q}{\omega}\right)^2 \Phi\right)^{1/2} - n_o \Phi = -n_o \left(\frac{\omega}{q}\right)^2 \quad (16)$$

where the integration constant has been chosen to yield a localized solution with the derivative going to zero at infinity. Periodic solutions also exist. This relatively simple equation has been integrated numerically for Φ to yield the pulse shape of the self-consistent solitary waves as a function of the parameter ω/q , the pulse velocity in units of v_p . For the potential, the pulse shape is reminiscent of the sech^2 behavior of solutions of the KdV equation. In fact, a standard small parameter expansion [5] of these fluid equations with $(\omega/q - 1)$ as the small parameter yields the KdV equation. The peak values of Φ , n , and v are given by

$$\Phi_{\text{peak}} = 2\left(\frac{\omega}{q} - 1\right) \quad (17)$$

$$n_{\text{peak}} = \frac{n_o}{2\frac{q}{\omega} - 1} \quad (18)$$

$$v_{\text{peak}} = 2\left(\frac{\omega}{q} - 1\right) \quad (19)$$

where the parameter $\sqrt{\alpha}$, v_p , and mv_p^2 provide the length, velocity, and potential scales, respectively. Approximate full width half maximum values for a few sample ω/q are given in Table 1 in units of $\sqrt{\alpha}$.

Table 1
Widths of Perturbations

Velocity ω/q	FWHM widths		
	Φ	v	n
1.25	4.6	4.1	3.2
1.50	3.0	2.3	1.3
1.75	2.4	1.5	0.34
1.85	2.2	1.2	0.14

Note that the potential Φ tends to the width 2.0 as the strength parameter ω/q tends to 2.0. The velocity also remains finite, but the density becomes singular in this limit and indicates that breaking and shock formation may occur. This is in contrast to the KdV equation where the third-derivative term does not provide a high frequency rolloff due to dispersion.

4 TOWARD MORE REALISTIC CONFIGURATIONS

In Section 3, a particularly simple model was chosen because it allowed a formulation in terms of an ordinary differential equation. For more general Green's functions, one must expect an integral or integrodifferential equation to result. As a first step in seeing whether soliton-like behavior is expected for a realizable particle beam, we consider to what extent the form of expression (6) is a reasonable approximation to the field behavior.

Consider a transversely uniform beam of radius a in a beam pipe of radius b . The Fourier-transformed potential at radius r is given by

$$\tilde{\Phi} = \frac{4\tilde{n}(k)}{a^2 k^2} [1 - I_o(kr) [kaK_1(ka) + kaI_1(ka) \frac{K_o(kb)}{I_o(kb)}]] \quad (20)$$

where the K and I are the usual modified Bessel functions. As is well known, for small k , the field on axis reduces to the long wavelength limit

$$(1 + 2\log(b/a))\tilde{n}(k) \quad (21)$$

and for large k to

$$\frac{4}{k^2 a^2} \tilde{n}(k) \quad (22)$$

This suggest that on axis the exact expression (20) can be approximated by

$$\tilde{\Phi} \cong \frac{1 + 2\log(b/a)}{1 + \frac{k^2 a^2}{4} (1 + 2\log(b/a))} \tilde{n}(k) \quad (23)$$

In fact on axis, for $a/b=0.5$, the functions match within several percent for all k . The corresponding length scale parameter is given by

$$\sqrt{\alpha} = \frac{a}{2} \sqrt{1 + 2\log(b/a)} \quad (24)$$

Off axis, one expects the $(1 + 2 \log(b/a))$ term to be modified by an additional $(-r^2/a^2)$. In addition, it is found necessary to introduce extra k dependence in the denominator of the form

$$1 + \alpha k^2 (1 + \kappa \exp(-\gamma|k|)) \quad (25)$$

to yield fits at the several percent level, where κ and γ are fitting parameters. The $\exp(-\gamma|k|)$ will induce spatial smearing with a factor $1/(x^2 + \gamma^2)$. If the r -dependence is averaged over the assumed uniform beam profile, the equation analogous to (15) has Φ'' replaced by $(\Phi'' + \kappa(\Phi_{\text{smear}})')$. For $a/b = 0.5$, $\kappa \approx 0.3$ and the

smearing length $\gamma \approx b/10$. In this case, pulses of the order of a should remain well described by the simple differential equation model. The KdV equation remains the weak dispersion limiting equation. However, it is clear that a rigorous description requires dealing with integral or integrodifferential equations.

A second question involves the general appropriateness of averaging over transverse motion. The observable desired is a large amplitude density pulse traveling undistorted at velocities exceeding the linear wave velocity. From Table 1 and equation (24), it is expected to have a length of the order of the beam dimension. A typical time scale for averaging would then be a/v_p , the time necessary for exchange of longitudinal energy along the beam as the pulse moves to a new region. In this time, particles in the beam will have undergone $(v_b/L)(a/v_p)$ betatron oscillations where L is the betatron wavelength and v_b is the beam velocity. Averaging then is reasonable if there has been significant betatron motion as the pulse moves its width, or

$$a \geq L \frac{v_p}{v_b} \quad (26)$$

For example, for a beam size of 1 cm and $(v_p/v_b) = 10^{-3}$ a betatron period less than 10 meters is necessary. Note that the ratio (v_p/v_b) is of the order of the required longitudinal momentum spread for coherent longitudinal motion. Such values are not unreasonable, but do require a long transport system for observation. Clearly, the precise weighting of the average remains an open issue.

Wang, et al. [6] have observed that for space-charge dominated beams (where the beam size is not determined by the emittance, but by cancellation of the space-charge and external focusing forces), longitudinal waves propagate by increasing the beam radius rather than by increasing the local 3-D density. The resulting long wavelength limit is $g = \log(b/a)$ because there is no transverse variation of the longitudinal field within the beam. They have confirmed the conjecture experimentally. This effect suggests that transverse averaging may not be important in this regime. However, the pulse lengths of their experiment were relatively long compared to the beam diameter and thus not strongly dispersive. Also, the time scales involved are long relative to the plasma frequency, which for stable beam motion is comparable to the betatron frequency. More importantly, numerical evaluation of the associated Green's function shows that at short wavelengths the fields do in fact have radial dependence.

A contrasting regime exists where only a small fraction of a betatron oscillation occurs during motion of the pulse. The transverse motion is effectively frozen, and the appropriate equation for possible solitons would be of the form

$$\Phi = \int r' dr' h(x, r; x', r') \left(\frac{n_0}{\sqrt{1 - 2(\frac{q}{\omega})^2 \Phi(x', r')}} - n_0 \right) \quad (27)$$

where h is the potential Green's function. If a particular solution Φ implies large transverse fields, transverse oscillations may be important for a complete picture.

Other machine impedances such as resistive wall and chamber discontinuities should be included to complete the picture of nonlinear wave propagation. For small wall resistance, a slow exponential growth of the pulses is expected [7, 8]. In addition, sequences of solitons may form [9]. A series of localized resonators will generate a $(1/k)^2$ short wavelength behavior of the potential, but since the resistive effects are no longer small, extrapolations from the results presented are not as straightforward.

5 CONCLUSIONS

A case has been made that longitudinal collective effects, particularly those generated by space charge, can exhibit solitary waves and possibly solitons. Conversely, the solitary wave concept should be a useful tool in understanding nonlinear waves on particle beams. Equations (24) and (26) delineate a likely regime for observing such phenomena.

ACKNOWLEDGMENTS

This work derives in part from a collaboration with Kurt Riedel more than a decade ago when he was a summer college intern at Lawrence Berkeley Laboratory. I also would like to thank a more recent College of William and Mary student, Craig Kruschwitz, for performing some of the numerical experiments. This work was supported by the U.S. Department of Energy under contract number DE-AC05-84ER40150.

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